1.2.3 Classification of States

Definition 3 *In a MC, a state j is said to be* ***accessible*** *from state i if there is a value for $n \geq 0$ s.t. $P_{ij}^{(n)} > 0$.*

A straightforward corollary of this definition is the following:

Theorem 1 *If j is NOT* ***accessible*** *from state i, then given that the process starts in i, the probability that the process will ever enter j is zero.*

Proof.

$$\begin{aligned} P(\text{ever enter } j|\text{start in } i) &= \lim_{n\to\infty} P(\bigcup_{k=1}^{n} X_k = j|X_0 = i) \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{n} P(X_k = j|X_0 = i) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} P_{ij}^{(k)} \\ &= 0 \end{aligned}$$

Definition 4 *If state j is accessible from state i and state i is accessible from state j, then i and j are said to* ***communicate****, denoted $i \leftrightarrow j$.*

Show that, if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

If all states communicate in a MC, this MC is said to be ____________________.

Ex: Consider the Markov Chain models for the 2 Umbrella Problem, the 202 Problem, the Spider-and-Insect Problem, and the Random Walk. Which of these chains are irreducible?

Definition 5 *A state i is called recurrent if, starting in i, the probability of ever returning to i is one. Otherwise, state i is transient.*

Properties:

- Let $A(i)$ be the set of states accessible from i. State i is recurrent only if i is accessible from any $j \in A(i)$.
- If i is recurrent, the process will reenter state i *infinitely often*.
- If state i is transient, the total number of times that i is visited, until it is left for good, is a Geometric random variable (and thus it is finite with probability 1) (Why).
- If i is recurrent, then all $j \in A(i)$ are recurrent. $\{i\} \cup A(i)$ forms a **recurrent class**.
- In a Markov Chain with a finite set of states, not all states can be transient.
- The set of states can be partitioned as $S = T \bigcup C_1 \bigcup C_2 \ldots$ where T is the set of transient states, and $C_i's$ are recurrent classes.

- If a finite-state MC is irreducible, it consists of a single recurrent class.

Ex: In the MC examples considered above, list transient states and recurrent classes.

Ex: Consider the random walk on the set of integers, with probability q of going to the right, and $1-q$ of going to the left. Note that all states are either recurrent, or all states are transient (Why?). (For those who are interested: What is the condition for all states to be transient? There is an argument that uses the solution of the "Cliff hanger" problem).

Definition 6 *Let* $d_i = gcd\{n,\ s.t.\ P_{ii}^{(n)} > 0\}$. *If* $d_i > 1$, *state* i *is periodic with period* d_i.

Corollary 1 *All states in a recurrent class have the same period.*

Ex: Provide examples of Markov Chains with periods 2 and 3.

Argue that, if $P_{ii} > 0$ for and $i \in S$, the MC cannot be periodic.

Ex: Consider the Markov Chain given by the following transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$$

Study P^n for several values of n. What do you observe? Do you think the entries of P^n will converge as n increases?

Ex: Now consider the Markov Chain given by the following transition probability matrix:

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix}$$

Study P^n for several values of n. Do you think the entries of P^n will converge as n increases? What does each column converge to?

Ex: Now consider the Markov Chain belonging to the "Taking 202" problem. Do you think the entries of P^n will converge as n increases? What does each column converge to?