Birth-death Chains:

Ex: Random Walk with Reflecting Barriers:

Ex: Geo/Geo/1 Queue:

Mean First Passage and Recurrence Times

Imagine getting a unit reward whenever we visit state i. Every time state i is visited, the process is renewed. Let us name the duration (that is, the number of transitions) between any two consecutive reward instants an inter-reward epoch. As the process is renewed after each reward, the durations of these epochs are independent. By the same argument, they each have the same distribution. In other words, they are IID. Then, the average of epoch durations satisfies the Strong Law of Large Numbers. This means that the long term average of the time between rewards is equal to its expectation, with probability 1. This expected time is the Expected Recurrence Time for state i. With this argument, we obtain:

$$lim_{n\to\infty}\frac{R_{ij}(n)}{n} = \frac{1}{M_i}, \text{ with probability 1.}$$

Combining this with (1.2), we obtain:

$$\pi_i = \frac{1}{M_i}$$

In the rest, we will explore how to compute the mean recurrence time of a recurrent state.

Ex: Consider the Rain/No Rain example, with $P_{11} = 0.8$, and $P_{22} = 0.4$. Compute the mean number of dry days between any two rainy days.

Ex: Consider the given Markov Chain containing a single recurrent class, and some transient states.

(a) Compute the mean first passage time to state 1, starting at any given state i at time $n = 0$. (Hint: Turn state 1 into an absorbing state, and write a set of equations for the expected time for absorption, starting from each state other than 1)

(b) Compute the mean recurrence time for state 1. (Hint: Use your answers to the previous problem. Why do you not need to include any of the transient states in the computation?)

1.3 The Exponential Distribution

An exponential r.v. has the following PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0, & \text{o.w.} \end{cases},$$

where λ is a positive parameter.

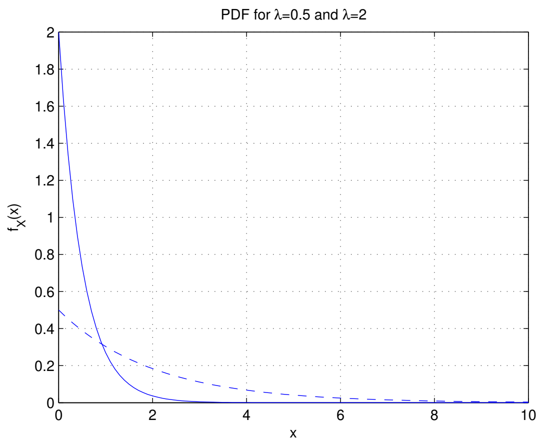

Figure 1.5: Exponential PDF

Memorylessness: Show that the exponential random variable is memoryless. That is, let X be an exponential with parameter λ. Given $X > t$, find $\mathrm{P}(X > t + x)$.)

Ex: Suppose you are waiting at the bus stop to get onto the next bus. Would you rather the bus inter-arrival times be exponential with mean 10 minutes, or exactly 10 minutes?

Racing Exponentials: Show that the minimum of k independent exponential random variables is an exponential random variable, with parameter ....

Ex: Suppose you arrive at a bank that has two tellers, and suppose that both tellers are busy when you arrive, but . there is no-one else in queue. Somebody else arrives a right after you, while you are waiting. If service times for teller i are exponential with rate λ_i, i = 1,2, find the expected time you will spend at the bank, under each of the following strategies:

(a) You pick a teller at random and wait until the customer being served by that teller leaves.

(b) You will queue up in front of both tellers, and join the first teller that gets freed up.