2.2 Counting Processes

Consider arrivals (of busses, customers, photons, e-mails, etc) occurring at random points in time. A Counting Process is a model for such an arrival process, that counts the number of arrival events by time t.

A counting process $\{N(t), t \geq 0\}$ is an integer valued random process, such that $N(0) = 0$, $N(\tau) \geq N(t)$ for all $\tau \geq t$.

Ex: Sketch a sample path of a counting process and notice that it looks like a staircase.

Let $X_i, i \geq 1$ be the times between arrival events. We will refer to these as "inter-arrival times". We can alternatively define the counting process $N(t)$ by specifying these inter-arrival epochs.

We can also express the process in terms of the times of the arrivals, S_1, S_2, ... such that $S_i \leq S_{i+1}$. Let $S_0 = 0$ for completeness.

Ex: Mark the S_i's in the sketch you made above. Also mark the inter-arrival times $X_{i+1} = S_{i+1} - S_i$. If these are continuous random variables, notice that each jump in the staircase has height 1, with probability 1.

Definition 7 *Let $\{S_n, n \geq 1\}$ be the arrival times in the counting process. Note that $S_n = \sum_{i=1}^{n} X_i$.*

Definition 8 *A counting process is a sequence of non-negative random variables $\{X_i, i \geq 1\}$ where $X_i \geq i \; \forall \; i$.*

Hence, the process is alternatively characterized by $\{N(t), t \geq 0\}$, or $\{S_n, n \geq 1\}$, or

$\{X_i, i \geq 1\}$. We make the following observations which will help in solving problems using counting process models.

Properties

- $\{S_n \leq t\} \Rightarrow \{N(t) \geq n\}$, and $\{N(t) \geq n\} \Rightarrow \{S_n \leq t\}$
  - Show that this is true.

- $\{S_n > t\} \Rightarrow \{N(t) < n\}$, and $\{N(t) < n\} \Rightarrow \{S_n > t\}$
  - This is a corollary of the above, because

$$\{S_n > t\}^c \equiv \{S_n \leq t\} \Leftrightarrow \{N(t) \geq n\} \equiv \{N(t) < n\}^c$$

- $\{S_n \geq t\} \Rightarrow \{N(t) \leq n\}$, but $\{N(t) \leq n\}$ does not imply $\{S_n \geq t\}$
  - Counter example:

- $\{S_n < t\} \Rightarrow \{N(t) \geq n\}$, but $\{N(t) \geq n\}$ does not imply $\{S_n < t\}$
  - Note that $\{N(t) \geq n\} = \{S_n < t\} \bigcup \{S_n = t\}$