2.3 The Poisson Process

Definition 1 of the Poisson Process: *A counting process is called a Poisson Process if the times between arrivals are IID, Exponential random variables.*

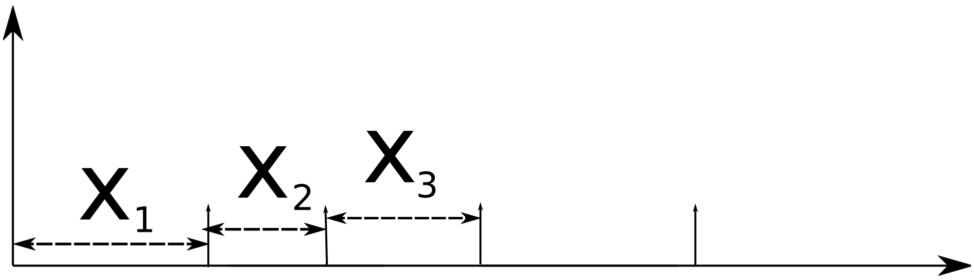

Figure 2.2: Inter-arrival times X_1, X_2, X_3 in a Poisson process

In other words, in a Poisson process of rate λ, the inter-arrival times $\{X_i\}$, $i \geq 1$ are independent and Exponential with rate λ. Note that the mean time between two arrivals is $\frac{1}{\lambda}$.

Ex: I am waiting for the bus, and bus arrivals are known to be a Poisson process at rate 1 bus per 10 minutes. Starting at time $t = 0$, what is the expected time until the arrival of the third bus?

The memorylessness property of the exponential distribution carries on to the Poisson process, which means it starts fresh at any point in time. We make this precise in the following exercise.

Ex: Distribution of residual time: Following an arbitrary amount of time $t > 0$ after an arrival in the Poisson process, let R be the duration until the next arrival. This is called the "residual time". Show that R has the same distribution as a regular inter-arrival time. This is referred to as the "fresh-start" property of the Poisson process.

Ex: Given that I arrive at the bus stop at $t = 19$ and learn that I have missed the second bus by two minutes, how much do I expect to wait?

As a consequence of the fresh-start property, one can easily see that the numbers of arrivals in any two disjoint intervals are independent. We say that the Poisson process has the "independent increments" property.

Definition 9 *A counting process is said to have "independent increments" if, for any* $t_1 < t_2 \leq t_3 < t_4$, $N(t_2) - N(t_1)$ *is independent of* $N(t_4) - N(t_3)$.

Another useful property of the Poisson process (that follows from the fresh-start property) is the following:

Definition 10 *A counting process is said to have "stationary increments" if the number of arrivals in a time interval depends only on the length of the interval (and not on where the interval is). That is, for any* $\tau > 0$, *the distribution of* $N(t + \tau) - N(t)$ *does not depend on the value of* t, *and is the same as the distribution of* $N(\tau) - N(0) = N(\tau)$.

Ex: Compute $P(N(t) < 1)$. Compute $P(N(t + \delta) - N(t) < 1)$. Use this to compute the probability that there is at least one arrival in an interval of size δ. How does this probability scale with δ, as $\delta \to 0$?

Definition 11 *A function $f(\cdot)$ is said to be $o(\delta)$ if it decays faster than linearly as its argument goes to zero, that is,*

$$\lim_{\delta \to 0} \frac{f(\delta)}{\delta} = 0$$

Ex: Which of the following functions are $o(\delta)$?

(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3 + cx$, where c is a real valued constant

(d) $f(x) = ah(x) + bg(x)$, where $h(\cdot)$ and $g(\cdot)$ are both $o(\delta)$, a and b are constants.

We can use the observations above to make an alternative definition for the Poisson process. We will show later that this definition is equivalent to Definition 1 of the Poisson process.

Definition 2 of the Poisson Process: *A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if $\{N(0) = 0\}$, if the process has stationary and independent increments, and also satisfies the following:*

(i) $P(N(\delta) = 1) = \lambda\delta + o(\delta)$, and

(ii) $P(N(\delta) \geq 2) = o(\delta)$.

We can use the above to discover that $N(t)$ is Poisson distributed with rate λt. We can do this through writing the distribution of the number of arrivals in a large interval, and using the Poisson approximation for the Binomial, or through computing the MGF as we will do below. First, let's recall the Poisson distribution:

Definition 12 *A random variable Y is said to have the Poisson Distribution with mean a, if*

$$\mathbf{P}(M = k) = \frac{a^k e^{-a}}{k!},\ k = 0, 1, \ldots$$

Show that the Moment Generating function of a Poisson random variable Y, with mean a, is given by

$$M_Y(s) = e^{a(e^{-s}-1)}$$

Ex: Compute the MGF of the number of arrivals, $N(t)$, by time t, in the Poisson process whose rate is λ.

We have found that $E[\exp(-sN(t)] = e^{\lambda t(e^{-s}-1)}$. This is the MGF of a Poisson random variable with mean λt. Recalling that a MGF fully determines the corresponding distribution, we conclude that in a Poisson process with rate λ, the number of arrivals by time t is Poisson with mean λt. This explains why λ is the "rate" of the process. Now, by the stationary increments property, we conclude that the number of arrivals in any time interval of size t is Poisson with mean λt. This brings us to the following alternative definition for the Poisson process.

Definition 3 of the Poisson Process: *A counting process* $\{N(t), t \geq 0\}$ *is said to a Poisson process with rate* $\lambda > 0$ *if* $\{N(0) = 0\}$*, if the process has stationary and independent increments, and, for any* $s, t \geq 0$

$$\mathbf{P}(N(t+s) - N(s) = k) = e^{\lambda t}\frac{(\lambda t)^k}{k!}, \; k = 0, 1, 2, \ldots$$

Ex: Show that Definition 3 implies Definition 2. As Def. 3 was already derived from Def.2, we conclude that the two definitions are equivalent.

Ex: Show that Definition 3 implies Definition 1: Let X_1 be the time until the first arrival in the process in Definition 3. Compute the CDF of X_1. How about the CDF of $X_i, i > 1$?

Ex: I get email according to a Poisson process at rate $\lambda = 1.5$ arrivals per minute. If I check my email every hour, what is the expected number of new messages I find in my inbox when I check my email? What is the probability that I find no messages? One message? Repeat for an e-mail checking period of two hours. (Note that we find very small values, because we are looking for the probability of getting exactly 0 or 1 messages, in a relatively long time period. Repeat the computations for a time period of 1 and 2 minutes.)

Waiting Times: Let $S_n = \sum_{i=1}^{n} X_i$, the time of the n^{th} arrival in the Poisson process with rate λ, $n \geq 1$. Show that S_n obeys the Erlang distribution of order n, with parameter λ:

$$f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{(n-1)}}{(n-1)!}$$

(Hint: Note that $\{S_n \leq t\} \equiv \{N(t) \geq n\}$.)

Ex: Suppose, starting at time $t = 8:25$, students start entering Building A according to a Poisson process at a rate of 1 student per minute.

(a) Find the distribution of the time elapsed up to and including the arrival of the 10^{th} student.

(b) Let T be the elapsed time between the second and fourth arrivals. What is the probability that $T > 3$ minutes?

(c) Write an expression to compute the third moment of T.

Ex: An alternative way to obtain the density of the n^{th} inter-arrival time is the following: Write the probability $\mathbf{P}(t < S_n < t + \delta)$, divide by δ and take the limit $\delta \to 0$.

The time-reversed process is also Poisson: We can show that the reverse residual time distribution is the same as the inter-arrival time distribution.

Ex: In the bus problem, what is the expected number of people on the bus that I get on? (Hint: Consider the people that arrived in the two minutes before I arrived, as well as the people that arrive while I am waiting.)

The random incidence "paradox": Note from the above example that when I arrive at random, the inter-arrival time I sample is expected to have twice the duration of the average inter-arrival duration. Consider the following: if bus arrivals are a Poisson process, in order to understand whether buses are too crowded or not, should planners interview people, or bus drivers?

Linking the Poisson Process to the Bernoulli process: Take a Poisson process at rate λ and discretize time finely, in chunks of size δ. Show that as $\delta \to 0$, the Poisson process can be approximated by a Bernoulli process.