# Brief Information on Fourier Transform:

In this course, we will use the following definition of Fourier transform.

$$x(t) \longleftrightarrow X(f)$$

$\leftarrow f$ (in Hertz)

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \leftarrow \text{Forward F.T.}$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df \quad \leftarrow \text{Inv. F.T.}$$

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The main advantage of this definition is the absence of $\frac{1}{2\pi}$ factor in the inv. F.T. expression. This leads to several simplifications in transform pairs and eases the application of duality.

$$\boxed{x(t) \xleftrightarrow[F.T]{} X(f)}$$

① $x(t-t_0) \longleftrightarrow X(f)e^{-j2\pi f t_0}$ (Time shift)

② $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$ (Scaling)

③ $x(t)e^{j2\pi f_0 t} \longleftrightarrow X(f-f_0)$ (Freq. shift or modulation)

④ $X(t) \longleftrightarrow x(-f)$ (Duality)

⑤ $x_1(t) * x_2(t) \longleftrightarrow X_1(f)X_2(f)$ (Convolution)

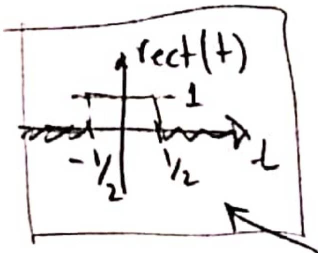
⑥ $x_1(t) \cdot x_2(t) \longleftrightarrow X_1(f) * X_2(f)$ (Time-multiplication)

## Some Fourier Transform Pairs:

$$\boxed{x(t) \longleftrightarrow X(f)}$$

$$1 \longleftrightarrow \delta(f)$$

$$e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$$

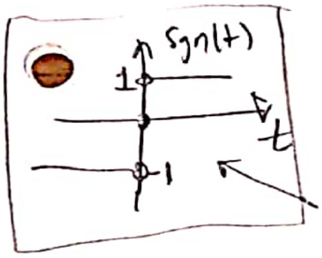


$$\text{rect}(t) \longleftrightarrow \text{sinc}(f)$$

$$\boxed{\text{sinc}(f) \triangleq \frac{\sin(\pi f)}{\pi f}}$$

$$\text{sinc}(t) \longleftrightarrow \text{rect}(f) \quad \text{(By duality)}$$

$$u(t) \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$



$$\text{sgn}(t) \longleftrightarrow \frac{1}{j\pi f}$$

$$\frac{1}{\pi t} \longleftrightarrow -j\,\text{sgn}(f) \quad \leftarrow \left(\begin{array}{c}\text{Hilbert Transform} \\ \text{related}\end{array}\right).$$

## Power Spectral Density:

We have observed that LTI processing of stochastic WSS processes yields an output process with the auto-correlation function of

$$r_y(\tau)=r_x(\tau)*h(\tau)*h^*(-\tau).$$

Considering the relation of convolution operation and Fourier transforms, the idea of utilizing Fourier Transforms for output auto-correlation calculation immediately pops up. This is indeed correct, but there is more into this topic than the computational considerations.

Given a WSS process with auto-correlation function $r_x(\tau)$, the Fourier transform of $r_x(\tau)$, i.e.

$$S_x(f)=\int_{-\infty}^{\infty} r_x(\tau)\, e^{-j2\pi f\tau}\, d\tau ,$$

is called the power spectral density of $\underset{\sim}{x}(t)$.

Similarly for WSS processes with power spectral density $S_x(f)$; it is possible to get $r_x(\tau)=\int_{-\infty}^{\infty} r_x(\tau)\, e^{j2\pi f\tau}\, d\tau$.

Inverse Fourier Transform

Hence, we define the following as a Fourier Transform pair:

$$r_x(\tau) \xleftrightarrow[F.T.]{} S_x(f)$$

auto-cor. func.

Power spectral density func.

A careful eye must have noticed the word "density" in the "power spectral density function" title given to the Fourier transform of $r_x(\tau)$. The first few properties will reveal, why this function is called a density function.

## Properties of Power Spectral density:

① Area under psd function:

$$r_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df$$

This equation immediately follows from $r_x(\tau) \xleftrightarrow[F.T]{} S_x(f)$ relation. That's, the area of the function in one of the domains is identical to the value of the function in the other domain at the origin.

Since, $r_x(0) = E\{|\underset{\sim}{x}(t)|^2\}$ is the avg. power of the process, we can say that the area under $S_x(f)$ is the ensemble average power of the process.

② $S_X(f)$ is real valued.

Proof We will use Hermitian-symmetry of $r_X(z)$ for the proof.

$$S_X(f) = \int_{-\infty}^{\infty} r_X(z)\, e^{-j2\pi f z}\, dz$$

Taking conjugate of both parts,

$$S_X^*(f) = \int_{-\infty}^{\infty} r_X^*(z)\, e^{j2\pi f z}\, dz$$

$z' = -z$

$$= \int_{-\infty}^{\infty} r_X^*(-z')\, e^{-j2\pi f z'}\, dz'$$

$r_X(z) = r_X^*(-z)$

$$= \int_{-\infty}^{\infty} r_X(z')\, e^{-j2\pi f z'}\, dz'$$

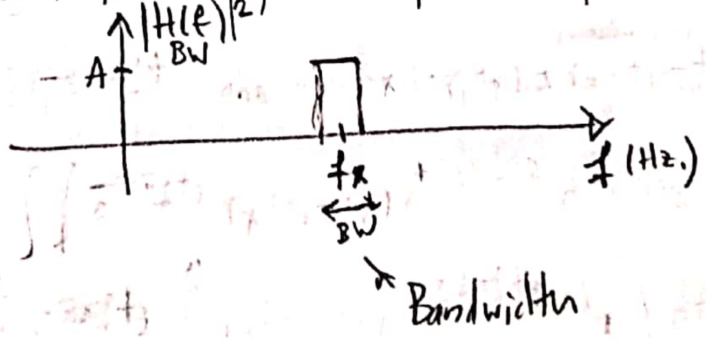
$$= S_X(f).$$

③ $S_X(f) \geq 0$.

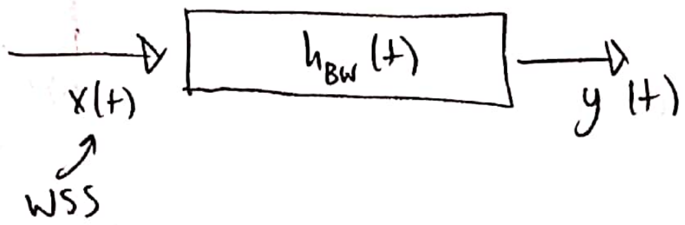
(This property is closely related positive semi definitiness of $r_X(z)$)

Proof: Let's assume that $S_X(f_x) < 0$ for some $f_x$.

Then, we construct a bandpass filter with center freq. $f_x$ and denote the impulse response of this filter as $h_{BW}(t)$.



NOW,



$x(t) \leftrightarrow X(f)$.
$x^*(-t) \leftrightarrow X^*(f)$

We know that

$$r_y(\tau) = r_x(\tau) * h_{BW}(\tau) * h^*_{BW}(-\tau).$$

$$\to S_y(f) = S_x(f) . H_{BW}(f) . H^*_{BW}(f)$$
$$= S_x(f) . |H_{BW}(f)|^2$$

We know that $r_y(0) = \int_{-\infty}^{\infty} S_y(f)\, df \geq 0$

total avg. power of output process

$$= \int_{f_x - \frac{BW}{2}}^{f_x + \frac{BW}{2}} S_x(f) . A \geq 0$$

$$\approx S_x(f_x) . A . BW$$

(Note that the approximation to the integral becomes more and more accurate as $BW \to 0$ and $S_x(f_x)$ is continuous at $f_x$.).

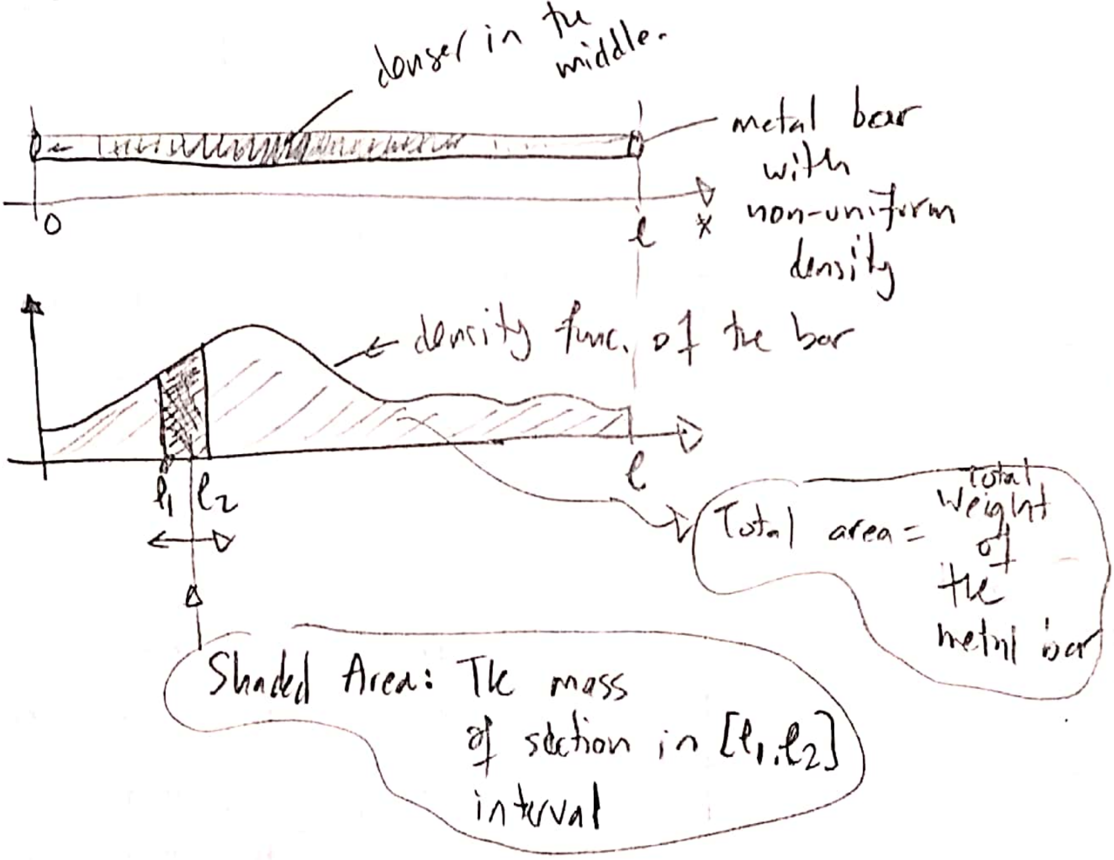
At the begining, we have claimed that $S_x(f_x) < 0$, but this contradicts with $r_y(0) \geq 0$.

Hence, there is no frequency value $f_x$ for which $S_x(f_x) < 0$.

Summarizing:

$S_X(f)$ is a real, non-negative valued function whose area under it is equal to the average power of the WSS process $\underset{\sim}{x}(t)$.

Hence, $S_X(f)$ shows the distribution of power across the spectrum. It is indeed a density function, i.e. a function showing the distribution of "mass" along an axis.



**Ex:** ④ $\underset{\sim}{x}(t) = A\cos(2\pi f_x t + \underset{\sim}{\theta})$ (Random phase cosine)

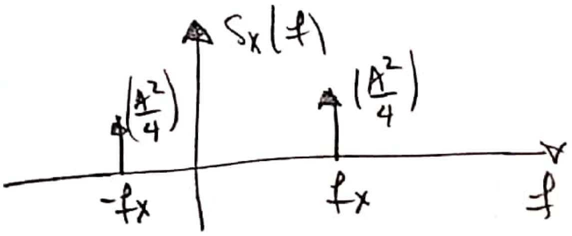
$\hookrightarrow \underset{\sim}{\theta}$ : Unif. $[0, 2\pi)$

We know that $\underset{\sim}{x}(t)$ is WSS with $r_x(\tau) = \frac{A^2}{2}\cos(w_x \tau)$.

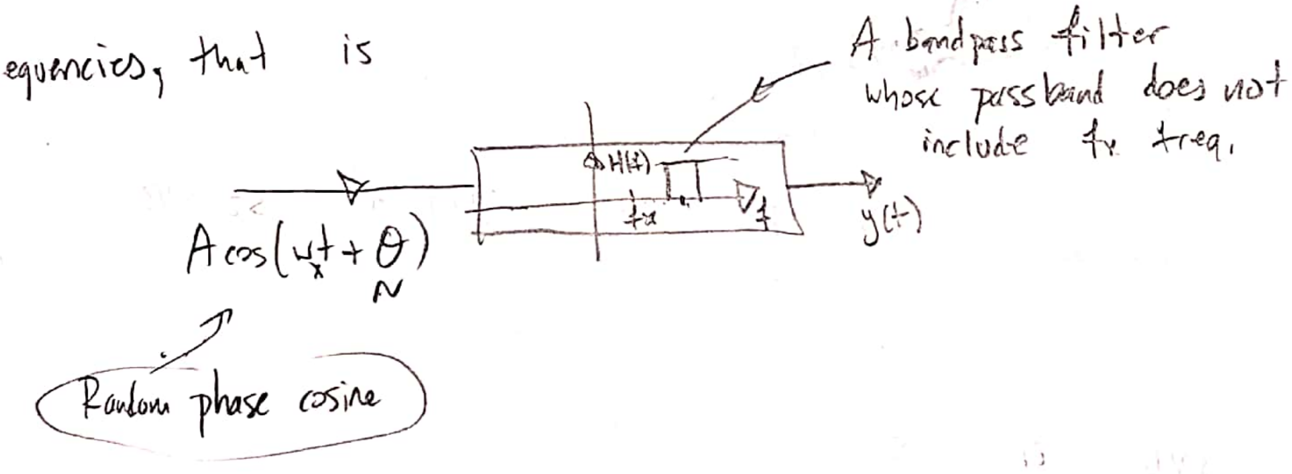
$w_x = 2\pi f_x$

Then $S_x(f) = \mathcal{F}\{\frac{A^2}{2}\cos(2\pi f_x \tau)\}$

$$= \frac{A^2}{4}\left[\delta(f-f_x) + \delta(f+f_x)\right]$$



Hence, the power of this process is concentrated to $\mp f_x$ frequencies, that is



The output p.s.d $S_y(f) = S_x(f)|H(f)|^2 = 0$. Hence, there is no output power if filter passband does not include frequency $f_x$. This should make sense considering the realizations of the random phase cosine process.

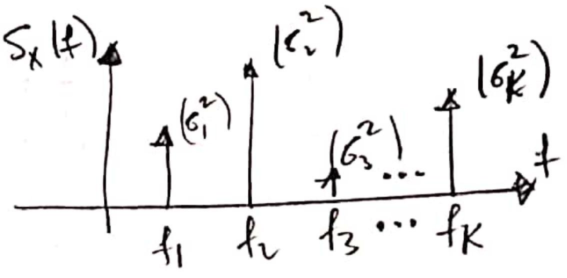
**Ex:** $\underset{\sim}{x}(t) = \sum_{k=1}^{K} \underset{\sim}{a}_k e^{j2\pi f_k t}$

$\underset{\sim}{a}_k$: zero-mean and uncorrelated r.v.'s with variance $\sigma_k^2$

Then $R_x(\tau) = \sum_{k=1}^{N} \sigma_k^2 e^{j2\pi f_k \tau} \xleftrightarrow{F.T.} S_x(f) = \sum_{k=1}^{N} \sigma_k^2 \delta(f-f_k)$

Note that $R_x(0) = \sum_{k=1}^{K} \sigma_k^2$

total power



distribution of total power in the spectrum.

Notes: ① LTI filtering of white-noise (WSS) with a filter having magnitude response $|H(f)|$ results in an output process with power spectral density shaped in $|H(f)|^2$.

Hence, by designing filters, we can synthesize random processes with a desired $S_x(f)$ or $R_x(\tau)$.

In many practical modeling problems, $\hat{r}_x(\tau)$ or $\hat{S}_x(f)$ is estimated from collected data and an LTI filter with white-noise input is designed s.t. the filter output process has approximately the observed $\hat{r}_x(\tau)$ or $\hat{S}_x(f)$.

② Previously, we have studied some necessary cond. for auto-correlation func. Here, we have found another necessary condition

$$\underset{\text{a valid auto-cor. func.}}{r_X(\tau)} \longrightarrow \underbrace{S_X(f) \geq 0 \quad \forall f.}_{\text{new necessary cond.}}$$

Different from other necessary conditions, the non-negativity of $S_X(f)$ is also <u>sufficient</u> for the validity of auto-cor. func. That is,

$$\underset{\text{a valid auto-corr func.}}{r_X(\tau)} \longleftarrow \underbrace{S_X(f) = \mathcal{F}\{r_X(\tau)\} \geq 0 \quad \forall f.}_{\text{Sufficient cond.}}$$

Hence, we have

$$\underset{\text{a valid auto-cor. func}}{r_X(\tau)} \longleftrightarrow \underbrace{S_X(f) = \mathcal{F}\{r_X(\tau)\} \geq 0 \quad \forall f}_{\text{necessary and sufficient cond. for a valid } r_X(\tau).}$$

The sufficiency cond. is proved by constructing a WSS process for an arbitrary $g(f) \geq 0 \ \forall f$ function whose power spectral density is $g(f)$, see Papoulis (Sec. 10.24 p.322).

③ Power spectral density of a r.p. allows us to classify the processes as low pass, High pass, Bandpass processes similar to the classification of filters.

Different from filters, if a significant amount of process power resides, say, in low freq. band (a freq. band containing zero) that process is a low-pass process.

For filters, the impulse response of the filter has finite energy and if depending on the distribution of energy across the spectrum, filters are classified.
(This is called energy spectrum density (ESD).)

## Bandpass Processes, Bandpass Process Representation:

Before we discuss the bandpass signal representation, let's focus on another tool that will be needed along this road, the Hilbert transform.

$$x(t) \longrightarrow \boxed{h(t) = \frac{1}{\pi t}} \longrightarrow \hat{x}(t) = H.T.\{x(t)\}$$

Hence,

$$\hat{x}(t) = \int_{-\infty}^{\infty} x(z) \frac{1}{\pi(t-z)} dz .$$

(Hilbert Transform of x(t))

If $x(t) \xleftrightarrow[F.T.]{} X(f)$ and $\frac{1}{\pi t} \xleftrightarrow[F.T.]{} -j\,\mathrm{sgn}(f)$

a known F.T. pair

$\mathrm{sgn}(x) = \begin{cases} 1, x>0 \\ 0, x=0 \\ -1, x<0 \end{cases}$

Then, $\hat{X}(f) = X(f).-j\,\mathrm{sgn}(f)$ by convolution theorem of F.T.

$F\{H.T.\{x(t)\}\}$

Now, it is easy to see that,

Observe $\left(-j\,\mathrm{sgn}(f)\right)^2 = -1$ $(f \neq 0)$

Hence,

$$\hat{x}(t) = \int_{-\infty}^{\infty} x(z) \cdot \frac{1}{\pi(t-z)}\, dz$$

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(z) \cdot \frac{1}{\pi(z-t)}\, dz$$

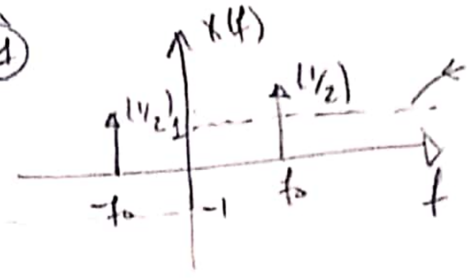
Hilbert Transform Forward and Inverse relation expression.

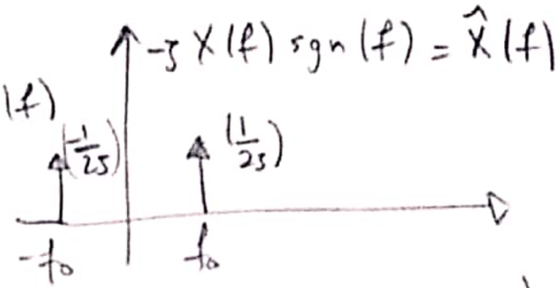
Note that $h(t) = \frac{1}{\pi t}$ is not absolutely summable impulse response, Hence, the H.T. as implemented by filtering results in a filtering operation with an unstable filter!

The H.T. forward and inverse expression are valid in the distribution sense which is not very important for our purposes; Since we will using the H.T. as a tool to facilitate other calculations.

**Ex:** Let $x(t) = \cos(2\pi f_0 t)$. Find $\hat{x}(t)$.

**Solu.** Sln ①





$$\hat{x}(t) = \sin(2\pi f_0 t)$$

Sln ②
$$X(f) = \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

Assum. ($f_0 > 0$)

$$\hat{X}(f) = \frac{1}{2}(-j\,\mathrm{sgn}(f))\left[\delta(f-f_0) + \delta(f+f_0)\right]$$

$$= \frac{1}{2j}\underbrace{\mathrm{sgn}(f_0)}_{1}\delta(f-f_0) + \frac{1}{2j}\underbrace{\mathrm{sgn}(-f_0)}_{-1}\delta(f+f_0)$$

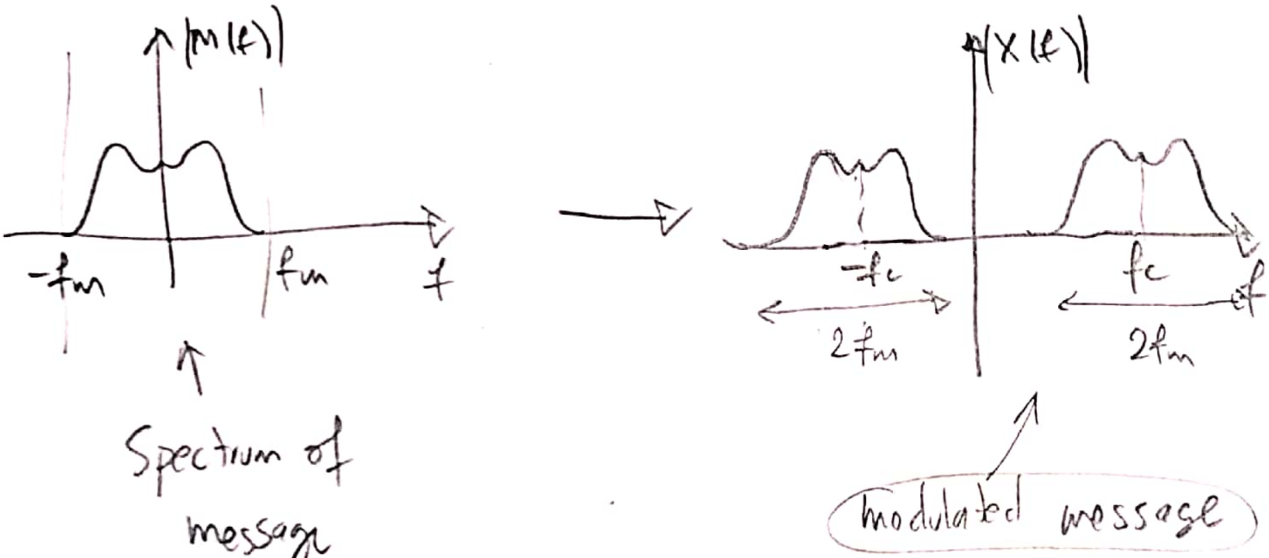
$$= \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

$\mathcal{F}^{-1}\{\}$

$\rightarrow \hat{x}(t) = \sin(2\pi f_0 t)$.

## Single Side Band Modulation

In communications applications, the message signal is modulated to a carrier frequency suitable for the propagation of the message. The message can be a song and this message can be modulated to 6 MHz to reach far away listeners (of Voice of Turkey radio) say in China.



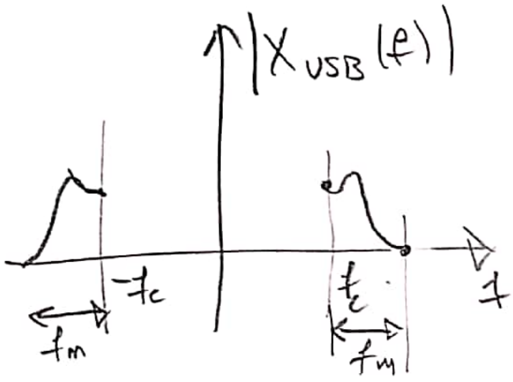
$$x(t) = m(t)\cos(2\pi f_c t)$$

Note that, since ① $m(t)$ is real valued → $|M(f)|$ symmetric around zero

② $|X(f)|$ is then symmetric around $f_c$.

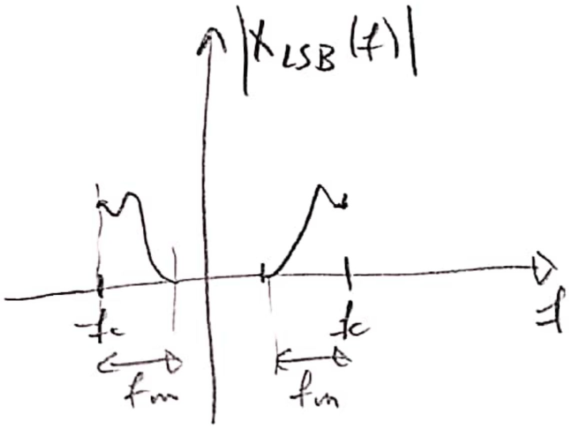
Hence, there is redundancy in the band $[f_c - f_m, f_c + f_m]$, that is we know that $X(f_c + v) = X^*(f_c - v)$ (why) $|v| < f_m$.

We can improve the spectral efficiency, if we can only send one of the side-bands, i.e the one below or above fc.



Upper Side Band Modulation

$\boxed{*}$



Lower Side Band Modulation

Single Side Band Modulation Schemes

<u>Claim</u>:

$$X_{USB}(t) = m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)$$

$$X_{LSB}(t) = m(t)\cos(2\pi f_c t) + \hat{m}(t)\sin(2\pi f_c t).$$

<u>Proof</u>:

$$X_{USB}(f) = \frac{M(f)}{2} * \left[\delta(f-f_c) + \delta(f+f_c)\right]$$

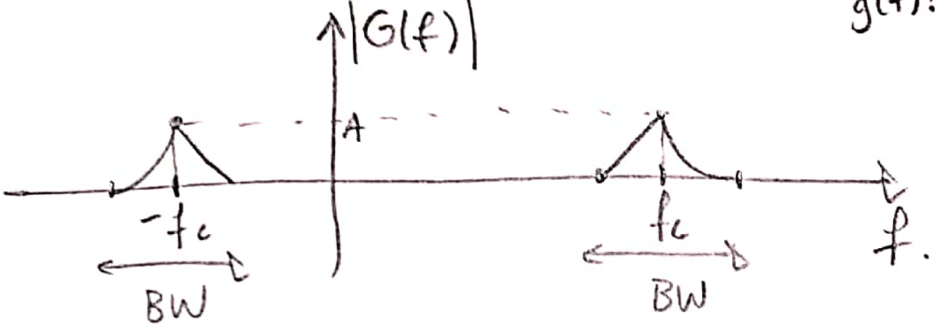
$$- \frac{M(f).(-j\,\mathrm{sgn}(f))}{2j} * \left[\delta(f-f_c) - \delta(f+f_c)\right]$$

$$= \frac{M(f-f_c)}{2}\underbrace{\left(1+\mathrm{sgn}(f-f_c)\right)}_{=\begin{cases}2, & f>f_c\\ 0, & f<f_c\end{cases}} + \frac{M(f+f_c)}{2}\underbrace{\left(1-\mathrm{sgn}(f+f_c)\right)}_{=\begin{cases}2, & f<-f_c\\ 0, & f>-f_c\end{cases}}$$

(Compare $X_{USB}(f)$ expression with the sketch in $\boxed{*}$ for $|X_{USB}(f)|$.

# Representation of Band Pass Signals:

A pass band signal, typically a message signal modulated to a high frequency, occupies a bandwidth around a chosen frequency "$f_c$".



g(t): real valued band-pass signal

The information is clearly contained in the frequency band/interval shown by the BW (Bandwidth) above.

Our first goal is to represent this real-valued bandpass signal as an equivalent low pass (but complex valued) signal.

## ① Bandpass Signal ⟶ Low Pass Equivalent:

Step ①: Form the analytic/pre-envelope signal

$$g_+ \triangleq g(t) + j\,\hat{g}(t)$$

analytic signal (pre-envelope)

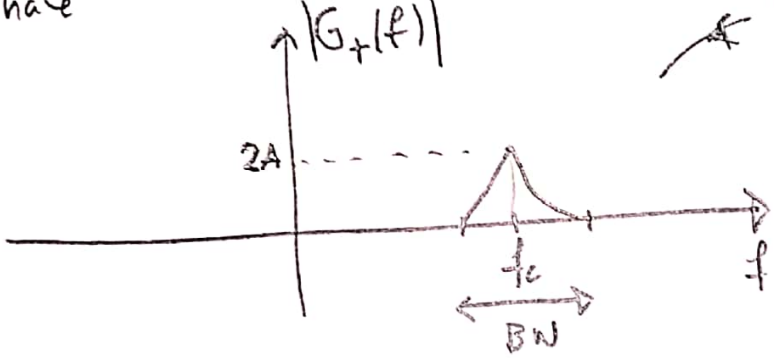
H.T. $\{\hat{g}(t)\}$.

To understand $g_+(t)$, let's evaluate is F.T.

$$\hat{G}(f) = -j\,\mathrm{sgn}(f)\,G(f)$$

$$\begin{aligned} G_+(f) &= G(f) + j\hat{G}(f) \\ &= (1+\mathrm{sgn}(f))\,G(f) \\ &= \begin{cases} 2\,G(f), & f>0 \\ 0, & f<0 \end{cases} \end{aligned}$$

Hence, we have



Negative freq. components are eliminated!

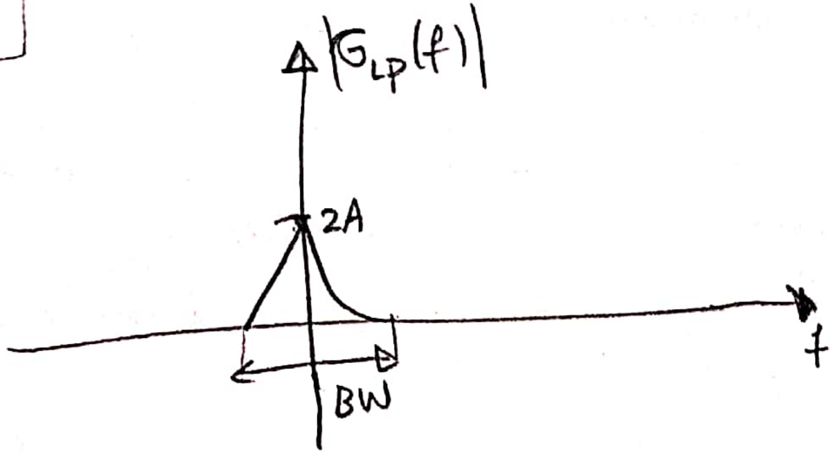
**Step ②:** Shift $G_+(f)$ to DC frequency to get the low pass equivalent signal:

$$g_{LP}(t) \triangleq g_+(t)\,e^{-j2\pi f_c t}$$

Freq. Shift towards the negative frequencies ($f_c>0$)

Low pass equivalent signal

$$G_{LP}(f) = G_{LP}(f+f_c)$$



**Note:** ① $g_{LP}(t)$ is a complex valued signal in general, since $|G_{LP}(f)|$ does not have even symmetry in spectrum. (This should also explain why we focus also on negative frequencies or complex valued signals in signal processing. The answer is such signals are related to the low pass equivalents of band-pass signals.)

② The "center" frequency doesnot have to be in the middle of the bandwidth of the band-pass signal. The center frequency is arbitrary in this process. It can even be outside of the bandwidth!

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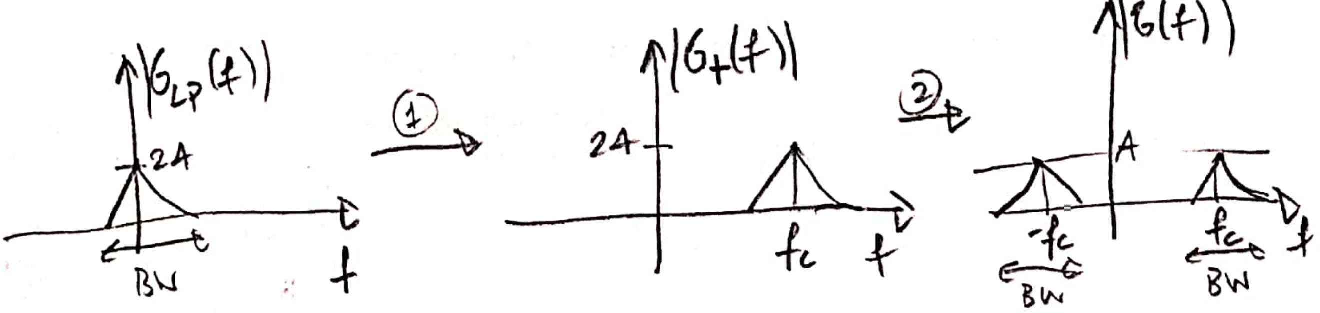
## Low-Pass Equivalent → Bandpass Signal:

Band-pass signal

Given $g_{LP}(t)$ and $f_c$, how can we get $g(t)$?

**Answer:** By following the same steps in reverse!



Let's do this algebraically:

Real part of $g_{LP}(t)$

$$\boxed{g_{LP}(t) = g_I(t) + j\,g_\varphi(t)}$$

Complex value

imag. part

$$g_+(t) = g_{LP}(t)\, e^{j2\pi f_c t}$$

① analytic signal retrieved

$$g(t) = \mathrm{Re}\{g_+(t)\}$$

Remember that ②
$$g_+(t) = g(t) + j\hat{g}(t)$$

Hence,

$$g(t) = \mathrm{Re}\{\overbrace{g_{LP}(t)}^{g_I(t)+jg_\varphi(t)}\, e^{j2\pi f_c t}\}.$$

$$\boxed{g(t) = g_I(t)\cos(2\pi f_c t) - g_\varphi(t)\sin(2\pi f_c t)}$$

where $g_I(t)$, $g_\varphi(t)$ are inphase, quadrature phase components of low pass equivalent signal $g_{LP}(t)$, respectively.

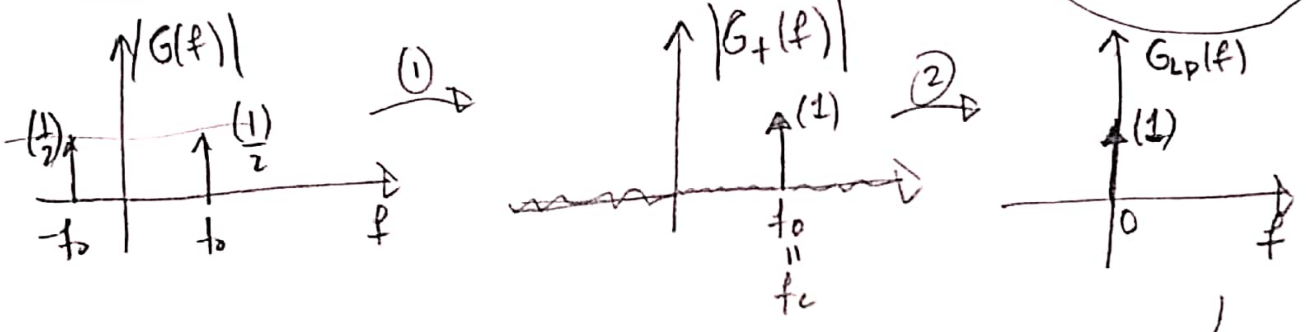
The expression $\boxed{g(t) = g_I(t)\cos(2\pi f_c t) - g_\varphi(t)\sin(2\pi f_c t)}$ is called the canonical representation of band-pass signals and it is very important in both theory and practice.

Ex: $g(t) = \cos(2\pi f_0 t)$

i) Find the lowpass equivalent of g(t) for $f_c = f_0$.

representation freq. for band-pass signal

Graphically:



$$\boxed{g_{LP}(t) = 1}$$

From cannonical representation:

$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$$

any band-pass signal

In this case, $g(t) = \cos(2\pi f_0 t)$ and $f_c = f_0$ $\longrightarrow$ $\left.\begin{matrix} g_I(t) = 1 \\ g_Q(t) = 0 \end{matrix}\right\}$

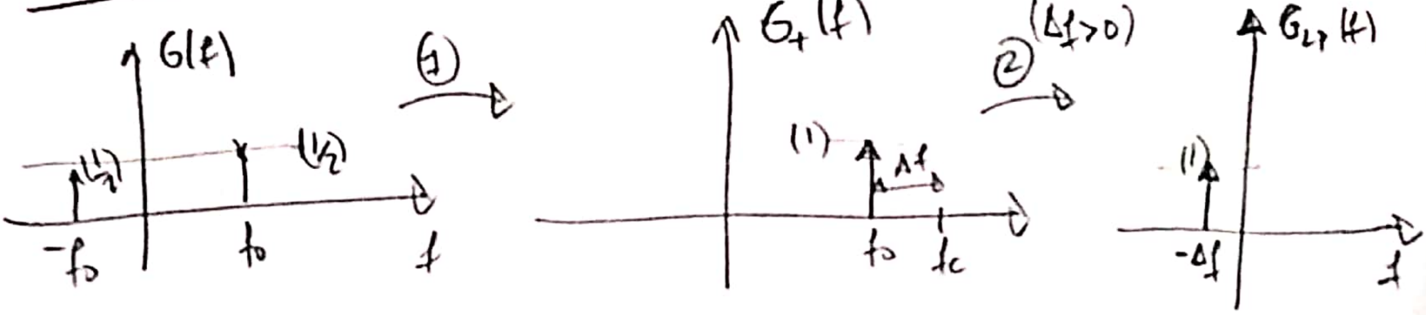
$$\boxed{\begin{aligned} g_{LP}(t) &= g_I(t) + j g_Q(t) \\ &= 1 \end{aligned}}$$

ii) Repeat the same for $f_c = f_0 + \Delta f$ $\quad \Delta f > 0$

Typically $\Delta f$ is very small in comparison to $f_0$

It may correspond to a mismatch in frequency to $f_0$

## Graphically:



$$g_{Lp}(t) = e^{-j2\pi\Delta_f t}$$

$$= \underbrace{\cos(2\pi\Delta f t)}_{g_I(t)} - j\underbrace{\sin(2\pi\Delta f t}_{-g_Q(t)}$$

i.e. $g_I(t) = \cos(2\pi\Delta f t)$

$g_Q(t) = -\sin(2\pi\Delta f t)$

### From canonical rep.

$$g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \leftarrow \text{canonical rep.}$$

our band pass signal

$$\text{Let, } g(t) = \cos(2\pi f_0 t) = \cos(2\pi\overbrace{(f_c - \Delta f)}^{f_0} t)$$

$$= \underbrace{\cos(2\pi\Delta f t)}_{g_I(t)}\cos(2\pi f_c t) + \underbrace{\sin(2\pi\Delta f t)}_{-g_Q(t)}\sin(2\pi f_c t$$

By comparing with the canonical rep., we get the same result. (Remember $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$).

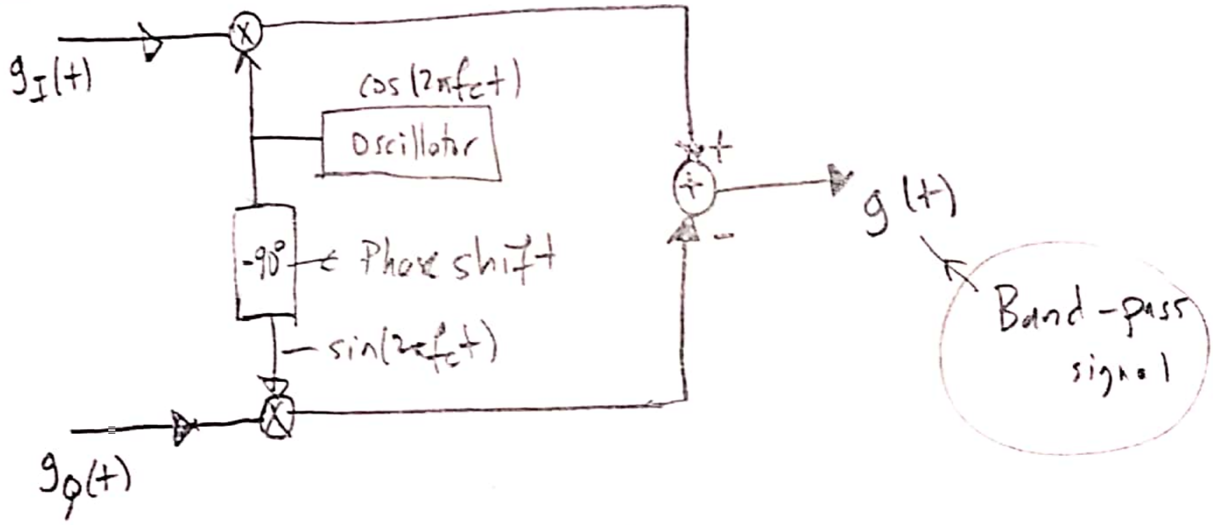
## From practical view point:

### Generation of Bandpass Signals
(i.e modulation)

Given $g_I(t)$ : inphase signal

$g_Q(t)$ : quadrature phase signal

Q. How to generate bandpass signal?

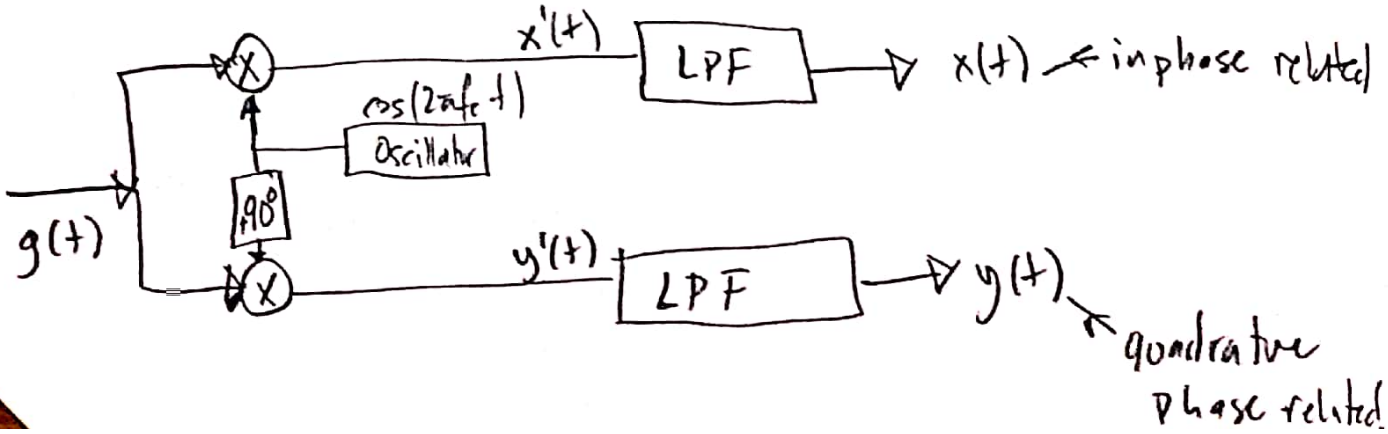


The above is a direct implementation of canonical representation.

### Generation of I/Q signals
(i.e. demodulation)

Given $g(t)$ a band-pass signal,

Q. How to generate I/Q signals?



$$x'(t) = g(t)\cos(2\pi f_c t)$$

$$= g_I(t)\underbrace{\cos^2(2\pi f_c t)}_{\frac{1+\cos(4\pi f_c t)}{2}} - g_Q(t)\underbrace{\sin(2\pi f_c t)\cos(2\pi f_c t)}_{\frac{\sin(4\pi f_c t)}{2}}$$

$$= \frac{g_I(t)}{2} + (\text{High freq. components})$$

Low pass filtering $x'(t)$ gives $\frac{g_I(t)}{2} = x(t)$

Similarly,

$$y'(t) = g(t)\Big[\overset{\cos(2\pi f_c t + 90^\circ)}{-\sin(2\pi f_c t)}\Big]$$

$$= g_I(t)\cos(2\pi f_c t)\sin(2\pi f_c t) + g_Q(t)\underbrace{\sin^2(2\pi f_c t)}_{\frac{1-\cos(4\pi f_c t)}{2}}$$

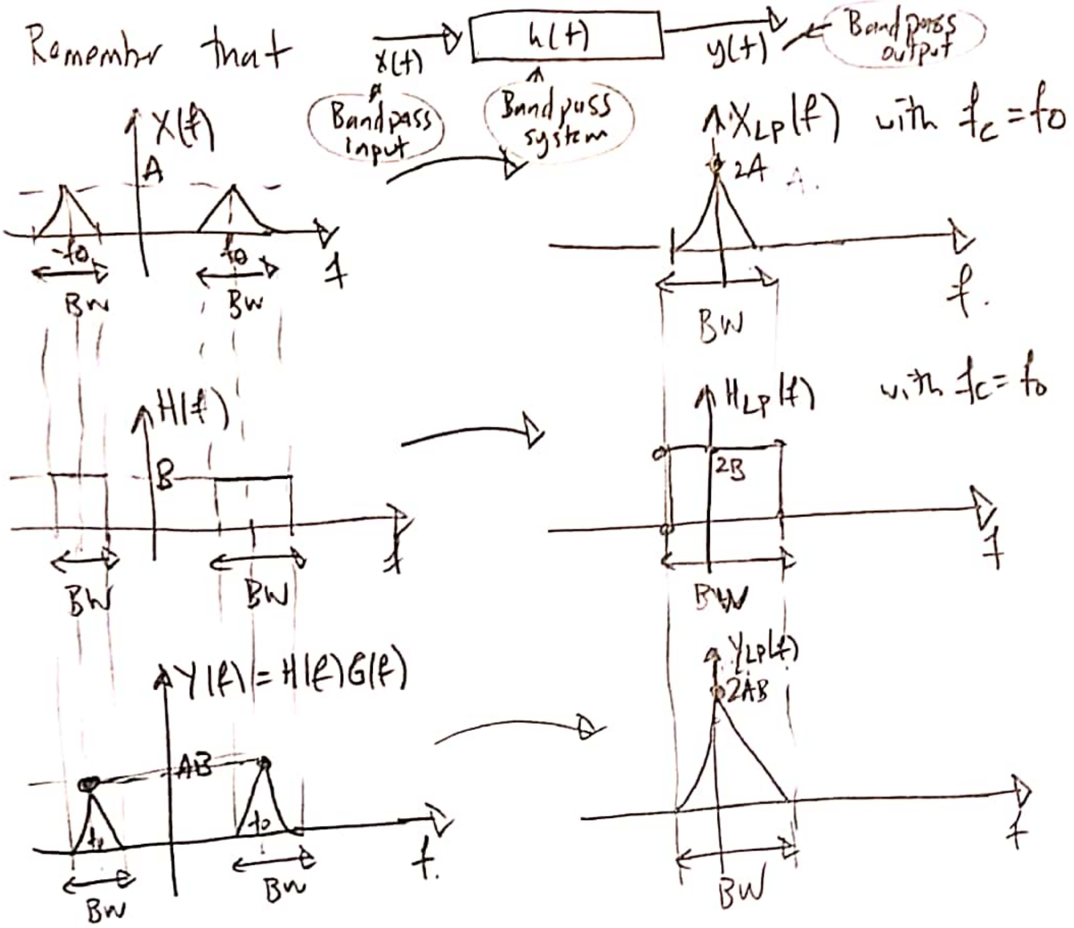
$$= \frac{g_Q(t)}{2} + (\text{High freq. terms})$$

## Representation of Bandpass Systems

A band-pass system is an LTI system whose frequency response is centered around a non-zero carrier frequency.

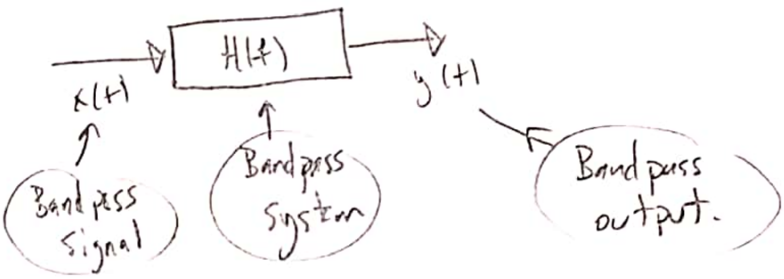
We extend our earlier results for bandpass signal representation to band pass systems by considering the impulse response of a band-pass system and its low pass equivalent.

Remember that



From above graphs, we can observe that

The relation $Y(f) = H(f) X(f)$, that is



results in the following for the low pass equivalents:

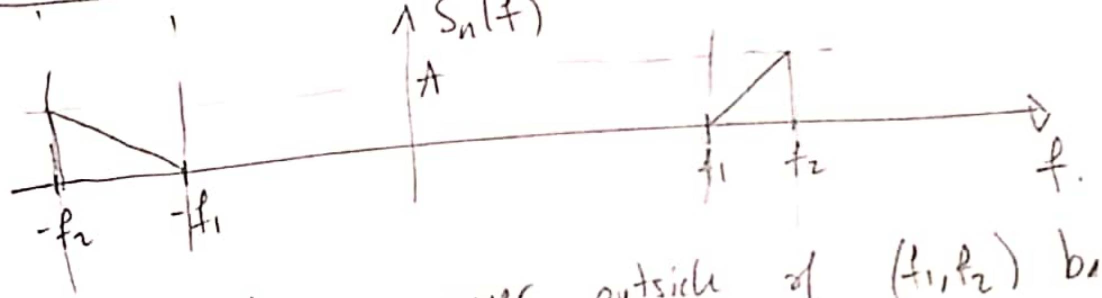
$$\boxed{Y_{LP}(f) = \frac{1}{2} H_{LP}(f) . X_{LP}(f)}.$$

The factor of ½ is required, since the spectrum of low-pass equivalent is "doubled" (multiplied by two) when compared with the corresponding spectrum of the band-pass signal. $H_{LP}(f) X_{LP}(f)$ correctly gets the "shape" of low pass equivalent for $Y(f) = H(f) X(f)$; but scaling by 2 is done twice when $H_{LP}(f) X_{LP}(f)$ is evaluated; Hence, a factor of $\frac{1}{2}$ is needed. to get $Y_{LP}(f)$ from $H_{LP}(f) X_{LP}(f)$ product.

## Representation of Bandpass WSS Processes:

Let $n(t)$ be a zero mean, band-pass WSS process with power spectral density $S_n(f)$ given as follows.

(*)



Note that $n(t)$ has no power outside of $(f_1, f_2)$ band in the spectrum. Our goal is to represent this r.p. with an equivalent low-pass r.p.

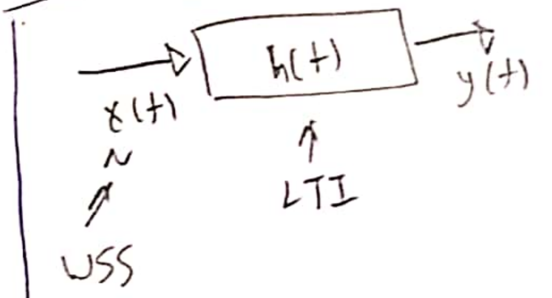
**Step ①:** Generate analytic process, $\underset{\sim}{n}_+(t) = \underset{\sim}{n}(t) + j\underset{\sim}{\hat{n}}(t)$ analytic process

**Step ②:** Generate low-pass equivalent process, $\underset{\sim}{n}_{LP}(t) = n_+(t) e^{-j2\pi f_c t}$ (with a choice of $f_c$)

Note that those steps are the same as generating the low-pass equivalents of deterministic signals. Since, a realization of a band-pass process is a deterministic signal, we may think that we are applying the low-pass equivalent generation process to the realizations of a band-pass process and generating a new process called the low-pass equivalent process.

We would like to answer questions like whether $\underset{\sim}{n}_{LP}(t)$ is WSS, the auto-correlation/power spectral density of $\underset{\sim}{n}_{LP}(t)$, the auto-cor. of $\underset{\sim}{n}_I(t) = Re\{n_{LP}(t)\}$; the cross-correlation of $n_I(t)$ and $n_Q(t)$ etc.

Before this discussion, let's remember some basic facts:



① $y(t)$ is also WSS

② $x(t), y(t)$ are jointly WSS

$$r_y(\tau) = r_x(\tau) * h(\tau) * h^*(-\tau)$$

③
$$r_{xy}(\tau) = r_x(\tau) * h^*(-\tau).$$
$$r_{yx}(\tau) = r^*_{xy}(-\tau) = h(\tau) * r_x(\tau)$$

F.T.

④
$$S_y(f) = S_x(f)\,|H(f)|^2$$
$$S_{xy}(f) = S_x(f)\,H^*(f)$$
$$S_{yx}(f) = H(f).\,S_x(f)$$

**Basic Facts:**

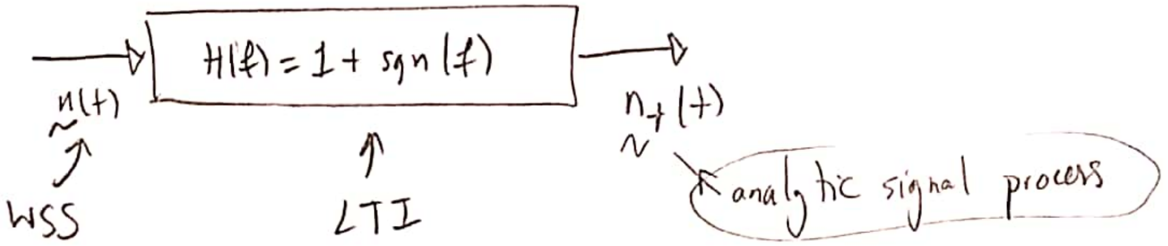
Now, let's start with step-1

Step ①: Generate the analytic process $\underset{\sim}{n}_+(t) = \underset{\sim}{n}(t) + j\hat{n}(t)$

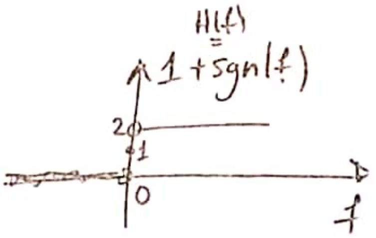
F.T.

$$N_+(f) = N(f) + j(-j\,\text{sgn}(f)\,N(f))$$

$$\boxed{N_+(f) = (1+\text{sgn}(f))\,N(f)}$$

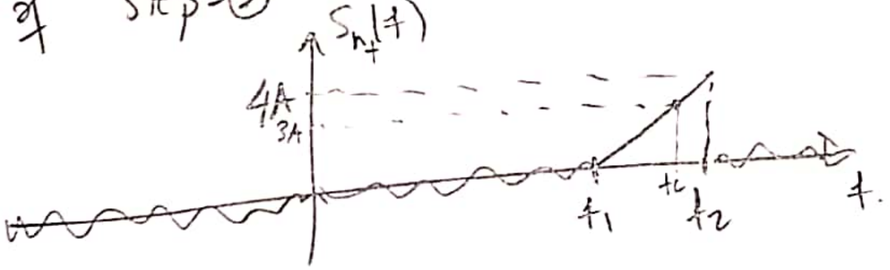


Since $\tilde{n}(t)$ is LTI filtering of $\tilde{n}(t)$; it is WSS and has the power spectrum density of



$$S_{n_+}(f) = |H(f)|^2 S_n(f)$$
$$= \begin{cases} 4 S_n(f), & f>0 \\ 0, & f<0 \end{cases}$$

Then, sketch (*) in page 23 has the following p.s.d. sketch at the end of step-①



(**)

Step-②: Generation of low-pass equivalent process

$$\tilde{n}_{LP}(t) = \tilde{n}_+(t)\, e^{-j2\pi f_c t}$$

($f_c$: a chosen representation freq. for low-pass equivalent rep.)

Let's calculate the auto-correlation of $\tilde{n}_{LP}(t)$:

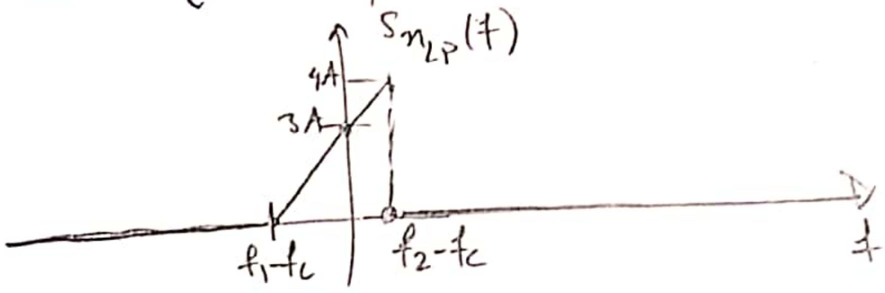
$$r_{n_{LP}}(\tau) = E\{n_{LP}(t)\, n_{LP}^*(t-\tau)\}$$
$$= E\{n_+(t)\, n_*(t-\tau) \boxed{e^{-j2\pi f_c \tau}}\}$$
$$= r_{n_+}(\tau)\, e^{-j2\pi f_c \tau}$$

Not a func. of "t", so $n_{LP}(t)$ is stat. in auto-correlation.

Then its p.s.d is

$$S_{n_{LP}}(f) = F\left\{ r_{n_+}(\tau)\, e^{-j2\pi f_c \tau} \right\}$$
$$= S_{n_+}(f+f_c)$$

Then, sketch (**) in p.25 after step ② becomes



So, we have seen that $n_{LP}(t)$ is a WSS process with zero mean ( it is easy to check this given the information that the band-pass r.p. is also zero-mean) WSS process with p.s.d as given/can be found, as shown above.

Next, we would like to find the auto-corr/power spectral density of I/Q components of low-pass equivalent signal.

Remember that, $\underset{\sim}{n}_{LP}(t) = \underset{\sim}{n}_I(t) + j\underset{\sim}{n}_Q(t)$. We, now, know that $\underset{\sim}{n}_{LP}(t)$ is WSS. How about $\underset{\sim}{n}_I(t)$, $\underset{\sim}{n}_Q(t)$ processes.

This is our next question.

$$\underset{\sim}{n}_I(t) = Re\{\underset{\sim}{n}_{LP}(t)\} = Re\{\underbrace{n_+(t)}_{n(t)+j\hat{n}(t)}\, e^{-j2\pi f_c t}\}$$

$$\boxed{\underset{\sim}{n}_I(t) = \underset{\sim}{n}(t)\cos(2\pi f_c t) + \underset{\sim}{\hat{n}}(t)\sin(2\pi f_c t)}$$

Similarly, $\rightarrow$ $$\boxed{\underset{\sim}{n}_Q(t) = \underset{\sim}{\hat{n}}(t)\cos(2\pi f_c t) - \underset{\sim}{n}(t)\sin(2\pi f_c t).}$$

Note that:

$$\begin{bmatrix} \underset{\sim}{n}_I(t) \\ \underset{\sim}{n}_Q(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(2\pi f_c t) & \sin(2\pi f_c t) \\ -\sin(2\pi f_c t) & \cos(2\pi f_c t) \end{bmatrix}}_{\underline{\underline{R}}(t)} \begin{bmatrix} \underset{\sim}{n}(t) \\ \underset{\sim}{\hat{n}}(t) \end{bmatrix}$$

$$= \underline{\underline{R}}(t) \cdot \begin{bmatrix} \underset{\sim}{n}(t) \\ \hat{n}(t) \end{bmatrix}$$

NOW,

$$E\left\{ \underbrace{\begin{bmatrix} n_I(t) \\ n_Q(t) \end{bmatrix}}_{\underline{\underline{R}}(t)\begin{bmatrix} n(t) \\ \hat{n}(t) \end{bmatrix}} \underbrace{[n_I(t-z) \;\; n_Q(t-z)]}_{\left(R(t-z)\begin{bmatrix} n_I(t-z) \\ n_Q(t-z) \end{bmatrix}\right)^T} \right\} = \left[\begin{array}{c|c} R_{n_I}(z) & R_{n_I n_Q}(z) \\ \hline R_{n_Q n_I}(z) & R_{N_Q}(z) \end{array}\right]$$

$$= \underline{\underline{R}}(t)\, E\left\{ \begin{bmatrix} \underset{\sim}{n}(t) \\ \underset{\sim}{\hat{n}}(t) \end{bmatrix} [n(t-z) \; \hat{n}(t-z)] \right\} \underline{\underline{R}}^T(t-z)$$

$$\left[\begin{array}{c|c} R_{n_I}(z) & R_{n_I n_Q}(z) \\ \hline R_{N_Q N_I}(z) & R_{N_Q}(z) \end{array}\right] = \underline{\underline{R}}(t) \left[\begin{array}{c|c} R_n(z) & R_{n\hat{n}}(z) \\ \hline R_{\hat{n}n}(z) & R_{\hat{n}}(z) \end{array}\right] R^T(t-z)$$

From $\boxed{\text{Basic Facts}}$ (p.24)

Since $$n(t) \longrightarrow \boxed{h(t) = \frac{1}{\pi t}} \longrightarrow \hat{n}(t)$$

$$H(f) = -j\,\mathrm{sgn}(f)$$

$$S_{\hat{n}}(f) = S_N(f)\,|-j\,\mathrm{sgn}(f)|^2 = S_N(f) \quad (f \neq 0)$$

$$S_{\hat{N}N}(f) = -j\,\mathrm{sgn}(f)\, S_N(f)$$

$$S_{N\hat{N}}(f) = S_N(f)\,[-j\,\mathrm{sgn}(f)]^*$$

$$\rightarrow R_{\hat{n}}(\tau) = F^{-1}\{S_n(f)\} = R_n(\tau)$$

$$\rightarrow R_{\hat{n}n}(\tau) = F^{-1}\{S_{\hat{n}n}(f)\} = H.T\{R_n(\tau)\} = \hat{R}_n(\tau)$$

$$\rightarrow R_{n\hat{n}}(\tau) = F^{-1}\{S_{n\hat{n}}(f)\} = -H.T.\{R_n(\tau)\} = -\hat{R}_n(\tau).$$

Hence,

$$\left[\begin{array}{c|c} R_{n_I}(\tau) & R_{N_I N_\varphi}(\tau) \\ \hline R_{N_\varphi N_I}(\tau) & R_{N_\varphi}(\tau) \end{array}\right] = \underline{R}(t) \left[\begin{array}{c|c} \overset{R_n(\tau)}{R_n(\tau)} & \overset{R_{n\hat{n}}(\tau)}{-\hat{R}_n(\tau)} \\ \hline \underset{R_{\hat{n}n}(\tau)}{\hat{R}_n(\tau)} & \underset{R_{\hat{n}}(\tau)}{R_n(\tau)} \end{array}\right] \underline{R}^T(t-\tau)$$

$$= \begin{bmatrix} \cos(2\pi f_c t) & \sin(2\pi f_c t) \\ -\sin(2\pi f_c t) & \cos(2\pi f_c t) \end{bmatrix} \left[\begin{array}{c|c} R_n(\tau) & -\hat{R}_n(\tau) \\ \hline \hat{R}_n(\tau) & R_n(\tau) \end{array}\right] \left[\begin{array}{c|c} (c) & -(s) \\ \hline \underset{\sin 2\pi f_c(t-\Delta)}{(s)} & \underset{\cos 2\pi f_c(t-\Delta)}{(c)} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \overset{\leftarrow R_{NI}(\tau)}{\begin{matrix} R_n(\tau)\cos(2\pi f_c \tau) \\ + \\ \hat{R}_n(\tau)\sin(2\pi f_c \tau) \end{matrix}} & \begin{matrix} R_n(\tau)\sin(2\pi f_c \tau) \\ - \\ \hat{R}_n(\tau)\cos(2\pi f_c \tau) \end{matrix} \leftarrow R_{N_I N_\varphi}(\tau) \\ \hline \underset{R_{N_\varphi N_I}(\tau)}{\begin{matrix} -R_n(\tau)\sin(2\pi f_c \tau) \\ + \\ \hat{R}_n(\tau)\cos(2\pi f_c \tau) \end{matrix}} & \underset{R_{N\varphi}(\tau)}{\begin{matrix} R_N(\tau)\cos(2\pi f_c \tau) \\ + \\ \hat{R}_N(\tau)\sin(2\pi f_c \tau) \end{matrix}} \end{array}\right]$$

This concludes the derivation of

$$(*)\begin{cases} ① \quad R_{n_I}(\tau) = R_{n_\varphi}(\tau) = R_N(\tau)\cos(2\pi f_c \tau) + \hat{R}_n(\tau)\sin(2\pi f_c \tau) \\ ② \quad R_{N_I N_\varphi}(\tau) = -R_{N_\varphi N_I}(\tau) = R_n(\tau)\sin(2\pi f_c \tau) - \hat{R}_n(\tau)\cos(2\pi f_c \tau). \end{cases}$$

when $R_n(\tau)$ is the auto-cor. of band-pass r.p.

$\hat{R}_n(\tau) = H.T.\{R_n(\tau)\}$

and

$f_c$: representation freq. for low pass equivalent.

We see that $R_{n_I}(\tau)$, $R_{N_\varphi}(\tau)$ and $R_{N_I N_\varphi}(\tau)$ expressions have no dependency on "$t$". Hence $\underset{\sim}{n_I}(t)$ and $\underset{\sim}{n_\varphi}(t)$ are jointly WSS.

---

The expressions ① and ② in (*) above are easier to keep in mind in terms of its Fourier transform. Let's calculate the p.s.d of $r_{N_I}(\tau)$ (or $r_{n_\varphi}(\tau)$).

$$S_{N_I}(f) = F\{r_{N_I}(\tau)\} = \left(S_N(f) * \left(\frac{\delta(f-f_c)+\delta(f+f_c)}{2}\right)\right) + \left((-j\,\mathrm{sgn}(f)S_n(f)) * \frac{\delta(f-f_c)-\delta(f+f_c)}{j2}\right)$$

$$= \frac{1}{2}\left[\underbrace{(1-\mathrm{sgn}(f-f_c))}_{=\begin{cases}0, & f>f_c\\ 2, & f<f_c\end{cases}} S_N(f-f_c) + \underbrace{(1+\mathrm{sgn}(f+f_c))}_{=\begin{cases}2, & f>-f_c\\ 0, & f<-f_c\end{cases}} S_N(f+f_c)\right]$$

We assume $f_c \gg BW$ *(BW of band pass process)

$$= \begin{cases} S_n(f-f_c)+S_n(f+f_c), & -f_c<f<f_c \\ 0, & \text{other}. \end{cases}$$

Similarly,

$$S_{N_I N_Q}(f) = \mathcal{F}\{R_{N_I N_Q}(\tau)\}$$

$$= S_N(f) * \frac{(\delta(f-f_c) - \delta(f+f_c))}{j2} - (j\,\mathrm{sgn}(f) S_N(f)) * \left(\frac{\delta(f-f_c) + \delta(f+f_c)}{2}\right)$$

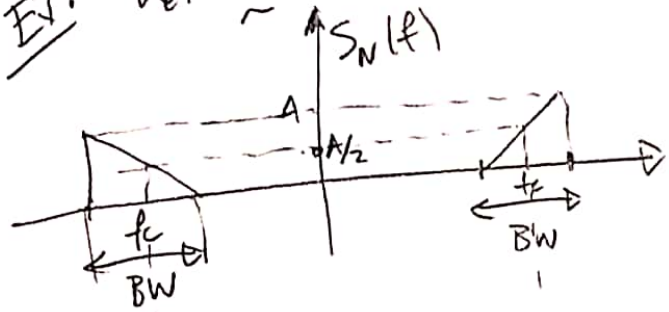
$$= \frac{1}{j2}\left[\underbrace{[1-\mathrm{sgn}(f-f_c)]}_{=\begin{cases}0, & f>f_c\\ 2, & f<f_c\end{cases}} S_N(f-f_c) - \underbrace{[1+\mathrm{sgn}(f+f_c)]}_{=\begin{cases}2, & f>-f_c\\ 0, & f<-f_c\end{cases}} S_N(f+f_c)\right]$$

$$= \frac{1}{j}\cdot\begin{cases} S_N(f-f_c) - S_N(f+f_c), & -f_c<f<f_c \\ 0, & \text{oth.}\end{cases}$$

$S_Q(f) = S_I(f)$.

Also, $S_{N_Q N_I}(f) = -S_{N_I N_Q}(f)$.

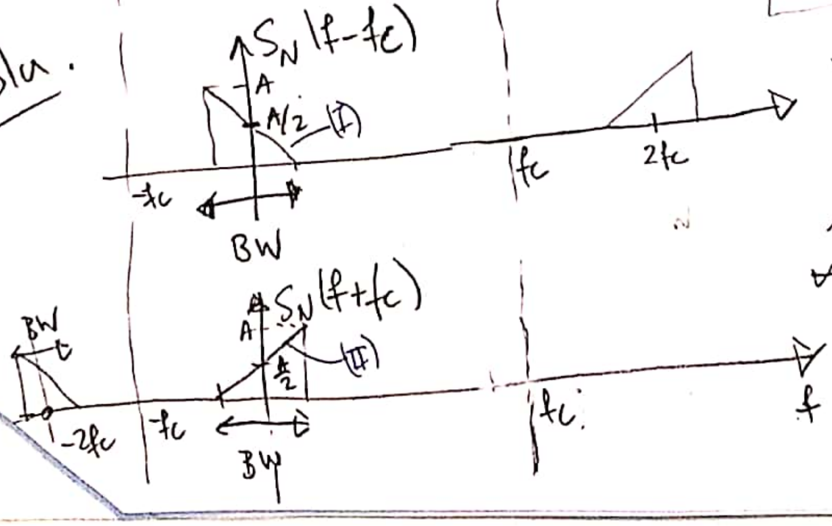
Ex: Let $\underset{\sim}{x}(t)$ be zero-mean WSS band-pass process with $S_N(f)$

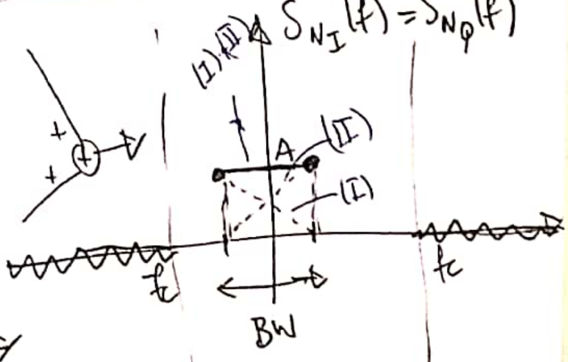


Find, $S_{N_I}(f)$, $S_{N_Q}(f)$, $S_{N_I N_Q}(f)$.

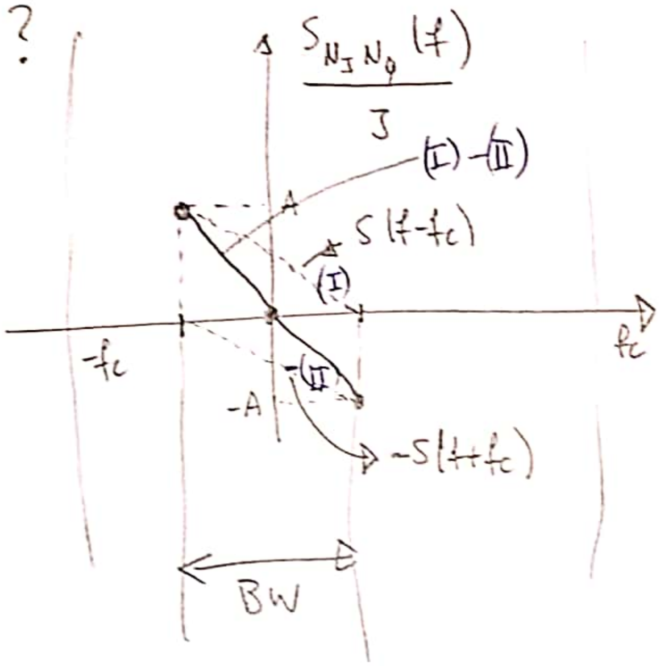
Soln.

$$\boxed{S_{N_I}(f) = S_N(f-f_c)\overbrace{u(f_c-f)}^{\begin{cases}1, & f<f_c\\ 0, & f>f_c\end{cases}} + S_N(f+f_c)\overbrace{u(f+f_c)}^{\begin{cases}1, & f>-f_c\\ 0, & f<-f_c\end{cases}}}$$





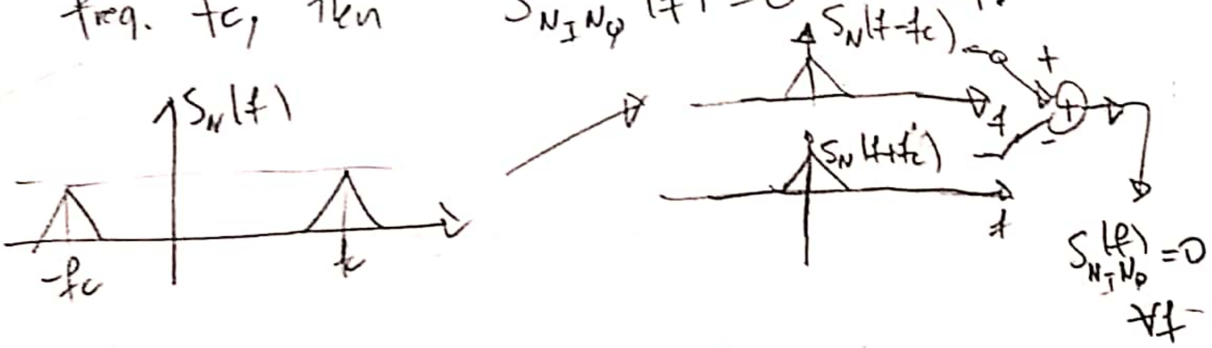
$S_{N_I N_Q}(f) = ?$



So, $S_{N_I}(f) = S_{N_Q}(f)$ is formed by sum of +/- freq. components of $S_N(f)$ after modulating to low pass region $(-f_c < f < f_c)$.

$S_{N_I N_Q}(f)$ is formed by dif. of +/- freq. componets of $S_N(f)$ after modulating to the low-pass region.

**Notes:** ① If $S_N(f)$ is symmetric around representation freq. "$f_c$", then $S_{N_I N_Q}(f) = 0 \quad \forall f$.



② If $\underset{\sim}{n}(t)$ is a band-pass $\overset{\text{zero-mean}}{\wedge}$ Gaussian process

→ $n_I(t)$ and $n_Q(t)$ are jointly WSS Gaussian processes, (Since linear operations on Gaussian processes preserves Gaussianity).

③ If $\underset{\sim}{n}(t)$ is a band-pass $\overset{\text{zero-mean}}{\wedge}$ Gaussian process and $S_n(f)$ is symmetric around $f=f_0$ and $f_c=f_0$, i.e representation freq. for low pass equivalent is the symmetry center (as in Note ①) then $\underset{\sim}{n_I}(t)$ and $\underset{\sim}{n_Q}(t)$ are Gaussian and independent and identically distributed (iid).

Since, $S_{N_I N_Q}(f) = 0$ ← (from Note-①)

→ $r_{N_I N_Q}(\tau) = 0$

↓

$\underset{\sim}{n_I}(t)$ and $\underset{\sim}{n_Q}(t-\tau)$ are uncorrelated

↓

$n_I(t_1)$, $n_Q(t_2)$ independent (since Gaussian) $\forall t_1, t_2$

---

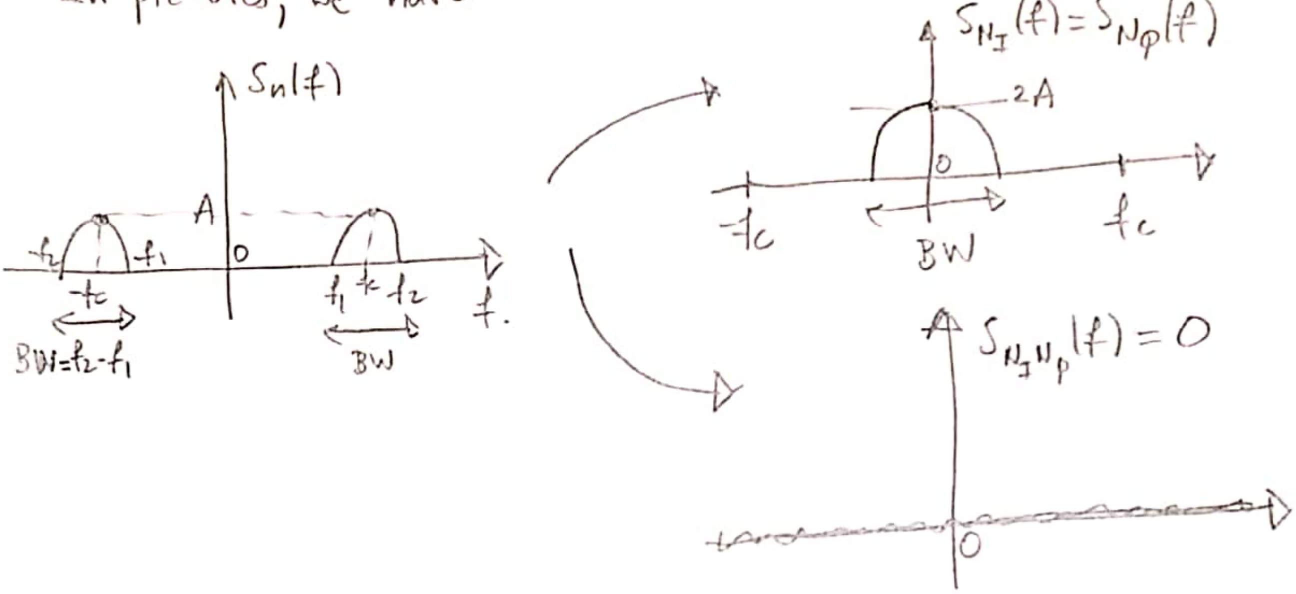
Also, $S_{N_I}(f) = S_{N_Q}(f) \Rightarrow r_{N_I}(\tau) = r_{N_Q}(\tau) \Rightarrow n_I(t)$ and $n_Q(t)$ have the same auto-cor. (and zero-mean)

⇓

$n_I(t)$ and $n_Q(t)$ are identically dist. ← $n_I(t), n_Q(t)$ are Gaussian dist. with identical autocor. func.

In many practical applications, the low pass equivalent of noise process is assumed be Gaussian with iid $n_I(t_1)$ and $n_Q(t_2)$ parts for all $t_1$ and $t_2$.

In pictures, we have

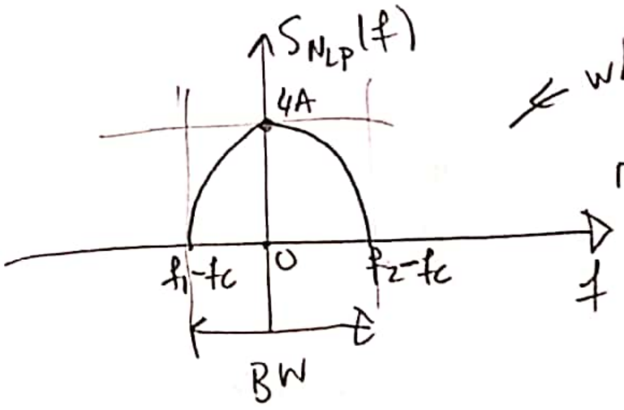


$$\underset{\sim}{n}_{Lp}(t) = \underset{\sim}{n}_I(t) + j\underset{\sim}{n}_Q(t) \rightarrow r_{n_{Lp}}(\tau) = E\{n_{Lp}(t)\, n_{Lp}^*(t-\tau)\}$$

$$= r_{n_I}(\tau) + \underbrace{r_{n_Q}(\tau)}_{r_{N_I}(\tau)} + j\left(\cancelto{0}{r_{N_Q N_I}(\tau)} - \cancelto{0}{r_{n_I n_Q}(\tau)}\right)$$

$$\left(\text{Since } S_{N_I N_Q}(f)=0\right)$$

$$S_{n_{Lp}}(f) = 2S_{N_I}(f) = 2S_{N_Q}(f)$$



which is the result we expect from p. 25-26.