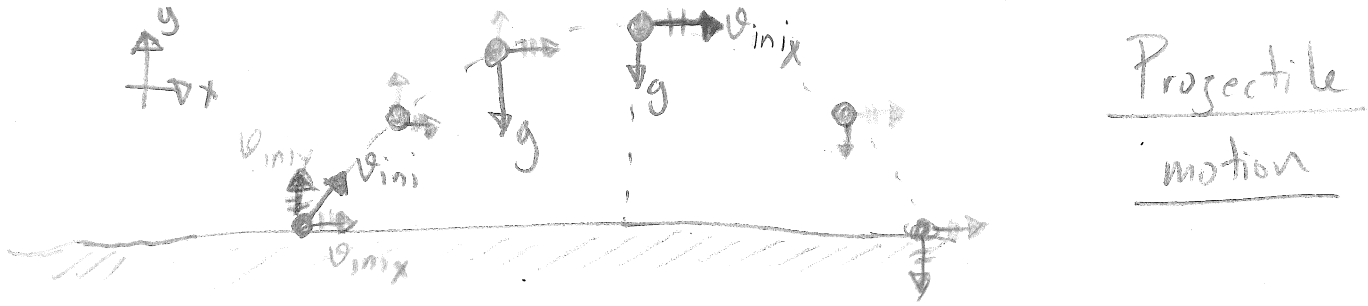
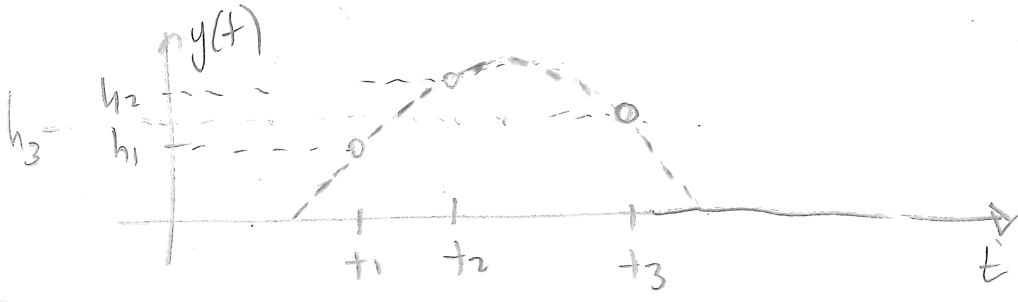
EE 306: Random Processes

To understand (or get a feeling) about random processes, we start with deterministic signals. In signal processing, a major approach is to use the properties of the signals involved in an application to improve the performance of that application. Let's assume that we are tracking the position of a ball under projectile motion:



The height of the ball above the ground at time "t", that is $y(t)$, is a quadratic function of time. Assume that, we observe the height by a radar (why?) system and get the height measurements/observations at 3 different time instants.



$y(t_1) = h_1$
$y(t_2) = h_2$
$y(t_3) = h_3$

The question is whether we can find the height at any other instant. For example, can we predict the time that ball will hit the ground?

Kinematics tells us that $y(t)$ is quadratic function of time, that is $y(t) = at^2 + bt + c$ $\quad a, b, c \in \mathbb{R}$.

If we can (a, b, c) then we can identify the quadratic and then evaluate the height at any time by $y(t_x) = a t_x^2 + b t_x + c$.

Since we have
$$\left.\begin{aligned} y(t_1) &= h_1 \\ y(t_2) &= h_2 \\ y(t_3) &= h_3 \end{aligned}\right\} \rightarrow \text{We can get } (a, b, c)$$

from the measurements and we can do the height prediction as intended.

(Actually

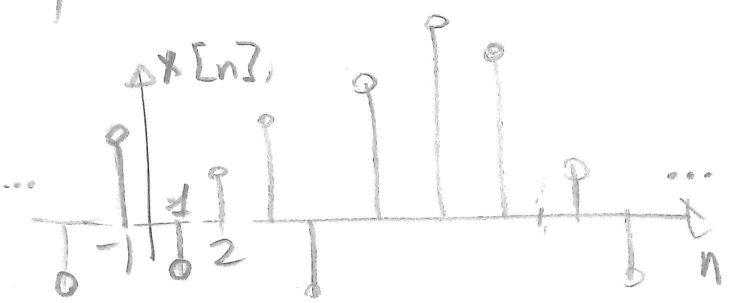
$$y(t) = h_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + h_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + h_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

(Why?)

(Lagrange interpolator))

Forgeting the application details, what we have done is the fitting the observations to a quadratic function. In this case, we are sure that the signal class is 2^{nd} order polynomials due to basic physics.

Let's assume that we observe the samples of a band-limited function. The question is whether, we can have the value of the function at an unobserved point or not?



The sampling theorem says that if we sample $x(t)$ bandlimited to f_{max} (Hz) at $2f_{max}$ samples/sec. or higher rate (Nyquist rate or higher) we can reconstruct the continous-time function $x(t)$ from its samples without any error.

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\mathrm{sinc}\left(\frac{t}{T_s} - n\right) \quad \Bigg\} \text{ Sampling Theorem}$$

where $f_s = \frac{1}{T_s} \geq 2f_{max}$

Hence, $\underbrace{x(t_x)}_{\text{"prediction" for } t=t_x \text{ (reconstruction in this case)}} = \sum_{n=-\infty}^{\infty} \underbrace{x(nT_s)}_{\text{observations}} \operatorname{sinc}\left(\frac{t_x}{T_s} - n\right)$.

Again, in this case, we can predict an unobserved time value from the observations, since we have assumed at the beginning that the function is an element of band-limited functions and collected sufficiently many samples to uniquely identify it. (In this case, we need infinite of samples collected above Nyquist rate.)

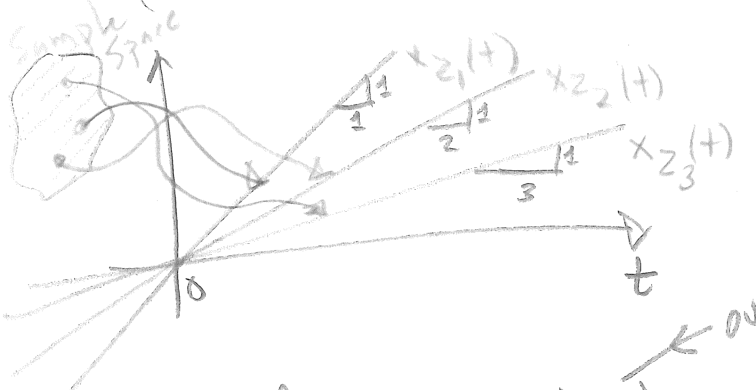
In these two problems, there is a <u>deterministic mapping</u> between observed and unobserved quantities.

In many problems of interest, the mapping observed and unobserved quantities is not deterministic, but random. For example, the weather prediction for tomorrow given today's and yesterday's temperature is a problem of this kind. We would like to model such random experiments and use these models in applications as we did for deterministic signals.

Random Processes:

We introduce the random processes with a simple example.

Ex: (Random slope signal)



outcome ∈ Sample space

random variable

we define $\quad x(t, z) = a(z)\, t = \underset{\sim}{a}\, t$. $\qquad (1)$

process variable

Here, we have functions at as the function class, that is linear functions passing through origin. In (1), "t" is a deterministic quantity (process variable), while slope comes from sample space. (To emphasize random variables further, I may use $\underset{\sim}{a}$ instead of A)

Let's assume that $f_A(a) \sim \text{Unif}\,[0,1]$, that is slope is uniformly distributed in $[0,1]$. In the figure above "3 realizations" are shown with slopes $1, \frac{1}{2}, \frac{1}{3}$.

Every realization is formed by an outcome selection from sample space (fixing z to z_1 or z_2 or z_3).

You may consider that every realization is constructed by selecting an element of sample space, i.e. outcome of random experiment. Hence, when z is fixed/selected we have deterministic functions such as

$$x(t) = t$$
$$x(t) = \frac{1}{2}t$$
$$x(t) = \frac{1}{3}t$$

At this point, you may have feeling that since $\underset{\sim}{A} \sim \text{Unif}\,[0,1]$, each of these functions are equally likely; This is correct but we clearly need more accurate modeling.

Joint Pdf Description of Random processes:

Given r.p. $x(t)$

1) 1^{st} Order pdf Description

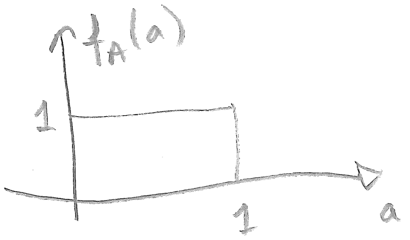
The density of the sample of the r.p. at $t = t_1$, that is

$$\underbrace{X_1}_{\text{r.v.}} = \underbrace{x(t_1)}_{\text{r.p.}} \quad \leftarrow \text{sampling time} \qquad \text{and} \qquad f_{X_1(t_1)}(x_1) \text{ is needed for the description.}$$

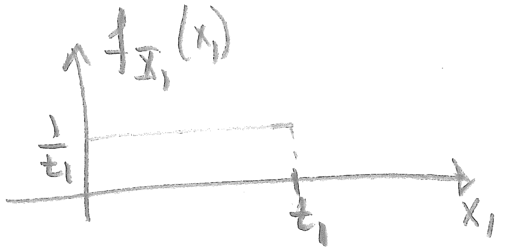
Ex: (Random slope signal)

$$X_1 = x(t_1) = \underset{\sim}{a}\, t_1 \qquad \underset{\sim}{a} \sim \text{Unif } [0,1]$$

$(t_1 > 0)$

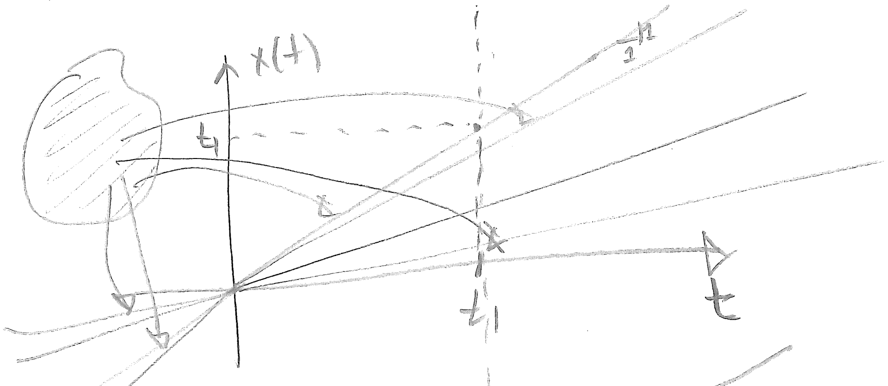


1-func of 1-r.v.

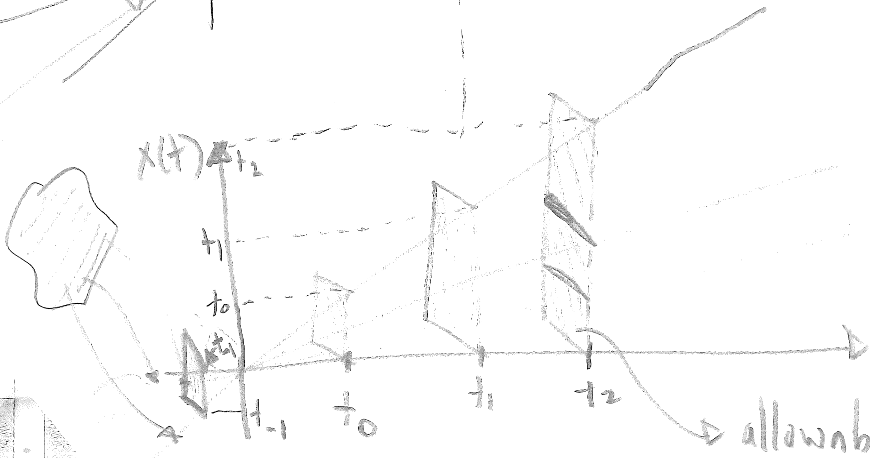


$$\left(f_A(a) = \begin{cases} 1, & 0 \le a \le 1 \\ 0, & \text{other} \end{cases} \right) \qquad \left(f_{X_1}(x_1) = \frac{f_A(x_1/t_1)}{t_1} \right)$$

Let's think about the realizations for this process



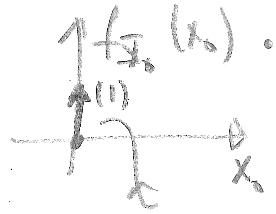
If we focus only on $t = t_1$ instant $x(t_1)$ is limited to $[0, t_1]$ values and $X_1 = x(t_1)$ is Uniformly distributed in this interval.



→ allowable range and p.d.f.

The second figure shows the p.d.f.'s for different "t" instants. The allowable range or support of the density function is $[0, t_x]$ where t_x is the time instance. (Note that the p.d.f for $x(t)$ at $t=0$ is

$$\underset{\sim}{X}_0 = x(0) = 0 \longrightarrow f_{\underset{\sim}{X}_0}(x_0) = \delta(x)$$



(Dirac delta!).

Hence,

$\underset{\sim}{x}(t)$: r.p. or $\underset{\sim}{x}(t, z)$ (more specifically);

→ When $t = t_1$ (fixed to t_1), $x_1 = \underset{\sim}{x}(t_1)$: r.v.

→ When $z = z_k$ (Outcome fixed to z_k), $\underset{\sim}{x}(t, z_k)$: deterministic function

→ When $z = z_k$, $t = t_1$, $\underset{\sim}{x}(t_1, z_k)$: scalar (real/complex number).

2nd Order Pdf. Description

Given r.p. $\underset{\sim}{x}(t)$, the joint density of r.v.'s

$$\left.\begin{array}{l} \mathbb{X}_1 = \underset{\sim}{x}(t_1) \\ \mathbb{X}_2 = \underset{\sim}{x}(t_2) \end{array}\right]$$

that is, $f_{\mathbb{X}_1,\mathbb{X}_2}(x_1,x_2)$ or $f_{\mathbb{X}(t_1),\mathbb{X}(t_2)}(x_1,x_2)$.

is the required description.

Ex: (Random Slope Signal)

$$\mathbb{X}_1 = \underset{\sim}{x}(t_1) = \underset{\sim}{a}\, t_1$$
$$\mathbb{X}_2 = \underset{\sim}{x}(t_2) = \underset{\sim}{a}\, t_2 \longrightarrow f_{\mathbb{X}(t_1),\mathbb{X}(t_2)}(x_1,x_2) = ?$$

$$\begin{pmatrix} t_1 > 0 \\ t_2 > 0 \end{pmatrix}$$

$$\underset{\sim}{a} \sim \text{Unif}\,[0,1]$$

Let's write the joint density as

$$f_{\mathbb{X}(t_1),\mathbb{X}(t_2)}(x_1,x_2) = \underbrace{f_{\mathbb{X}(t_1)}(x_1)}_{\substack{\text{Calculated above} \\ (1^{st}\text{ order density})}} \cdot \underbrace{f_{\mathbb{X}(t_2)|\mathbb{X}(t_1)}(x_2|x_1)}_{(?)}$$

$$f_{\mathbf{X}(t_2)|\mathbf{X}(t_1)}(x_2|x_1) = ?$$

Q: Find the density of $X_2 = \underset{\sim}{a} t_2$ given $x_1 = \underset{\sim}{a} t_1$

From conditioning event,

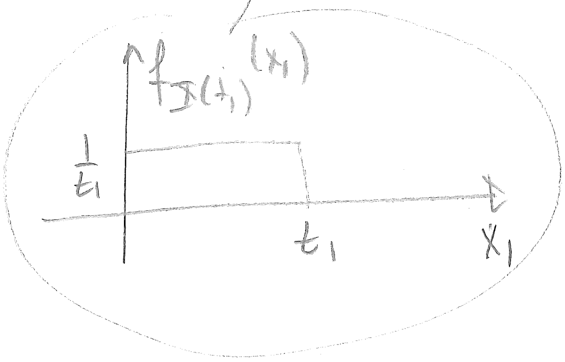
$$\underbrace{x_1}_{\substack{\text{observation at } t=t_1 \\ (\text{known scalar})}} = a \underbrace{t_1}_{\text{known time}} \longrightarrow \underbrace{a = \frac{x_1}{t_1}}_{\text{the specific realization is identified.}} \longrightarrow$$

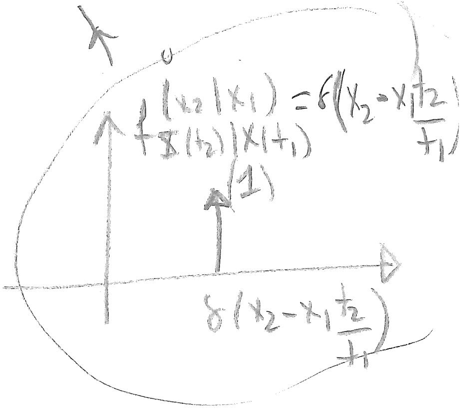
$$\longrightarrow \underbrace{a = \frac{x_1}{t_1}}_{\text{deterministic}} \longrightarrow f_{\mathbf{X}(t_2)=x_2|\mathbf{X}(t_1)=x_1}(x_2|x_1) = \delta\left(x_2 - x_1 \frac{t_2}{t_1}\right)$$

known

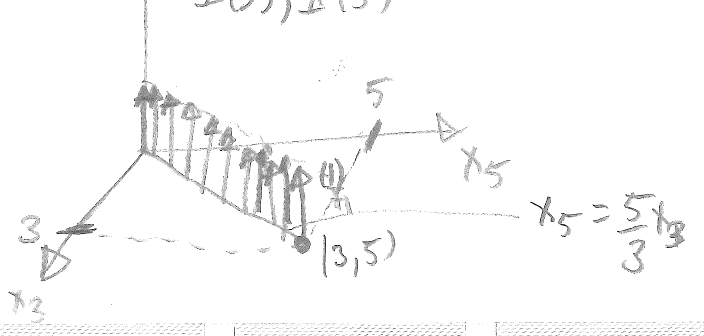
Hence,

$$f_{\mathbf{X}(t_1),\mathbf{X}(t_2)}(x_1, x_2) = f_{\mathbf{X}(t_1)}(x_1) \cdot f_{\mathbf{X}(t_2)|\mathbf{X}(t_1)}(x_2|x_1) \qquad \begin{pmatrix} t_1 > 0 \\ t_2 > 0 \end{pmatrix}$$





Sketch for $t_1 = 3$, $t_2 = 5$



Comments: ① 2^{nd} Order p.d.f description contains the information of 1^{st} order p.d.f description.

Since $f_{X_1(t_1)}(x_1) = \int_{-\infty}^{\infty} f_{X(t_1),X(t_2)}(x_1,x_2)\, dx_2$

② A r.p. is fully characterized with N^{th} order joint pdf description for $\forall N$ (or as $N\to\infty$)

$f_{X(t_1),X(t_2),\dots,X(t_N)}(x_1,x_2,\dots,x_N) \quad \forall (t_1,t_2,\dots,t_N)$

Ex: Calculate 1^{st} Order p.d.f from 2^{nd} Order pdf description for random slope signal

Solu: $f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_1)}(x_1)\, f_{X(t_2)|X(t_1)}(x_2|x_1) = \dfrac{f_A\left(\frac{x_1}{t_1}\right)}{t_1} \cdot \delta\left(x_2 - x_1\frac{t_2}{t_1}\right)$

① $f_{X(t_1)}(x_1) = \int_{-\infty}^{\infty} f_{X(t_1),X(t_2)}(x_1,x_2)\, dx_2 = \dfrac{f_A\left(\frac{x_1}{t_1}\right)}{t_1} \underbrace{\int_{-\infty}^{\infty} \delta\left(x_2 - x_1\frac{t_2}{t_1}\right) dx_2}_{1}$

② $f_{X(t_2)}(x_2) = \int_{-\infty}^{\infty} f_{X(t_1),X(t_2)}(x_1,x_2)\, dx_1 = \int_{-\infty}^{\infty} \dfrac{f_A\left(\frac{x_1}{t_1}\right)}{t_1} \underbrace{\delta\left(x_2 - x_1\frac{t_2}{t_1}\right)}_{\delta\left(x_2 - x_1\frac{t_2}{t_1}\right) = \delta\left(x_1\frac{t_2}{t_1} - x_2\right)} dx_1$

U

x_1

$= \int_{-\infty}^{\infty} \dfrac{f_A\left(\frac{U + x_2}{\cancel{t_1}} \cdot \frac{t_1}{t_2}\right)}{\cancel{t_1}} \delta(U) \cdot \dfrac{t_1}{t_2}\, dU$

dx_1.

$= f_A(x_2/t_2)/t_2.$

Note that For the random slope example, if $x(t)$ is observed at $t_1 \neq 0$ time instant, its value at any other time instant, say t_2, is simply $x(t_2) = x(t_1) \cdot \frac{t_2}{t_1}$ This is expected from this simple process and in general, we can only give $f_{X(t_2)|X(t_1)}(x_2|x_1)$, i.e. the density for the value of $x(t_2)$.

Moment Descriptions for Random Processes (Partial description):

For many random processes, the process description by joint p.d.f is not easy or analytically feasible. For such processes, we use a simpler description called partial description or moment description of r.v.'s.

1st Order Moment description:

$\underset{\sim}{X}(t)$: r.p.

$\mu_X(t) = E\{\underset{\sim}{X}(t)\}$: mean function

Not that $\mu_X(t_1) = \int_{-\infty}^{\infty} x_1 \underbrace{f_{X(t_1)}(x_1)}_{\text{1st order density description}} dx_1$

2nd Order Moment Description:

$$\left.\begin{array}{l} x_1 = \underset{\sim}{x}(t_1) \\ x_2 = \underset{\sim}{x}(t_2) \end{array}\right]$$

$\rightarrow R_x(t_1, t_2) = E\{x(t_1)\, x(t_2)\}$: Auto-correlation func.

$\rightarrow C_x(t_1, t_2) = E\{[x(t_1) - \mu_x(t_1)][x(t_2) - \mu_x(t_2)]\}$: Covariance function.

Note that $R_x(t_1, t_2)$ and $C_x(t_1, t_2)$ can be calculated from the joint density of $\underset{\sim}{x}(t_1)$ and $\underset{\sim}{x}(t_2)$; but, in many problems we can calculate the partial descriptions without calculating the joint densities by calculating expectations.

Ex: (Random slope signal)

$\underset{\sim}{x}(t) = \underset{\sim}{a}\, t$, $\underset{\sim}{a}$: Unif. $[0, 1]$

(1) Mean function : $\mu_x(t_1) = E\{\underset{\sim}{a}\, t_1\} = \dfrac{t_1}{2}$

(2) Correlation function : $R_x(t_1, t_2) = E_{\underset{\sim}{x}(t_1), \underset{\sim}{x}(t_2)}\{x(t_1)\, x(t_2)\}$ Joint density requiring calculation

$= E_{\underset{\sim}{a}}\{\underset{\sim}{a}^2 t_1 t_2\}$ density of $\underset{\sim}{a}$ is given! easy!!

$= E_{\underset{\sim}{a}}\{\underset{\sim}{a}^2\}\, t_1 t_2$

$= \left[\displaystyle\int_0^1 a^2 f_{\underset{\sim}{a}}(a)\, da\right] t_1 t_2 = \dfrac{t_1 t_2}{3}$

Covariance function : $C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$ (Why?)

$= \dfrac{1}{3} t_1 t_2 - \dfrac{t_1 t_2}{4} = \dfrac{t_1 t_2}{12}$.

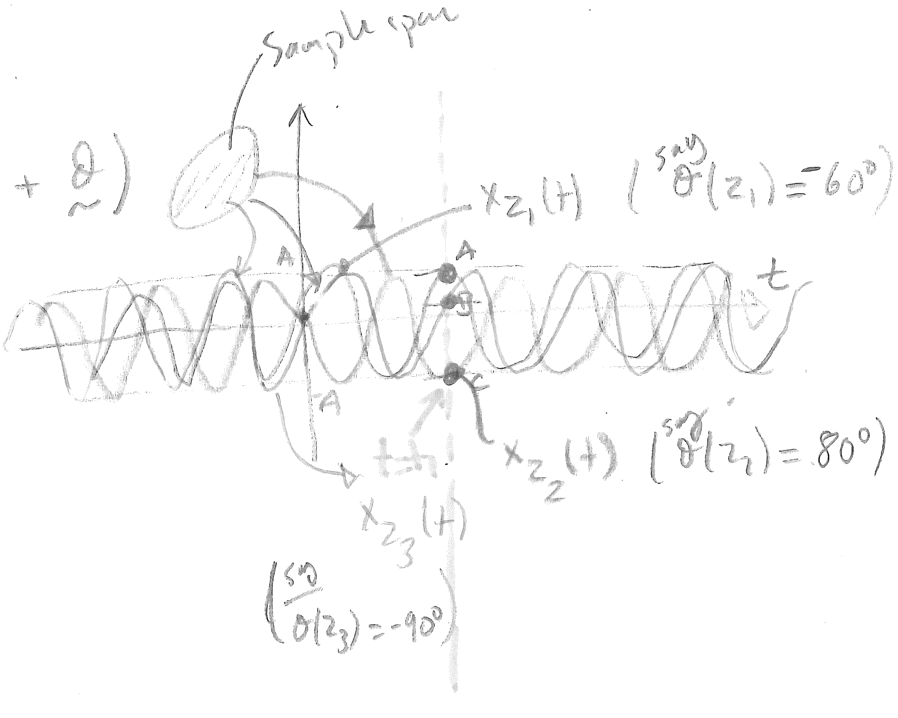
Ex: (Random phase cosine)

$$\underset{\sim}{x}(t) = A\cos(wt + \underset{\sim}{\theta}) \qquad \underset{\sim}{\theta} : \text{Unif. } [0, 2\pi)$$

Find joint p.d.f and moment descriptions of the process

a) Joint p.d.f description

$$X_1 = x(t_1) = A\cos(wt_1 + \underset{\sim}{\theta})$$



In the illustration given above, we observe that 3 realization of r.p. $\underset{\sim}{x}(t)$ [that is $x_{z_1}(t), x_{z_2}(t), x_{z_3}(t)$] end up with the values of A, B, C shown in the figure. These realizations correspond to say θ of $-60°$, $80°$ and $-90°$. Of course, θ is not limited to these values, but uniformly distributed in $[0, 2\pi)$.

1^{st} Order p.d.f:

$\underset{\sim}{\alpha} = \omega t_1 + \underset{\sim}{\theta} \sim \text{Unif.}[\omega t_1, \omega t_1 + 2\pi)$

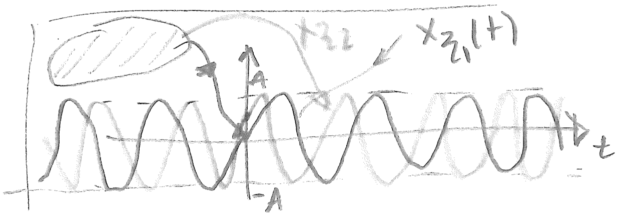
$$\underset{\sim}{x}_1 = \underset{\sim}{x}(t_1) = A\cos(\omega t_1 + \underset{\sim}{\theta}) \overset{(d)}{=} A\cos(\underset{\sim}{\alpha}) \overset{(d)}{=} A\cos(\Phi)$$

$\underset{\sim}{\theta} \sim \text{Unif.}[0, 2\pi]$

$\Phi \sim \text{Unif.}[0, 2\pi)$

Notes: (1) $\underset{\sim}{a} \overset{(d)}{=} \underset{\sim}{b}$ means that r.v.'s $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same density / distribution.

(2) In the last equality on RHS of the equation above, we make use of the fact that $\cos(\cdot)$ is a periodic function with 2π period.



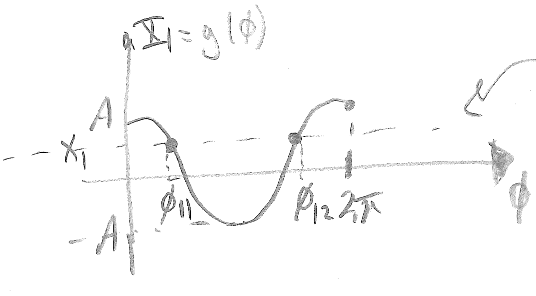
We see that $\underset{\sim}{x}_1 = x(t_1) \overset{(d)}{=} A\cos(\underset{\sim}{\phi})$,

$\underset{\sim}{\phi}$: Unif. $[0, 2\pi)$

hence the density of $\underset{\sim}{x}(t_1)$ does not depend on t_1.

Let's find the density.

$$\underset{\sim}{X}_1 = \underbrace{A\cos(\underset{\sim}{\phi})}_{g(\phi)}$$



Assume $|X_1| < A$

$$f_{X(t_1)}(x_1) = \frac{\overbrace{f_{\Phi}}^{\text{Unif.}[0,2\pi)}(\phi_{11})}{|g'(\phi_{11})|} + \frac{f_{\Phi}(\phi_{12})}{|g'(\phi_{12})|}$$

$\phi_{1k} = \pm\cos^{-1}\left(\frac{x_1}{A}\right)$

or

$x_1 = A\cos(\phi_{1k})$ $k = \{1, 2\}$

$$= \frac{1/2\pi}{|-A\sin(\phi_{11})|} + \frac{1/2\pi}{|-A\sin(\phi_{12})|}$$

$$= \frac{\frac{1}{2\pi}}{A\sqrt{1-\left(\frac{x_1}{A}\right)^2}} + \frac{\frac{1}{2\pi}}{A\sqrt{1-\left(\frac{x_1}{A}\right)^2}}$$

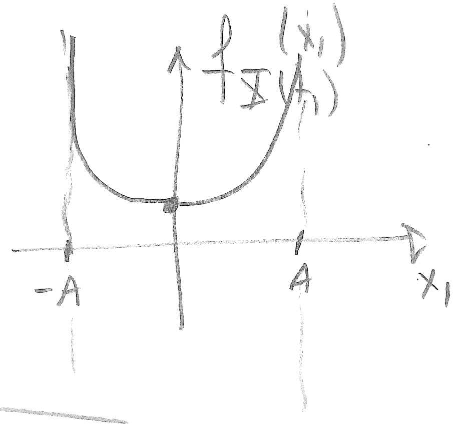
Since

$$\frac{x_1}{A} = \cos(\phi_{1k})$$

and

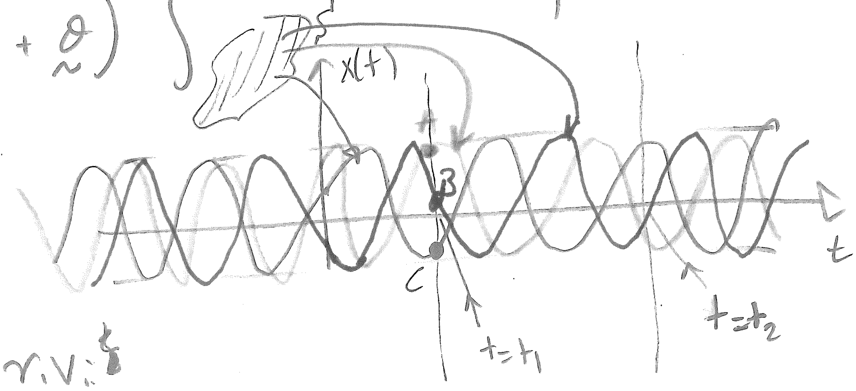
$$|\sin \phi_{1k}| = \sqrt{1-\cos^2\phi_{1k}}.$$

$$f_{X(t_1)}(x_1) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - x_1^2}}, & -A < x_1 < A \\ 0, & \text{o.t.w.} \end{cases}$$



2nd Order Density:

$$\left.\begin{aligned} x_1 &= x(t_1) = A\cos(wt_1 + \theta) \\ x_2 &= x(t_2) = A\cos(wt_2 + \theta) \end{aligned}\right\} \text{ 2 function of 1-r.v.}$$



$$f_{X(t_1),X(t_2)}(x_1, x_2) = ?$$

Let's use conditional r.v.:

$$f_{X(t_1),X(t_2)}(x_1, x_2) = \underbrace{f_{X(t_1)}(x_1)}_{\substack{\text{1st order density} \\ \text{(found before) } \checkmark}} \cdot \underbrace{f_{X(t_2)|X(t_1)}(x_2|x_1)}_{?}$$

Meaning:

Given that (we know) $X(t_1) = x_1$ (say can be either A, B, C in the figure for a 3 outcome experiment and we know that it is say A), what is $X(t_2)$ dist.?

$$f_{\underset{\sim}{X}(t_2)|\underset{\sim}{X}(t_1)}(x_2|x_1) = ?$$

This calculation should be done by the fact that x_1 value of $x_1 = A\cos(wt_1+\theta)$ is observed. Hence, x_1 is no more a r.v., but a fixed value in $[-A, A]$

Given this value of $x_1 = \underset{\sim}{X}(t_1)$, what can you say about $\underset{\sim}{X}_2 = A\cos(wt_2+\underset{\sim}{\theta})$ $(t_1 \neq t_2)$?

In the random slope experiment, the observation at time t_1, $x(t_1) = x_1 = \underset{\sim}{a}t$ was sufficient to "observe" the value of realization of a r.v. $\underset{\sim}{a}$ (Assume $t \neq 0$). Let's see what we have for the random phase cosine.

$$X_2 = \underset{\sim}{X}(t_2) = A\cos(wt_2+\underset{\sim}{\theta}) \qquad (\Delta t \triangleq t_2 - t_1)$$

$$= A\cos(wt_1 + \underset{\sim}{\theta} + w(t_2-t_1))$$

$$= \underbrace{A\cos(wt_1+\underset{\sim}{\theta})}_{x_1}\cos(w\Delta t) - \underbrace{A\sin(wt_1+\underset{\sim}{\theta})}_{\mp\sqrt{A^2-x_1^2}}\sin(w\Delta t)$$

$$= x_1\cos(w\Delta t) \pm \sqrt{A^2-x_1^2}\,\sin(w\Delta t)$$

two possible values of x_2

$$f_{\underset{\sim}{X}(t_2)|\underset{\sim}{X}(t_1)}(x_2|x_1) = \frac{1}{2}\delta(x_2-\hat{x}_{2,1}) + \frac{1}{2}\delta(x_2-\hat{x}_{2,2})$$

$$\hat{x}_{2,1} = x_1\cos(w\Delta t) + \sqrt{A^2-x_1^2}\,\sin(w\Delta t)$$

$$\hat{x}_{2,2} = x_1\cos(w\Delta t) - \sqrt{A^2-x_1^2}\,\sin(w\Delta t)$$

Note that $X_2 | X_1$ r.v. is a binary valued r.v., that is it takes either the value of $\hat{x}_{2,1}$ or $\hat{x}_{2,2}$.
We have not calculated the probability for $\hat{x}_{2,1}$ and $\hat{x}_{2,2}$ outcomes; but, since $\underset{\sim}{\theta}$ r.v. defined at the begining of problem is Unif $[0,2\pi)$ and we expect them to be equally likely which is the case.

Hence,

$$f_{X(t_2), X(t_1)}(x_2, x_1) = f_{X(t_1)}(x_1) \cdot f_{X(t_2)|X_1}(x_2|x_1)$$

$$= \left(\begin{cases} \frac{1}{\pi\sqrt{A^2 - x_1^2}} , & |x_1| < A \\ 0 , & \text{o/w} \end{cases} \right) \cdot \left(\frac{1}{2}\delta(x_2 - \hat{x}_{2,1}(x_1)) + \frac{1}{2}\delta(x_2 - \hat{x}_{22}(x_1)) \right)$$

<u>Partial description of Random Phase Cosine</u>

$x(t) = A\cos(\omega t + \underset{\sim}{\theta})$ → ① $M_x(t)$: Mean function

↳ $\underset{\sim}{\theta}$: Unif. $[0, 2\pi)$

$$M_x(t_1) = E_{\theta}\{x(t_1)\} = \frac{1}{2\pi}\int_0^{2\pi} A\cos(\omega t_1 + \theta)\, d\theta$$

$$= 0.$$

② <u>Auto-correlation func:</u> $R_x(t_1, t_2) = E_{\theta}\{x(t_1)x(t_2)\}$

$$= \frac{1}{2\pi}\int_0^{2\pi} A^2 \cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\, d\theta$$

↓.

$$= \frac{A^2}{2\pi}\int_0^{2\pi} \frac{\cos(w(t_1+t_2)+2\theta) + \cos(w(t_1-t_2))}{2}\, d\theta$$

$$= \frac{A^2}{2\pi}\left[\frac{\cos(w(t_1-t_2))}{2}\,2\pi + \cancel{\int_0^{2\pi}\frac{\cos(w(t_1+t_2)+2\theta)}{2}\,d\theta}^{\,0}\right]$$

$$= \frac{A^2}{2}\cos(w(t_1-t_2))$$

<u>Covariance function</u>: $C_X(t_1,t_2) = R_X(t_1,t_2) - \cancel{\mu_X(t_1)}^{0}\cancel{\mu_X(t_2)}^{0}$.

Note that ① partial description is very easy to determine, since it involves expectation calculations.

② For the random phase cosine $\mu_X(t)$ is constant (independent of time) and $R_X(t_1,t_2)$ is a function of a time difference between independent variables t_1 and t_2. That is $R_X(t_1,t_2) = \text{func}(t_1-t_2)$. Hence, say if we know $R_X(5,0)$ that gives us info for $R(6,1) = R(7,2) = R(5,0)$

or more generally $R_X(t_1+\ell, t_2+\ell) = R_X(t_1,t_2)$ $\forall \ell$ in this example. (This observation will be important in stationarity discussions.)

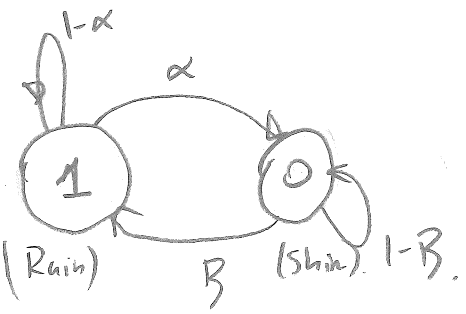
Discrete Random Processes:

The previous descriptions of r.p.'s with continuous variables applies with no change to the discrete setting. We may consider that a discrete r.p. is formed by sampling of the continuous r.p. that is $\tilde{x}[n] = \tilde{X}(nT)$. The joint p.d.f and moment descriptions of $\tilde{x}[n]$ can be given as before. The only change is that process variable "n" is integer valued.

Some random processes are inherently discrete such as markov chains. We can use our earlier knowledge of Markov chains for their joint p.d.f and moment descriptions.

Ex: (Rain or Shine Markov Chain)

$$x[0] = \begin{cases} 1 & \text{w.p. } p_0 \\ 0 & \text{w.p. } \underbrace{\bar{p}_0}_{1-p_0} \end{cases}$$



$$\underline{\underline{P}} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

i) 1^{st} Order density at $n = n_0$

$$\left[f_{\tilde{X}[n_0]}(x_{n_0}=1) \;\; f_{\tilde{X}[n_0]}(x_{n_0}=0) \right] = [p_0 \;\; \bar{p}_0] \cdot \underline{\underline{P}}^{n_0}$$

b) Joint density of $x[n_1]$ and $x[n_2]$.

<u>Remember</u> $f_{X(n_1),X(n_2)}(x_1,x_2) = \underbrace{f_{X(n_1)}(x_1)}_{\text{calculated before}} \cdot \underbrace{f_{X(n_2)|X(n_1)}(x_2|x_1)}_{?}$

conditional density

<u>Conditional density calculation:</u>

① : $f_{X(n_2)|X(n_1)=1}(x_2|x_1=1) = \left[f_{X(n_2)|X(n_1)=1}(x_2=1) \;\; f_{X(n_2)|X(n_1)=1}(x_1=0) \right] = \left[f_{X(n_1)}(x_1=1) \;\; f_{X(n_1)}(x_1=0) \right] \cdot \underline{\underline{P}}^{n_2-n_1}$

$\left[f_{X(n_1)}(x_1=1) \;\; f_{X(n_1)}(x_1=0) \right] = [1 \;\; 0]$

② $f_{X(n_2)|X(n_1)=0}(x_2|x_1=0) = \left[f_{X(n_2)|X(n_1)=0}(x_2=1) \;\; f_{X(n_2)|X(n_1)=0}(x_2=0) \right] = [0 \;\; 1] \cdot \underline{\underline{P}}^{n_2-n_1}$

<u>Example</u>

$$f_{X(5),X(8)}(x_5,x_8) = \begin{cases} \left[[p_0 \;\; \bar{p}_0] \cdot \underline{\underline{P}}^5 \right]_1 \cdot \left[[1 \;\; 0] \cdot \underline{\underline{P}}^3 \right]_1, & x_5=1,\; x_8=1 \\ \left[[p_0 \;\; \bar{p}_0] \cdot \underline{\underline{P}}^5 \right]_1 \cdot \left[[1 \;\; 0] \cdot \underline{\underline{P}}^3 \right]_2, & x_5=1,\; x_8=0 \\ \left[[p_0 \;\; \bar{p}_0] \cdot \underline{\underline{P}}^5 \right]_2 \cdot \left[[0 \;\; 1] \cdot \underline{\underline{P}}^3 \right]_1, & x_5=0,\; x_8=1. \\ \left[[p_0 \;\; \bar{p}_0] \, \underline{\underline{P}}^5 \right]_2 \cdot \left[[0 \;\; 1] \, \underline{\underline{P}}^3 \right]_2, & x_5=0,\; x_8=0 \end{cases}$$

$[x \;\; y]_1 \triangleq x$ (the 1^{st} entry of vector)

$[x \;\; y]_2 \triangleq y$ (the 2^{nd} entry of the vector)

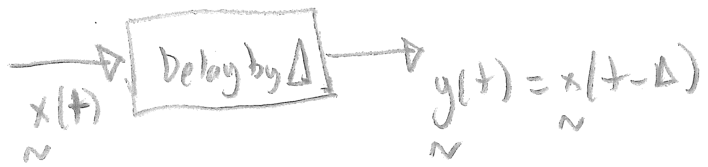
Stationary Random Processes:

The concept of steady-state in deterministic studies arise in linear system theory. In the study of LTI circuits, we know that there is DC and AC steady-state. For DC steady-state (1^{st} so far order circuits), we say that it is sufficient to wait 5τ (5 time constants, $\tau = \{RC, \frac{L}{R}\}$) seconds, to reach steady-state.
For AC steady-state, we assume that transients have died out and we have a response in the form $A\cos(\omega t + \phi)$.

The concept of AC-DC steady-state can be understood as the response of the circuit in a time-interval where the response does not "change" that is it has reached its "final form" (transients died) and starts "repeating".

We will extend this concept to random processes and define stationarity as follows

Stationarity:



If the statistics of $\underset{\sim}{x}(t)$ and its delayed version $\underset{\sim}{y}(t) = \underset{\sim}{x}(t-\Delta)$ are the same for all Δ values, $\underset{\sim}{x}(t)$ is said to be stationary.

Note that, this is indeed a reasonable extension of the familiar steady-state concept to the r.p.'s.

Stationarity in joint density:

① 1st Order Stationarity:

$\underset{\sim}{x}(t) \longrightarrow \boxed{\delta(t-\Delta)} \longrightarrow \underset{\sim}{x}(t-\Delta)$

For 1st order stationarity, we need to have

$f_{\underset{\sim}{X}(t_1)}(x_1)$ and $f_{\underset{\sim}{X}(t_2)}(x_2)$ $\quad t_2 = t_1 - \Delta$

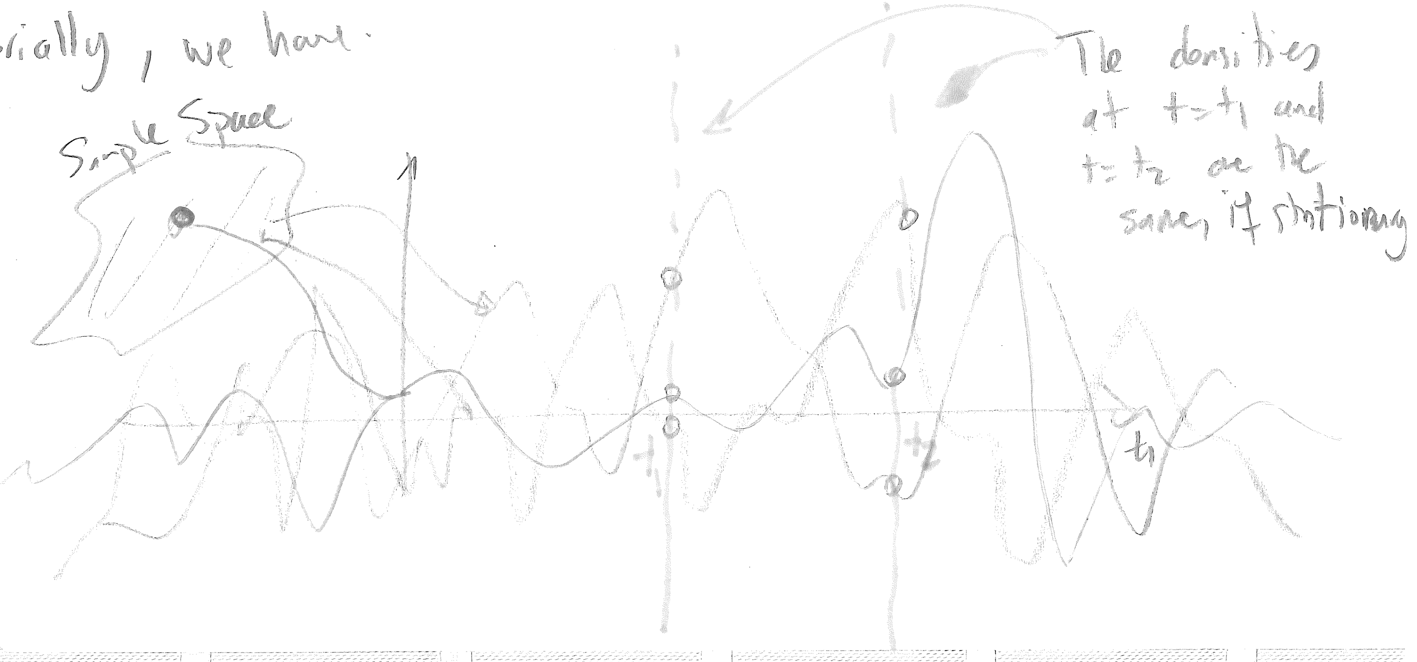
densities to be identical for all t_1 and t_2 values.

We can write this as

$$f_{\underset{\sim}{X}(t_1)}(x) = f_{\underset{\sim}{X}(t_2)}(x) \qquad \forall (t_1, t_2)$$

(Note that we write the equality above we use the same process variable "x" on LHS and RHS of the equation.)

Pictorially, we have.



Ex (Random slope signal)

$$\underset{\sim}{x}(t) = \underset{\sim}{a}\, t, \quad \underset{\sim}{a} \sim \text{Unif}\,[0,1]$$

$\underset{\sim}{X}_1 = x(1) \sim \text{Unif}\,[0,1]$

$\underset{\sim}{X}_5 = x(5) \sim \text{Unif}\,[0,5]$

$(X_{t_x} = x(t_x) \sim \text{Unif}\,[0, t_x])$

densities at different sampling times are different

$\Downarrow$

Not 1st Order Stationary

Random phase cosine

$$\underset{\sim}{x}(t) = A\cos(wt + \underset{\sim}{\theta}), \quad \underset{\sim}{\theta} \sim \text{Unif}\,[0, 2\pi)$$

$$f_{X(t_1)}(x_1) = \begin{cases} \dfrac{1}{\pi\sqrt{A^2 - x_1^2}}, & |x_1| < A \\ 0, & \text{o.w.} \end{cases}$$

density expression at time t_1 does not have any t_1.

It is clear that $f_{X(t_1)}(x) = f_{X(t_2)}(x)$.

$\Downarrow$

1st Order Stationary ✓

(2) 2nd Order Stationarity:

$$\underset{\sim}{x}(t) \longrightarrow \boxed{f(t-\Delta)} \longrightarrow \underset{\sim}{x}(t-\Delta)$$

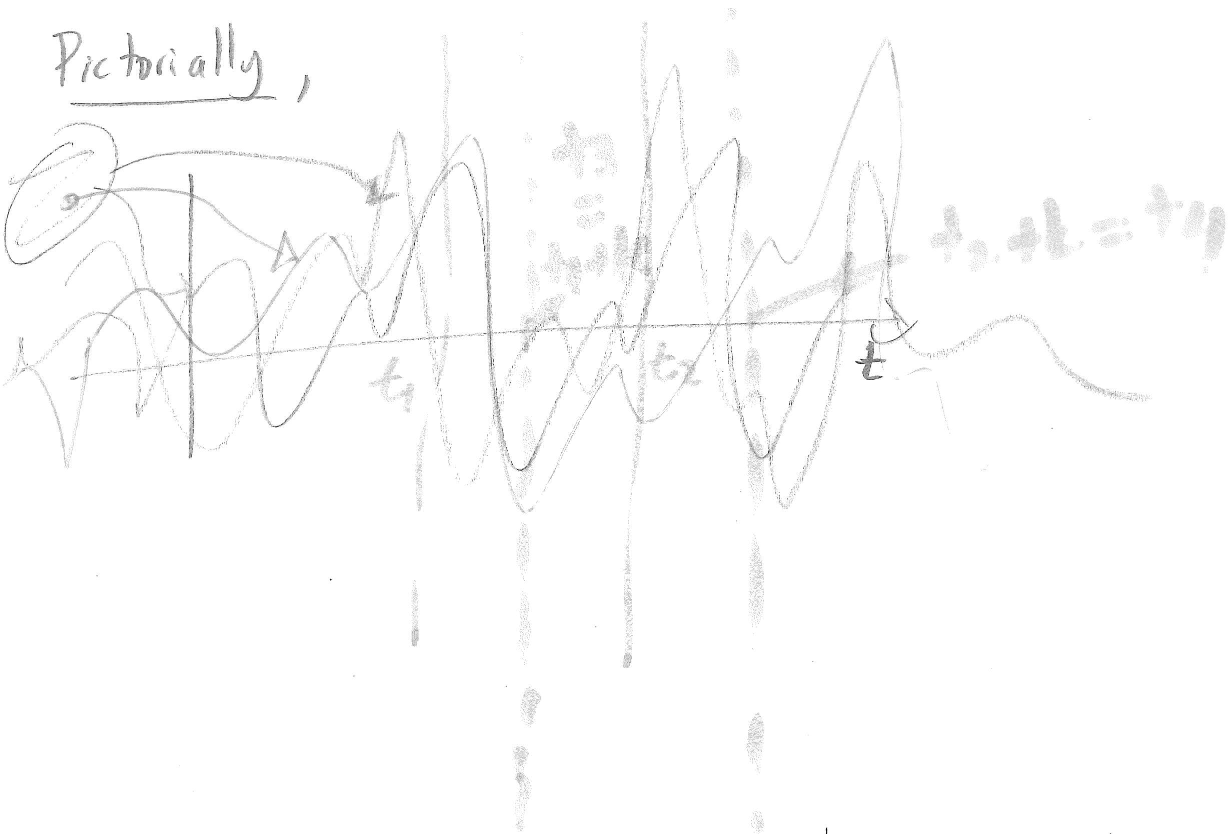
The 2nd Order stationary requires

$$f_{X(t_1), X(t_2)}(x_1, x_2) = f_{X(t_1-\Delta), X(t_2-\Delta)}(x_1, x_2)$$

for all (t_1, t_2) and Δ

Note that this requires the same joint density when both sampling instants t_1 and t_2 are shifted by the same amount Δ.

Pictorially,



$$\left.\begin{aligned} x_1 &= x(t_1) \\ x_2 &= x(t_2) \end{aligned}\right\} \qquad \left\{\begin{aligned} x_1 &= x(\overset{t_3}{t_1 + k}) \\ x_2 &= x(\underset{t_4}{t_2 + k}) \end{aligned}\right.$$

(t_3 and t_4 are offset from t_1 and t_2 by the same delay value!!)

If joint densities are identical for all "k" (time offset) → 2nd Order stationary.

Hence, for 2nd Order stationarity, the joint density of $\underset{\sim}{X}(t_1)$ and $\underset{\sim}{X}(t_2)$ should not depend on t_1 and t_2 parameters, but their difference $\tau = t_1 - t_2$.

time seperation between observations

(Since 2nd Order Stationarity requires

$$f_{\underset{\sim}{X}(t_1),\underset{\sim}{X}(t_2)}(x_1, x_2) \overset{\text{valid for all } \Delta}{=} f_{\underset{\sim}{X}(t_1-\Delta),\underset{\sim}{X}(t_2-\Delta)}(x_1, x_2)\Big|_{\Delta = t_2} = f_{\underset{\sim}{X}(t_1-t_2),\underset{\sim}{X}(0)}(x_1, x_2).$$

Random phase cosine is an example for 2nd Order stationary r.p.'s. (Check the joint density expression p. 18).

③ N^{th} Order Stationarity:

If $f_{\underset{\sim}{X}(t_1), \underset{\sim}{X}(t_2), \underset{\sim}{X}(t_3), \ldots, \underset{\sim}{X}(t_N)}(x_1, x_2, \ldots, x_N) = f_{\underset{\sim}{X}(t_1-\Delta), \underset{\sim}{X}(t_2-\Delta), \ldots, \underset{\sim}{X}(t_N-\Delta)}(x_1, x_2, \ldots, x_N)$ (*)

for all $(t_1, t_2, \ldots, t_N)$ and Δ $\longrightarrow$ $\underset{\sim}{X}(t)$ is N^{th} order stationary.

Notes: ① By marginalization operation, we can see that

If $\underset{\sim}{X}(t)$ is N^{th} order stationary, then it is also sationary of order $(N-1)$ and all smaller orders.

② Set $\Delta = t_1$ in (*) and observe that N^{th} order stationary processes depend not on $(t_1, t_2, \ldots, t_N)$; but on $(t_2 - t_1, t_3 - t_1, t_4 - t_1, \ldots, t_N - t_1)$.

③ For N^{th} order stationary processes, we may assume that $t_1 = 0$ ^first sampling time, by considering that we shift the process by t_1 seconds to the left (delay of $\Delta = -t_1$ seconds) without any loss of generality.

Ex Random phase cosine

$$x(t) = A\cos(wt + \underset{\sim}{\theta})$$

$\underset{\sim}{\theta}$: Unif $[0, 2\pi)$

$$x(t) \longrightarrow \boxed{\delta(t-\Delta)} \longrightarrow A\cos(wt \underbrace{- w\Delta + \underset{\sim}{\theta}}_{\underset{\sim}{\phi}}) = \underset{\sim}{y}(t)$$

Let's call $\underset{\sim}{\phi} = \underset{\sim}{\theta} - w\Delta$ ← constant → the density of ϕ is also uniform in $[-w\Delta, -w\Delta + 2\pi)$;

but since $\cos(x)$ is periodic with 2π, the shift of Uniform density to another interval does change the process definition

$$\underset{\sim}{x}(t) = A\cos(wt + \underset{\sim}{\theta}) = A\cos(wt + \underset{\sim}{\phi})$$

$\underset{\sim}{\theta}$: Unif. $[0, 2\pi)$ $\qquad$ $\underset{\sim}{\phi}$: Unif. $[-w\Delta, -w\Delta + 2\pi)$.

(Think about the realizations of the process.)

Hence, Random phase cosine is N^{th} order stationary for all N.

<u>Strict Sense Stationary:</u> If $\underset{\sim}{x}(t)$ is N^{th} order stationary for all N values → $\underset{\sim}{x}(t)$ is said to be strict sense stationary (SSS).

Random phase cosine is an example.

Summary:

* The stationarity in joint densities require invariance of the joint densities to the data collection starting time, say t_1.

* For joint density stationary processes, we may assume that the first data collection time $t_1 = 0$ without any loss of generality. This leads to

a) $f_{X(t_1)}(x_1) = \text{func}(x_1) \; \forall t_1$ for 1^{st} order stationarity

b) $f_{X(t_1), X(t_2)}(x_1, x_2) = \text{func}(x_1, x_2, t_1 - t_2)$ for 2^{nd} order stationary $\forall (t_1, t_2)$

* In many practical cases, the "time origin" of the process is randomized, that is the "starting time" becomes an arbitrary time via a randomization process. The random phase cosine is an example. The starting phase value, i.e phase of $A\cos(\omega t + \theta)$ at $t = 0$, is randomized. This can be seen as equivalent to process starting time randomization. [Since

$$A\cos(\omega t + \theta) = A\cos(\omega(t + \epsilon)).]$$

$\omega = 2\pi/T$ ← period

$\epsilon \sim \text{Unif.}[0, T)$

Stationarity in moment/partial description:

Stationarity in joint pdf requiring calculation of joint densities which can be practically difficult. An alternate form of stationarity can be given in terms of partial descriptions namely moment descriptions:

① Stationarity in the mean function / Stationarity in 1^{st} moment

$\underset{\sim}{x}(t)$: r.p. $\longrightarrow$ $M_x(t) = E\{x(t)\}$: mean function.

$$\underset{\sim}{x}(t) \longrightarrow \boxed{\delta(t-\Delta)} \longrightarrow \underset{\sim}{x}(t-\Delta) = \underset{\sim}{y}(t)$$

Stationarity requires the description of $\underset{\sim}{y}(t) = \underset{\sim}{x}(t-\Delta)$ to be invariant "Δ" (delay) parameter. From mean function characterization, this is equivalent to

$$M_x(t) = M_y(t) = M_x(t-\Delta) \qquad \forall \Delta$$

mean func. at input

mean func. at output.

Hence, $M_x(t) = M_x(t-\Delta) = c \leftarrow$ constant for stationarity in the mean func.

(2) Stationarity in 2^{nd} moment:

Remember that, we have

$$R_x(t_1,t_2) = E\{\underset{\sim}{x}(t_1)\,\underset{\sim}{x}(t_2)\} \quad \text{: auto-correlation func.}$$

$$C_x(t_1,t_2) = E\{[\underset{\sim}{x}(t_1)-\mu_x(t_1)][\underset{\sim}{x}(t_2)-\mu_x(t_2)]\} \quad \text{: auto-covariance func.}$$
$$= R_x(t_1,t_2) - \mu_x(t_1)\mu_x(t_2)$$

Considering the stationarity requirement for 2^{nd} order moments:

$$R_x(t_1,t_2) = \underbrace{R_y(t_1,t_2)}_{E\{x(t_1-\Delta)\,x(t_2-\Delta)\}} = R_x(t_1-\Delta, t_2-\Delta) \quad \forall \Delta$$

The requirement of

$$R_x(t_1,t_2) = R_x(t_1-\Delta, t_2-\Delta) \quad \forall \Delta$$

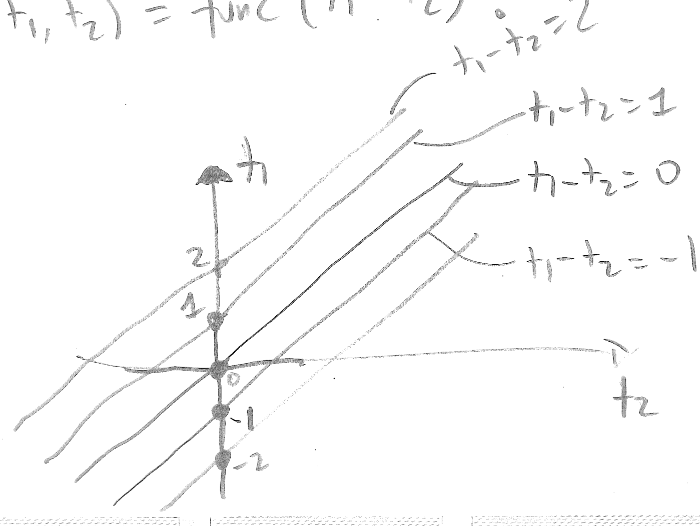
results in

$$R_x(t_1,t_2) = R_x(t_1-\Delta, t_2-\Delta)\Big|_{\Delta=t_2} = R_x(t_1-t_2, 0) = \text{func}(t_1-t_2)$$

that is

$$R_x(t_1,t_2) = \text{func}(t_1-t_2).$$

Pictorially,



$R_x(t_1,t_2)$ func. does not vary on straight lines shown!

Similarly, stationarity in covariance func. requires

$C_x(t_1,t_2) = \text{func}(t_1 - t_2)$.

Notes: ① Stationarity in higher order moments can also be defined; but not frequently used.

② Wide Sense Stationarity (WSS): A process is called wide sense stationary, if process is stationary in the mean and auto-correlation and covariance functions.

③

1st Order Stationarity $\xrightarrow{\text{implies}}$ Stationarity in 1st moment

↑   ↑

2nd Order Stationarity → Stationarity in 2nd moments

↑   WSS

3rd Order Stationarity → Stationarity in 3rd moments

↑ implies

⋮

Nth Order Stationary

↑

⋮

S.S.S

④ If $x(t)$ is complex valued, $R_x(t_1,t_2) \triangleq E\{x(t_1)\,x^*(t_2)\}$ ← conjugate!

so that

$R_x(t_1,t_1) = E\{|x(t_1)|^2\} \geq 0$

Ex

① $x_{①}(t) = a t$ (Random slope Signal), $a \sim$ Unif.$[0,1]$

$\mu_{x_①}(t) = E\{a\} t = \frac{t}{2} \neq c$

Not stationary in mean func
⇓
Not WSS.

② $x_{②}(t) = A\cos(wt + \theta)$, $\theta \sim$ Unif $[0, 2\pi)$ ⟶ we know that $x(t)$ is SSS; Hence, it should also be WSS.

Let's verify.

$$\mu_{x_②}(t) = E\{A\cos(wt+\theta)\} = A\, E_\theta\{\cos(wt+\theta)\} = 0 \quad ✓$$

$$R_{x_②}(t_1,t_2) = E\{A\cos(wt_1+\theta)\, A\cos(wt_2+\theta)\} = \frac{A^2}{2}\cos(w(t_1-t_2)) = \text{func}(t_1-t_2) \quad ✓$$

(p.19) (Lec. 30 8th min.)

③ $x_{③}(t) = \sum_{k=0}^{\infty} a_k e^{jw_0 k t}$ a_k: zero-mean independent r.v.'s (complex valued)

($x_③(t)$ is a periodic waveform whose Fourier series coefficients are random.)

i) $\mu_{x_③}(t) = E\left\{\sum_{k=0}^{\infty} a_k e^{jw_0 k t}\right\} = \sum_{k=0}^{\infty} \underbrace{E\{a_k\}}_{\mu_a} e^{jw_0 k t} = \cancel{\mu_a}^{\,0} \sum_{k=0}^{\infty} e^{jw_0 k t} = 0 \quad ✓$

ii) $R_{x_③}(t_1,t_2) = E\left\{\left(\sum_{k_1=0}^{\infty} a_{k_1} e^{jw_0 k_1 t_1}\right)\left(\sum_{k_2=0}^{\infty} a_{k_2} e^{jw_0 k_2 t_2}\right)^*\right\}$

$$= \sum_{k_1}\sum_{k_2} \underbrace{E\{a_{k_1} a_{k_2}^*\}}_{E\{|a_{k_1}|^2\}\delta[k_1-k_2]} e^{jw_0(k_1t_1 - k_2t_2)}$$

$$E\{|a_{k_1}|^2\}\delta[k_1-k_2] = \begin{cases} E\{|a_{k_1}|^2\}, & k_1 = k_2 \\ E\{a_{k_1}\}E\{a_{k_2}^*\}, & k_1 \neq k_2 \end{cases}$$

$$= \sum_{k_1} E\{|a_{k_1}|^2\} \underbrace{\sum_{k_2} \delta[k_2-k_1] e^{jw_0(k_1t_1-k_2t_2)}}_{= e^{jw_0 k_1 (t_1-t_2)}} = \sum_{k_1} E\{|a_{k_1}|^2\} e^{jw_0 k_1 (t_1-t_2)} \quad ✓$$

So, $x_③(t)$ is WSS

Gaussian Processes:

$\underset{\sim}{x}(t)$ is called a Gaussian process if N^{th} order joint density of the process can be written in the form:

$$f_{\underset{\sim}{x}(t_1), \underset{\sim}{x}(t_2), \dots, \underset{\sim}{x}(t_N)}(x_1, x_2, \dots, x_N) = \frac{1}{(\sqrt{2\pi})^N \underbrace{|\underline{\underline{C}}_x|}_{\det(\underline{\underline{C}}_x)}^{1/2}} \cdot e^{-\frac{1}{2}(\underline{x}-\underline{\mu}_x)^T \underline{\underline{C}}_x^{-1} (\underline{x}-\underline{\mu}_x)}$$

Remember: $\underline{X}_k = \underset{\sim}{x}_k \triangleq \underset{\sim}{x}(t_k)$

$$\underset{N\times 1}{\underline{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \underset{N\times 1}{\underline{\mu}_x} = E\left\{ \begin{bmatrix} \underset{\sim}{x}(t_1) \\ \underset{\sim}{x}(t_2) \\ \vdots \\ \underset{\sim}{x}(t_N) \end{bmatrix} \right\} = \begin{bmatrix} \mu_x(t_1) \\ \mu_x(t_2) \\ \vdots \\ \mu_x(t_N) \end{bmatrix}$$

mean vector

x_N

$\mu_x(t)$: mean function

$$\underset{N\times N}{\underline{\underline{C}}_x} = E\left\{ \begin{bmatrix} \underset{\sim}{x}(t_1) - \mu_x(t_1) \\ \underset{\sim}{x}(t_2) - \mu_x(t_2) \\ \vdots \\ \underset{\sim}{x}(t_N) - \mu_x(t_N) \end{bmatrix} \begin{bmatrix} x(t_1) - \mu_x(t_1) & \cdots & x(t_N) - \mu_x(t_N) \end{bmatrix} \right\}$$

Covariance matrix

Notes: ① $[\underline{\mu}_x]_k = \mu_x(t_k)$

k^{th} element of mean vector

mean function evaluated at $t = t_k$.

$$[\underline{\underline{C}}_x]_{k,l} = \mathrm{Cov}(\underset{\sim}{x}(t_k), \underset{\sim}{x}(t_l)) = E\{(X_k - \mu_x(t_k))(X_l - \mu_x(t_l))\}$$
$$= C_x(t_k, t_l)$$

$\underset{\sim}{x}(t_l)$

k^{th} row, l^{th} column of covariance matrix

Covariance function evaluated at (t_k, t_l)

(2) Observe that knowing 1^{st} and 2^{nd} moments, i.e. mean and covariance function is <u>sufficient to write</u> the joint p.d.f <u>description</u>.

(3) <u>Special case for $N=1$</u>

$$f_{X(t_1)}(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma_{x_1}} \cdot e^{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_{x_1}^2}}$$

$E\{x(t_1)\} = \mu(t_1)$

$\sigma_{x_1}^2 = Var\{x(t_1)\} = Cov(x(t_1), x(t_1))$

(4) If process is <u>WSS and Gaussian</u>, then

$\mu_x(t) = \text{constant} = c \leftarrow$ constant mean function

$C_x(t_k, t_e) = func(t_k - t_e) \leftarrow$ Covariance is only func. of "lag" variable $z = t_k - t_e$.

$\rightarrow$ the Gaussian process is also SSS.

(5) Gaussian processes are important since

i) linear operations on Gaussian processes results in processes which are also Gaussian. (Easy to prove from moment generating functions)

ii) By central limit theorem, Gaussian dist./process is "limit" process when many independent effects additively combined

Ex: $\underset{\sim}{x}(t)$: is a Gaussian process with zero mean and auto-correlation function, $R_x(t_k, t_e) = \dfrac{\sin(\pi(t_k - t_e))}{\pi(t_k - t_e)}$

$= \text{sinc}(z)$ $\quad z \triangleq t_k - t_e \quad \text{sinc}(x) \triangleq \dfrac{\sin \pi x}{\pi x}$

i) Is $x(t)$ WSS?

ii) Determine a sampling rate T s.t. $x[n] = x(nT)$ samples are independent.
Write the joint density for $x[1], x[2], x[3]$ for such a sampling period.

Soln i) $\mu_x(t) = 0$ and $R_x(t_k, t_e) = \text{sinc}(\overbrace{t_k - t_e}^{z}) = \text{sinc}(z)$

Hence, $\underset{\sim}{x}(t)$ is WSS.

ii) $x[n]$ and $x[m]$ are Gaussian distributed with

N=1 $x[n] \sim N(0, \sigma^2_{x_n})$ and $x[m] \sim N(0, \sigma^2_{x_m})$. (The case for $N = 1$). $\sigma^2_{x_n} = E\{(\underset{\sim}{x}(t_n) - 0)^2\} = \text{Cov}(x(t_n), x(t_n))$

$= C_x(t_n, t_N)$

$= R_x(t_n, t_n) + \cancel{\mu_x^2(t_n)}^{0}$

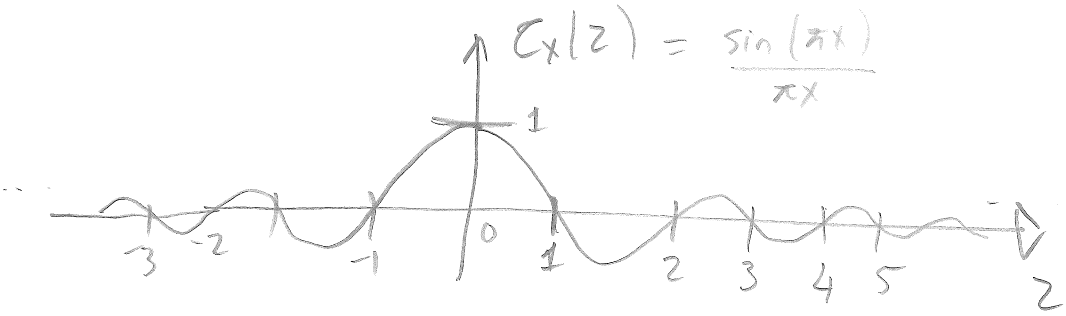
$= \text{sinc}(0)$

$= 1.$

Hence $x[n] \sim N(0, 1) \quad \forall n$

N=2 $x[n], x[m]$ are jointly Gaussian dist. pair of r.v.'s, the r.v.'s have the covariance func.

$C_x(t_k, t_e) = R_x(t_k, t_e) = \text{sinc}(t_k - t_e) = \text{sinc}(z)$



It is clear that $\underset{z=k}{C_x(z)} = 0$ $k \in I \leftarrow$ integer. $(k \neq 0)$

Hence, samples of the process $x(t)$ seperated by 1 second (or its integer multiples) are <u>uncorrelated</u>, i.e. have zero covariance. Since samples are jointly Gaussian distributed, this is equivalent to samples being independent.

Let's select $T=1$, $x[n] = \underset{\sim}{x}[nT]$. Then,

$$\underline{x} = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \end{bmatrix} \quad \underline{\mu}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\underline{C}}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f_{\underline{X}[1],\underline{X}[2],\underline{X}[3]}(x_1, x_2, x_3) = \frac{1}{(\sqrt{2\pi})^3 |\underline{\underline{C}}_x|^{1/2}} e^{-\frac{1}{2}(x-\mu_x)^T \underline{\underline{C}}_x^{-1} (x-\mu_x)}$$

$$= \frac{1}{(\sqrt{2\pi})^3 \cdot 1} e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)}$$

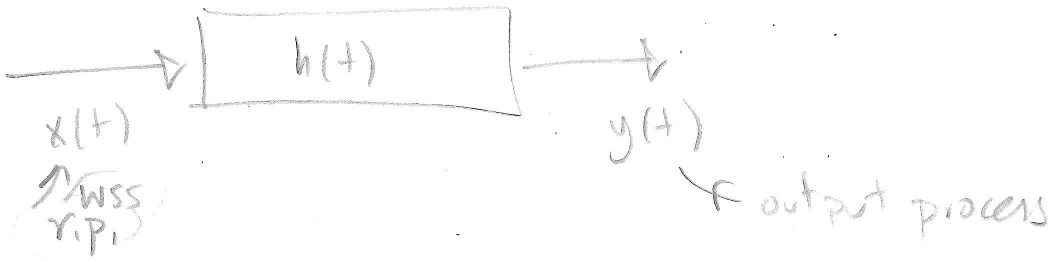
$$= \underbrace{f_{\underline{X}(1)}(x_1)}_{N(x_1;0,1)} \cdot \underbrace{f_{\underline{X}(2)}(x_2)}_{N(x_2;0,1)} \cdot \underbrace{f_{\underline{X}(3)}(x_3)}_{N(x_3;0,1)}$$

joint density factorized into marginals

definition of independence for r.v.s

LTI Processing of WSS Processes:

We will show that LTI processing of an WSS process results in another WSS process at the output.



where

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(t')\, x(t-t')\, dt'.$$

Before showing this important result, we define jointly WSS processes.

Jointly WSS processes:

$\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ are jointly WSS if

1) $\underset{\sim}{x}(t)$ is WSS
2) $\underset{\sim}{y}(t)$ is WSS
3) $R_{xy}(t_1, t_2) = E\{x(t_1)\, y(t_2)\} = \text{func}(\tau)$ $\quad (\tau \triangleq t_1 - t_2)$.

(cross-correlation of $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$)

Complex valued case: $\boxed{R_{xy}(t_1, t_2) \triangleq E\{x(t_1)\, y^*(t_2)\}}$

$$\underset{\text{WSS}}{x(t)} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$y(t)$ is WSS iff
- (1) $\mu_y(t) = c$
- (2) $R_y(t_1, t_2) = \text{func}(t_1 - t_2)$

(1) $\mu_y(t) \overset{?}{=}$ constant

$$y(t) = \int_{-\infty}^{\infty} h(t')\, x(t-t')\, dt'$$

$$E\{y(t)\} = \int_{-\infty}^{\infty} h(t')\, \underbrace{E\{x(t-t')\}}_{\mu_x}\, dt' = \mu_x \int_{-\infty}^{\infty} h(t')\, dt' = \mu_x \cdot H(0)$$

μ_x: constant since $x(t)$: WSS

$H(s) = \mathcal{L}\{h(t)\}$

Hence $\mu_y(t) = \mu_x \overbrace{H(0)}^{\text{D.C. gain of filter}} =$ constant.

(2) $R_y(t_1, t_2) \overset{?}{=} \text{func}(t_1 - t_2)$

Let's find $R_{xy}(t_1, t_2)$ first

$$R_{xy}(t, t-\tau) = E\{x(t)\, y^*(t-\tau)\}$$

$$= E\left\{x(t) \left[\int_{-\infty}^{\infty} h(t')\, x(t-\tau-t')\, dt'\right]^*\right\}$$

$$= \int_{-\infty}^{\infty} h^*(t')\, \underbrace{E\{x(\overbrace{t}^{t_1})\, x^*(\overbrace{t-\tau-t'}^{t_2})\}}_{r_x(\underbrace{\tau+t'}_{t_1-t_2}) = R_x(t_1, t_2)}\, dt'$$

$$= \int_{-\infty}^{\infty} h^*(t')\, r_x(\tau+t')\, dt'$$

Not a func. of "t" only τ

$$= \int_{-\infty}^{\infty} h^*(-t'') \, r_x(\tau - t'') \, dt'' \qquad (t'' = -t')$$

$$= h^*(-\tau) * r_x(\tau)$$

Hence,

$$R_{xy}(t, t-\tau) = h^*(-\tau) * r_x(\tau) = r_{xy}(\tau)$$

auto-correlation func. of $x(t)$

$R_y(t_1, t_2) = ?$

$$R_y(t, t-\tau) = E\{ y(t) \, y^*(t-\tau) \}$$

$$= E\left\{ \left(\int_{-\infty}^{\infty} h(t') \, x(t-t') \, dt' \right) y^*(t-\tau) \right\}$$

$$= \int_{-\infty}^{\infty} h(t') \, \underbrace{E\{ x(\overbrace{t-t'}^{t_1}) \, y^*(\overbrace{t-\tau}^{t_2}) \}}_{R_{xy}(t_1, t_2) = r_{xy}(\tau - t')} \, dt'$$

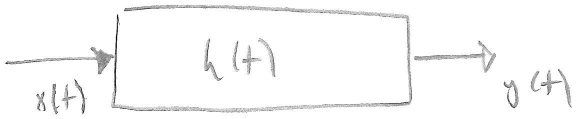
$$= \int_{-\infty}^{\infty} h(t') \, r_{xy}(\tau - t') \, dt'$$

Not a func. of "t"!

$$= h(\tau) * r_{xy}(\tau)$$

$$R_y(t_1, t_2) = r_y(\underbrace{t_1 - t_2}_{\tau}) = r_y(\tau) = h(\tau) * h^*(-\tau) * r_x(\tau) \, .$$

Ex:



$\underset{\sim}{x}(t)$: zero-mean WSS process with auto-correlation func. $r_x(\tau)$.

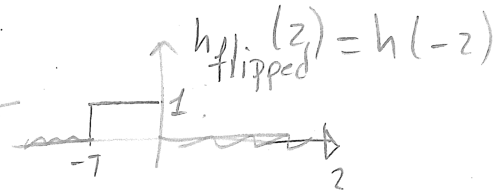
$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0, & \text{othw.} \end{cases}$$

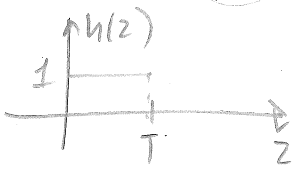
Find the moment characterization of the output process.

Soln. $y(t)$ is a WSS since $x(t)$ is WSS and processing is LTI.

i) $\mu_y(t) = H(0) \cdot \cancel{\mu_x}^{\,0} = 0$

ii) $R_y(t_1, t_2) = \underset{\substack{\uparrow \\ t_1 - t_2}}{r_y(\tau)} = r_x(\tau) * h(\tau) * h^*(-\tau)$





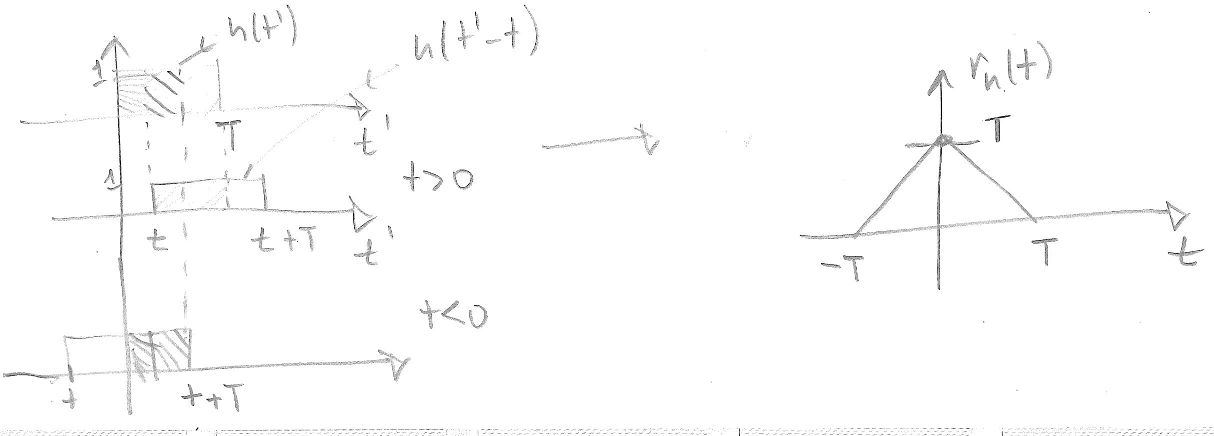
Notes:

(1) $h(t) * h^*(-t) \triangleq r_h(t)$ ← $\boxed{\text{deterministic auto-correlation func.}}$

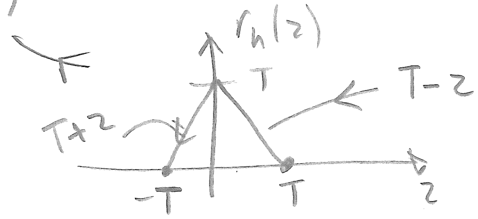
$$r_h(t) = \int_{-\infty}^{\infty} h(t')\, h^*(-(t - t'))\, dt'$$

$$\boxed{r_h(t) \triangleq \int_{-\infty}^{\infty} h(t')\, h^*(t' - t)\, dt'}$$

Note: $r_h(t) = r_h^*(-t)$



$r_y(\tau) = r_x(\tau) * r_h(\tau)$



(2) If $\underset{\sim}{x}(t)$ in the example is given as Gaussian process, $y(t)$ is also Gaussian with mean function $\mu_y(t) = 0$ and covariance function $R_y(t_1, t_2) = r_y(t_1 - t_2)$.

Let's assume that $r_x(\tau) = \delta(\tau)$, then $r_y(\tau) = \overbrace{\delta(\tau)}^{r_x(\tau)} * r_h(\tau)$
$= r_h(\tau)$.

Then the joint density of $\underbrace{y(4)}_{Y_A}, \underbrace{y(4+\frac{T}{2})}_{Y_B}, \underbrace{y(4+\frac{3T}{4})}_{Y_C}$ is

$$f_{Y_A, Y_B, Y_C}(y_a, y_b, y_c) = \frac{1}{(2\pi)^{3/2} |\underset{=}{C_y}|^{1/2}} e^{-\frac{1}{2} [y_a \; y_b \; y_c] \underset{=}{C_y}^{-1} \begin{bmatrix} y_a \\ y_b \\ y_c \end{bmatrix}}$$

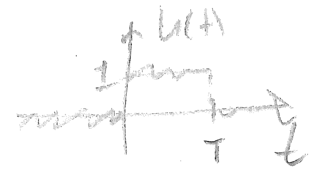
where.

$$\underset{=}{C_y} = \begin{bmatrix} E\{Y_A^2\} & E\{Y_A Y_B\} & E\{Y_A Y_C\} \\ E\{Y_B Y_A\} & E\{Y_B^2\} & E\{Y_B Y_C\} \\ E\{Y_C Y_A\} & E\{Y_C Y_B\} & E\{Y_C^2\} \end{bmatrix} = \begin{bmatrix} r_y(0) & r_y(-\frac{T}{2}) & r_y(-\frac{3T}{4}) \\ r_y(\frac{T}{2}) & r_y(0) & r_y(-\frac{T}{4}) \\ r_y(\frac{3T}{4}) & r_y(\frac{T}{4}) & r_y(0) \end{bmatrix}$$

$$= \begin{bmatrix} T & T/2 & T/4 \\ T/2 & T & 3T/4 \\ T/4 & \frac{3T}{4} & T \end{bmatrix}$$

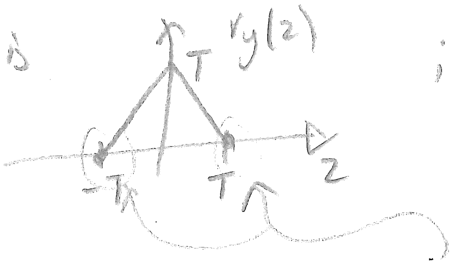
We see from the last example that the processing below

$$\underset{\sim}{x}(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow \underset{\sim}{y}(t)$$



where $\underset{\sim}{x}(t)$ is a zero-mean WSS Gaussian process with auto-correlation $r_x(z) = \delta(z)$;

yields an output process $y(t)$ which is also WSS Gaussian process with zero mean; but auto-correlation $r_y(z)$. Since $r_y(z)$ is



; all samples of $y(t)$ which are seperated by more than T seconds are uncorrelated; hence independent. But samples which lie in a T-second window are not independent.

Considering LTI filtering operation with $h(t)$, we note that $y(t_x)$ is nothing but sum/integral of all input in $[t_x - T, t_x]$; hence $y(t_x)$ and $y(t_y)$ do not process the same input sample $x(t)$ if $|t_y - t_x| > T$. This shows that $y(t_y)$ and $y(t_x)$ are independent from each other for this example.

The last example shows that by LTI filtering of WSS; we can "reshape" the input by changing its moment descriptions. We continue with the definition of white noise which is the simplest process that can be utilized for the synthesis of other WSS processes.

White-noise:

A process $x(t)$ is called white-noise, if

① $\mu_X(t) = 0$,

② $R_X(t_1, t_2) = f(t_1)\,\delta(t_1 - t_2)$.

Note that ② says that the samples of white-noise process is uncorrelated for any $t_1 \neq t_2$ pair. Note that $R_X(t_1, t_1) = C_X(t_1, t_1) = f(t_1) \leftarrow VAR\{x(t_1)\}$ is a function of t_1. Hence, the process variance can change from t_1 to t_2.

Special case, $x(t)$ is WSS white-noise

① $\mu_X(t) = 0$,

② $R_X(t_1, t_2) = \sigma_X^2\,\delta(t_1 - t_2)$. $\leftarrow$ The process variance is constant!

Properties of Auto-correlation function of WSS processes:

① Hermitian symmetry: $R_x(\tau) = R_x^*(-\tau)$

Since

$$R_x(\tau) = E\{x(t)\,x^*(t-\tau)\}$$

and

$$\left[R_x(-\tau)\right]^* = \left[E\{x(t)\,x^*(t+\tau)\}\right]^*$$
$$= E\{x(t+\tau)\,x^*(t)\} = R_x(\tau)$$

② Positive semi-definiteness:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y(t_1)\,E\{x(t_1)\,x^*(t_2)\}\,y^*(t_2)\,dt_1\,dt_2 \geq 0 \quad \forall y(t)$$

Since

conjugate

$$E\left\{\left[\int x(t_1)\,y(t_1)\,dt_1\right]\left[\int x^*(t_2)\,y^*(t_2)\,dt_2\right]\right\}$$

$$= E\left\{\left|\int x(t_1)\,y(t_1)\,dt_1\right|^2\right\} \geq 0 \quad \forall y(t)$$

Note: $x(t)$: WSS $\rightarrow$ $\iint y(t_1)\,R_x(t_1-t_2)\,y^*(t_2)\,dt_1\,dt_2 \geq 0, \ \forall y(t)$

③ $R_x(0) \geq 0$ (Since $E\{x(t)\,x^*(t)\} \geq 0$)

④ $R_x(0) \geq |R_x(\tau)| \quad \forall \tau$

Let's prove this for the real valued processes for simplicity.

Proof: $y(t) = x(t) - x(t-\Delta) = x(t) * \underbrace{(\delta(t) - \delta(t-\Delta))}_{h(t)}$

$$r_y(\tau) = r_x(\tau) * \underbrace{h(\tau) * h(-\tau)}_{\ } \quad (\text{Why?})$$

← $r_h(\tau)$: deterministic auto-correlation

$$r_y(z) = r_x(z) * r_n(z) = 2r_x(z) - r_x(z-\Delta) - r_x(z+\Delta)$$ 45

Since

$$r_y(0) \geq 0 \longrightarrow 2r_x(0) > r_x(-\Delta) + r_x(\Delta)$$

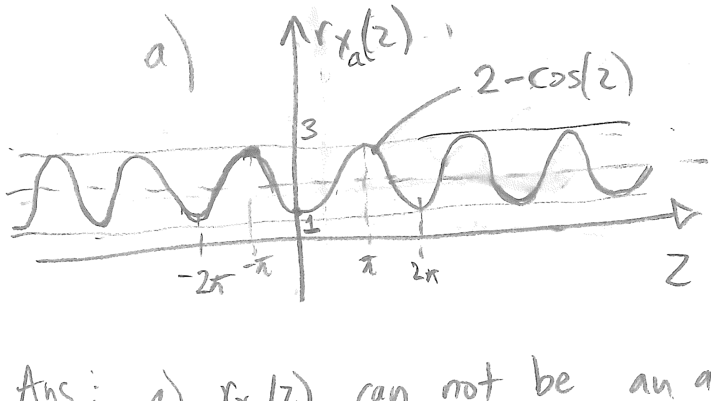
$r_x(\Delta) = r_x(-\Delta)$ (even symmetry)

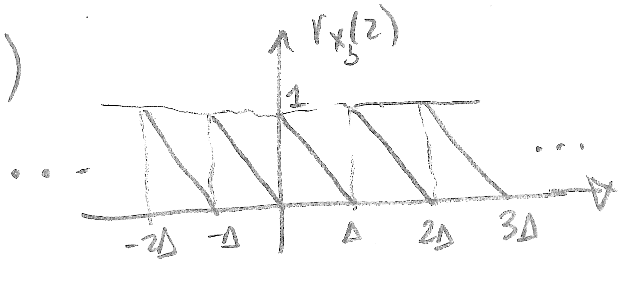
$$\boxed{r_x(0) > r_x(\Delta), \quad \forall \Delta}$$

Also need to show $r_x(0) > -r_x(\Delta)$ for the claim (try $h(t) = \delta(t) + \delta(t-\Delta)$!)

Hence, from these properties, we can see some of the conditions such as symmetry, positive definiteness etc. that are automatically satisfied by valid auto-correlation functions. Those properties enable us to eliminate some functions as potential auto-correlation functions.

Ex: $x(t)$ is a real valued WSS process. Determine whether each one of the following can be the auto-correlation function of this process

a) 

b) 

Ans: a) $r_{x_a}(z)$ can not be an auto-cor. function, since all auto-cor. functions satisfy $r_x(0) \geq r_x(\Delta)$ $\forall \Delta$; i.e. the peak value is the value at the lag $z=0$. This required property is not satisfied by $r_x(a)$.

b) $r_x(b)$ is not a valid auto-correlation func.

Since $r_x(b) \neq r_x(-b)$, i.e. does not satisfy the required symmetry property for valid auto-cor. functions.

<u>Note</u>: ① The properties given above are the necessary conditions for a valid auto-cor. func. That is (or properties)

valid auto-corr function $\xrightarrow{\text{implies}}$
① Symmetry
② Positive semi-definiteness
③ $R_x(0) \geq 0$
④ $R_x(0) \geq R_x(z) \ \forall z$.

If any one of the properties on the right side is not satisfied, say $R_x(0) = -5$!, Then we are sure that this $r_x(z)$ is not a valid/true auto-correlation func.

Hence, These properties are <u>necessary conditions</u> for a valid auto-cor. function.

② Let's assume that all <u>necessary conditions</u> for a valid auto-cor. func. is satisfied; can we say that $r_x(z)$ is indeed a valid auto-cor. function? The answer is <u>NO</u>.

Right now, we only know

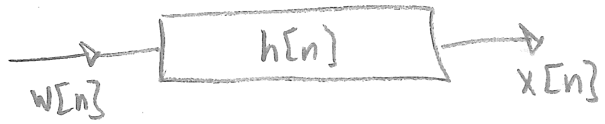
valid $r_x(z) \longrightarrow$ ① Symmetry ② Positive semi ③ ④ ⋮ , but not but not, the reverse implication

Valid $r_x(\tau)$ $\leftarrow$ ① Symmetry ② Positive semi definite ③ ; ④

Properties.

That is, we do not have any information that whether these properties are sufficient conditions for a valid auto-cor. seqn. (More on this after power spectral density) discussions.

Ex:



$w[n]$: WSS discrete-time r.p.

$h[n] = h_0\delta[n] + h_1\delta[n-1]$

Determine $R_x[n_1, n_2]$, $R_{wx}[n_1, n_2]$ and discuss WSS property of $x[n]$.

Soln: All previous results given for continuous valued r.p. and $h(t)$ function, is also valid in discrete time. The only change is that we do the convolutions in discrete-time!

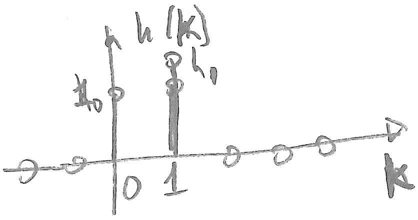
$k \triangleq n_1 - n_2$ ← lag variable

$$R_{wx}[n_1, n_2] = r_{wx}(k) = r_w[k] * h^*[-k]$$

$$R_x[n_1, n_2] = r_x(k) = r_w[k] * \underbrace{h[k] * h^*[-k]}_{r_h[k]}$$

and $x[n]$, $w[n]$ are jointly WSS.

In this example $h[n] = \delta[n] + \delta[n-1]$, that is



$$r_h[k] = h[k] * \underbrace{h^*[-k]}_{h_2^*[k]} = \sum_{\ell=-\infty}^{\infty} h[\ell]\, \cancel{h_2^*[k-\ell]} \quad h^*[\ell-k]$$

det. auto. cor.

$$= \sum_{\ell=-\infty}^{\infty} h[\ell]\, h^*[\ell-k]$$

$r_h[k]$ calculation

$$\begin{array}{lllllll}
 & \ell=0 & \ell=1 & & & & \\
h[\ell] \Rightarrow [& h_0 & h_1 & 0 & 0 & 0 &] \\
\hline
h^*[\ell] \Rightarrow [& h_0^* & h_1^* & 0 & 0 & 0 \ldots &] \\
h^*[\ell-1] \Rightarrow [& 0 & h_0^* & h_1^* & 0 & \ldots &] \\
h^*[\ell-2] \Rightarrow [& 0 & 0 & h_0^* & h_1^* & \ldots &]
\end{array}$$

inner product $\Rightarrow r_h(0) = |h_0|^2 + |h_1|^2$

inner product $\Rightarrow r_h(1) = h_1 h_0^*$

inner product $\Rightarrow r_h[2] = 0$

Then, since $r_x[k] = r_x^*[-k]$

$$r_h[k] = h_1^* h_0\, \delta[k+1] + (|h_0|^2 + |h_1|^2)\, \delta[k] + h_1 h_0^*\, \delta[k-1]$$

The answer is then

$$r_{wx}[k] = r_w[k] * \underbrace{h^*[-k]}_{(h_0^*\delta[k] + h_1^*\delta[k+1])} = h_0^*\, r_w[k] + h_1^*\, r_w[k+1]$$

$$r_x[k] = r_w[k] * \underbrace{h[k] * h^*[-k]}_{r_h[k] \text{ (given above)}}$$

$$= h_1^* h_0\, r_w[k+1] + (|h_0|^2 + |h_1|^2)\, r_w[k] + h_1 h_0^*\, r_w[k-1].$$

Revisiting Poisson Process:

Let $N(t)$ denote the number of arrivals in $(0, T]$.

$N(t) \sim \text{Poisson}(\lambda t)$

$$\left[Z \sim \text{Poisson}(B) \longrightarrow \underbrace{P(Z = n)}_{\text{pmf}} = e^{-B} \frac{B^n}{n!} \right.$$

variable of density func

① $E\{Z\} = B$

② $\text{Var}\{Z\} = B \quad \Big]$

and arrivals in disjoint intervals are independent and the number arrivals in interval $(t_1, t_2]$ is also $N(t_2) - N(t_1) \sim \text{Poisson}(\lambda \Delta t)$

$\Delta t = t_2 - t_1, \ (t_2 > t_1)$

($N(t)$ is Poisson and independent and stationary increments).

* Let's find joint density description of $N(t)$:

1st Order Density: $N(t_1) \sim \text{Poisson}(\lambda t_1)$ ✓

2nd Order Density: $\left.\begin{matrix} N_1 = N(t_1) \\ N_2 = N(t_2) \end{matrix}\right\} \quad f_{N_1, N_2}(n_1, n_2) = f_{N_1}(n_1) \cdot f_{N_2|N_1}(n_2|n_1)$

found above

$n_2 - n_1$ arrivals in $(t_1, t_2]$

Hence, $f_{N_1,N_2}(n_1, n_2) = \text{Poisson}_{n_1}(\lambda t_1) \cdot \text{Poisson}_{n_2 - n_1}\big(\lambda (t_2 - t_1)\big)$

variable of density func.

$$e^{-\lambda \Delta t} \cdot \frac{(\lambda \Delta t)^{n_2 - n_1}}{(n_2 - n_1)!}$$

Higher order densities can be found similarly.

Note: Not even 1st Order Stationary!

$N_1 \sim \text{Poisson}(\lambda t_1)$

* Moment Description:

1st order [**Mean function:** $N_1 = N(t_1)$: $M(t) = E\{N_1\} = \lambda t_1$

2nd order:

Variance function: $Var(N(t_1)) = \sigma^2_{N(t)} = E\lambda t_1$

Auto-correlation function: $R(t_1, t_2) = E\{N(t_1)N(t_2)\}$

increment from t_1 to t_2

$$\overset{(t_2>t_1)}{=} E\{N(t_1)[(N(t_2) - N(t_1)) + N(t_1)]\}$$

$$= E\{N(t_1)[\Delta_N^{t_1\to t_2}]\} + \underbrace{E\{N^2(t_1)\}}_{Var\{N_1\} + (E\{N_1\})^2}$$

$$\overset{\text{ind. increments}}{=} E\{N(t_1)\}E\{\Delta_N^{t_1\to t_2}\} + \overbrace{\lambda t_1 + (\lambda t_1)^2}$$

$$= \lambda t_1 \cdot \lambda(t_2 - t_1) + \lambda t_1 + (\lambda t_1)^2$$

$$= \lambda t_1 + \lambda^2 t_1 t_2$$

Hence

$$R(t_1, t_2) = \begin{cases} \lambda t_1 + \lambda^2 t_1 t_2 & , \ t_2 > t_1 \\ \lambda t_2 + \lambda^2 t_1 t_2 & , \ t_2 < t_1 \end{cases}$$

$$= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

(<u>Note</u>: For $t_2 < t_1$ analysis, you can interchange t_1 and t_2 variables in the $R(t_1, t_2)$ analysis for $t_2 > t_1$)