2015 #1

Ch. I Introduction

A dynamical system modeled by a finite number of (coupled) first-order ordinary differential equations:

$$\dot{x}_1 = f_1(t, x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_p)$$
$$\dot{x}_2 = f_2(t, x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_p)$$
$$\vdots$$
$$\dot{x}_n = f_n(t, x_1, x_2, \dots, x_n, u_1, u_2, \dots, u_p)$$

→ State variables : $x_1, x_2, \dots, x_n \in \mathbb{R}$

→ input variables : $u_1, u_2, \dots, u_p \in \mathbb{R}$

→ time : $t \in [0, \infty)$

Define vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}$, $f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $f : [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$

Then we write $\boxed{\dot{x} = f(t,x,u)}$ (1)

→ x: state (vector)

→ u: input (vector), forcing term

A nondifferential equation sometimes describes the output (measurement)

$$\left.\begin{aligned} y_1 &= h_1(t,x,u) \\ y_2 &= h_2(t,x,u) \\ &\vdots \\ y_q &= h_q(t,x,u) \end{aligned}\right\} \Rightarrow \boxed{y = h(t,x,u)} \quad (2)$$

unforced system equation: $\dot{x} = f(t,x)$

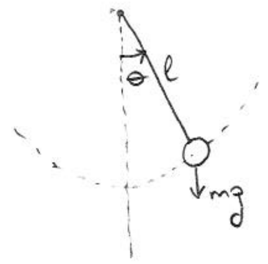
autonomous (time-invariant, unforced) system : $\dot{x} = f(x)$

An equilibrium point $x_{eq} \in \mathbb{R}^n$ of an autonomous system is a real root of eqn. $f(x) = 0$. that is, $f(x_{eq}) = 0$.

For linear systems (1) & (2) can be written as $\left\{ \begin{aligned} \dot{x} &= \overbrace{A(t)}^{n\times n} x + \overbrace{B(t)}^{n\times p} u \\ y &= \underbrace{C(t)}_{q\times n} x + \underbrace{D(t)}_{q\times p} u \end{aligned} \right.$

Note that $x(0) = x_{eq} \Rightarrow x(t) = x_{eq} \;\; \forall t$

Example (pendulum)



equation of motion: $m\ell\ddot{\theta} + mg\sin\theta + k\ell\dot{\theta} = 0$ (3)

$\hookrightarrow$ friction term

Note that eqn. (3) is time-invariant & unforced.

Let's put (3) into "$\dot{x} = f(x)$" form.

Choose states as $x_1 := \theta$ (angle)

$x_2 := \dot{\theta}$ (angular velocity)

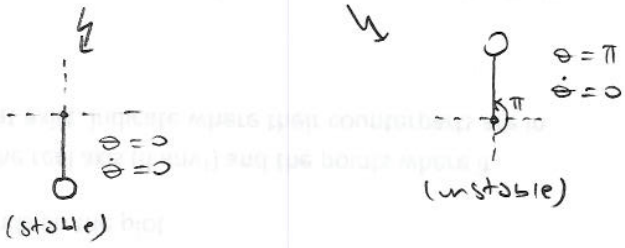
$$\Rightarrow \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{aligned}\right\} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ \& \ f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix}$$

Equilibrium point(s)?

$$f(x) = 0 \Rightarrow \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} &x_2 = 0 \\ &\& \ \sin x_1 = 0 \Rightarrow x_1 = 0, \mp\pi, \mp 2\pi, \dots \end{aligned}$$

$$\Rightarrow x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mp\pi \\ 0 \end{bmatrix}, \begin{bmatrix} \mp 2\pi \\ 0 \end{bmatrix}, \dots \quad \text{multiple equilibria}$$

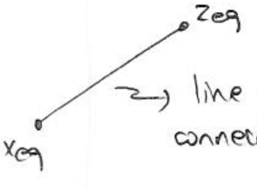
physically we have two equil. points: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$



Remark: The equil. points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$ are isolated. This is a nonlinear phenomenon. Linear systems cannot have multiple isolated equilibrium points.

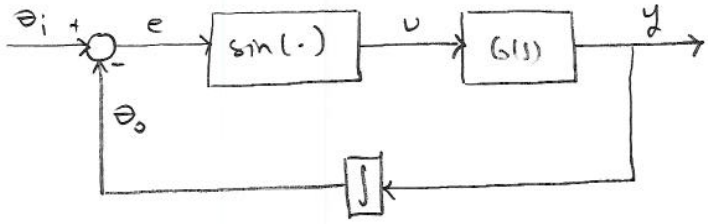
WHY? Because: let x_{eq} & z_{eq} be eq. points for the lin. sys. $\dot{x} = Ax$

$$\Rightarrow \left.\begin{aligned} Ax_{eq} &= 0 \\ Az_{eq} &= 0 \end{aligned}\right\} \Rightarrow$$



$\hookrightarrow$ line segment connecting x_{eq} & z_{eq}: $\left\{ y : \alpha x_{eq} + (1-\alpha) z_{eq} \right\}_{\alpha \in [0,1]} \Rightarrow Ay = 0$

Example A phase-locked loop can be represented by the block diagram



Let (A, B, C) be a minimal realization of the scalar, strictly proper TF G(s). Assume that all the eigenvalues of A have strictly negative real parts, $G(0) \neq 0$, and θ_i = constant. Let z be the state of the realization (A, B, C)

a) Show that the closed-loop system can be represented by the state eqn.

$$\dot{z} = Az + B\sin e$$
$$\dot{e} = -Cz$$

b) Find all equilib. points of the system

c) Show that when $G(s) = \frac{1}{\tau s+1}$, the closed-loop model coincides with the model of the pendulum eqn.

Sol'n: a) $\dot{z} = Az + Bu$, $y = Cz$ | $u = \sin e$ | $e = \theta_i - \int y \Rightarrow \dot{e} = -y = -Cz$

$$\Rightarrow \begin{cases} \dot{z} = Az + B\sin e \\ \dot{e} = -Cz \end{cases}$$

b) $0 = Az + B\sin e \Rightarrow z = -A^{-1}B\sin e$ (why A^{-1} exists?)

$0 = -Cz \Rightarrow CA^{-1}B\sin e = 0$

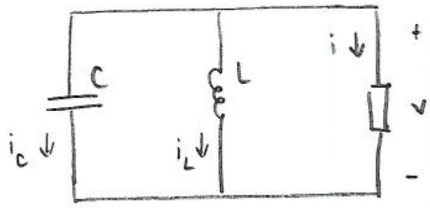
$\Rightarrow -C(sI-A)^{-1}B\Big|_{s=0} \sin e = 0 \Rightarrow G(s)\Big|_{s=0} \sin e = 0 \Rightarrow \sin e = 0$

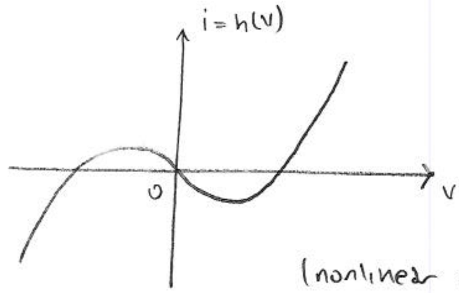
$\Rightarrow e = k\pi,\ k = 0, \mp 1, \mp 2, \dots \Rightarrow z = -A^{-1}B\sin e = 0$

$$\Rightarrow \begin{bmatrix} z \\ e \end{bmatrix}_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \pi \end{bmatrix}, \begin{bmatrix} 0 \\ -\pi \end{bmatrix}, \begin{bmatrix} 0 \\ 2\pi \end{bmatrix}, \begin{bmatrix} 0 \\ -2\pi \end{bmatrix}, \dots$$

c) $\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1} \Rightarrow \tau\dot{y} + y = u \Rightarrow \tau\dot{y} + y = \sin e$ & $\dot{e} = -y \Rightarrow \boxed{\tau\ddot{e} + \dot{e} + \sin e = 0}$

Example (Nonlinear oscillator)





(nonlinear resistor)

$$\text{KCL} \Rightarrow \quad i_c + i_L + i = 0 \quad \Rightarrow \quad C\frac{d}{dt}v + i_L + i = 0$$

$$\Rightarrow \quad C\frac{d^2}{dt^2}v + \frac{d}{dt}i_L + \frac{di}{dt} = 0$$

$$\Rightarrow \quad C\frac{d^2}{dt^2}v + \frac{1}{L}v + \underbrace{\frac{\partial}{\partial v}h(v)}_{h'(v)} \cdot \frac{dv}{dt} = 0$$

$$\Rightarrow \ddot{v} + \frac{h'(v)}{C} + \frac{1}{LC}v = 0$$

Let $LC = 1$ & $\varepsilon := \frac{1}{C}$

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0 \qquad (1)$$

For $h(v) = -v + \frac{1}{3}v^3$ (1) boils down to Van der Pol equation:

$$\boxed{\ddot{v} + \varepsilon(v^2-1)\dot{v} + v = 0} \qquad (\varepsilon > 0)$$

let $\left.\begin{aligned} x_1 &= v \\ x_2 &= \dot{v} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon(1-x_1^2)x_2 \end{aligned}$

Equilibrium point(s)?

$$\left.\begin{aligned} 0 &= x_2 \\ 0 &= -x_1 + \varepsilon(1-x_1^2)x_2 \end{aligned}\right\} \Rightarrow x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: VDP oscillator possesses a periodic solution that attracts every solution except the solution that sits at the equilibrium $x(t) = 0 \ \forall t$.

Hence, VDP oscillator is said to possess a stable limit cycle

Pause: Simulate VDP, play with ε, obtain the phase plot & $x_1(t)$, $x_2(t)$ separately.

$\boxed{\text{Ch. II}}$ Second-Order Systems

$$\dot{x} = f(x) \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad \& \quad f : \mathbb{R}^2 \to \mathbb{R}^2$$

OR

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \quad \text{where} \quad \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = f(x)$$

- why study second-order systems?
- Because : → we can visualize the solutions
  - → simple enough to study in detail
  - → yet rich enough to be useful in understanding higher-order phenomena

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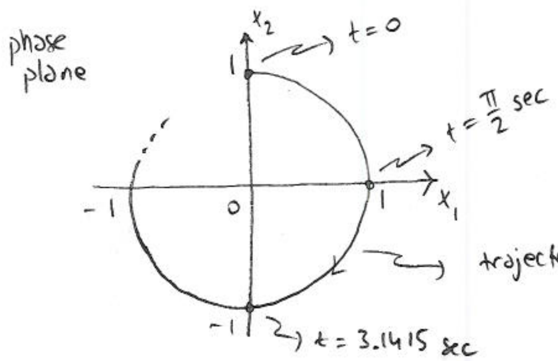
Let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be the solution of $\dot{x} = f(x)$ starting from an initial condition $x_0 \in \mathbb{R}^2$, that is, $x(0) = x_0$. The collection of points $(x_1(t), x_2(t))$ for $t \geq 0$ on $x_1 - x_2$ plane is called a trajectory or orbit of the system. The $x_1 - x_2$ plane is usually called the state plane or phase plane.

Ex : Let $\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned} \right\} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x = f(x)$

solution $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

phase plane



→ trajectory for the initial cond. $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(the direction of arrow indicates the direction of time)

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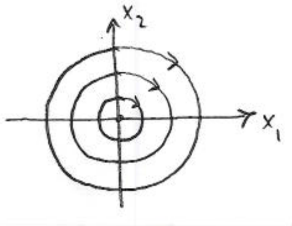
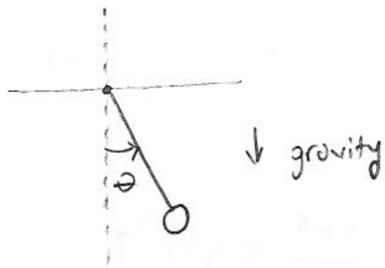
The family of all trajectories is called the phase portrait of the system

For our example :



Remark : $\dot{x} = f(x)$ & $\dot{x} = \alpha f(x)$ $(\alpha > 0)$ have the same phase portraits.

Example — Pendulum with friction



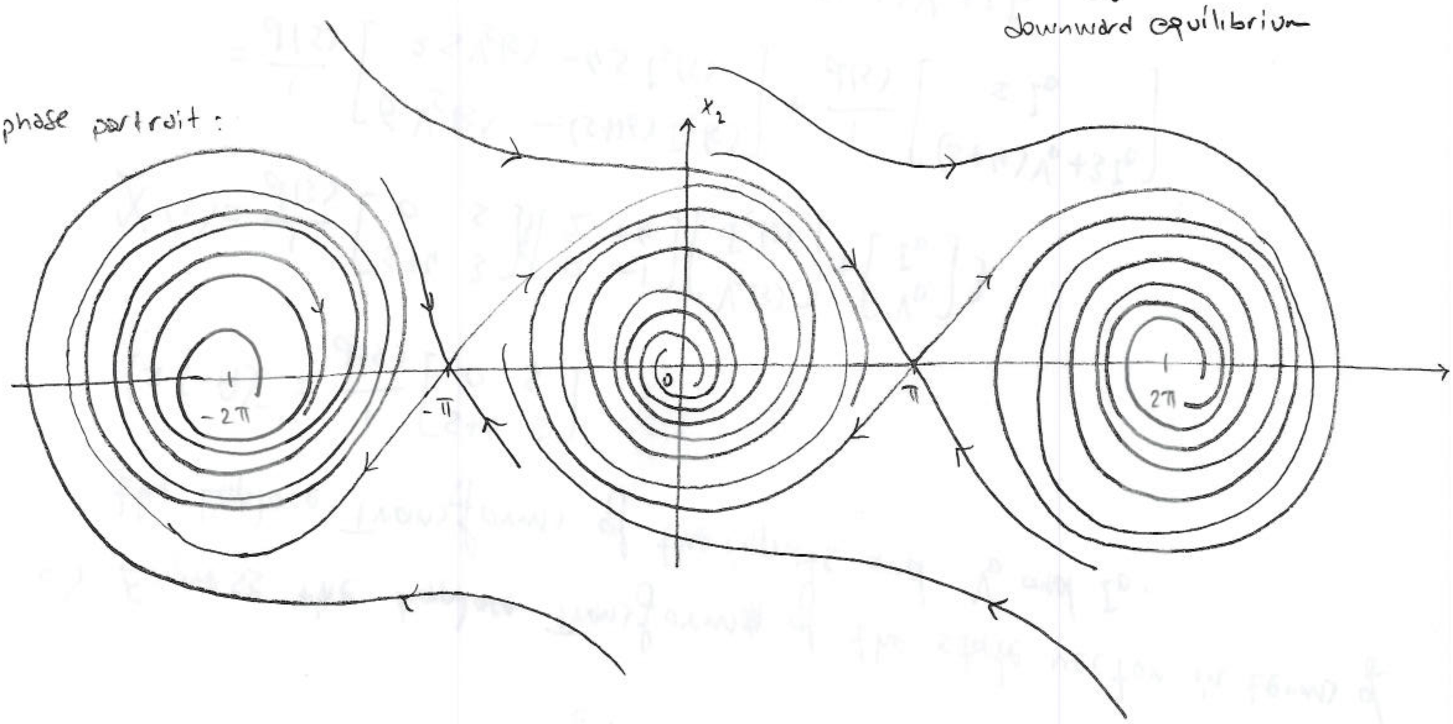
$x_1 = \theta$

$x_2 = \dot{\theta}$

model: $\dot{x}_1 = x_2$, $\dot{x}_2 = -10\sin x_1 - x_2$ — equilibrium points?

$$x_e = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \begin{bmatrix} \mp 2\pi \\ 0 \end{bmatrix}}_{\text{downward equilibrium}}, \qquad \overbrace{\begin{bmatrix} \mp \pi \\ 0 \end{bmatrix}, \ \begin{bmatrix} \mp 3\pi \\ 0 \end{bmatrix}}^{\text{upward equilibrium}}, \ \dots$$

phase portrait:

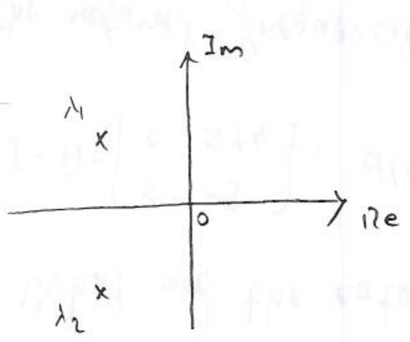


Question: How about the frictionless case? $\dot{x}_1 = x_2$ & $\dot{x}_2 = -10\sin x_1$

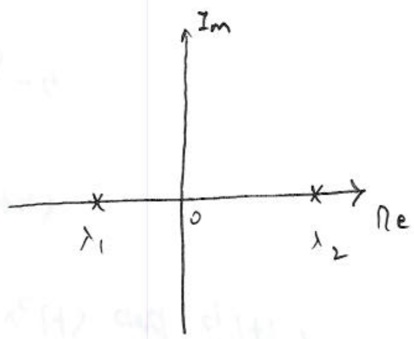
Remark: We could have used linearization to obtain the local behavior of the system around equilibria.

Phase portrait suggests the following eigenvalue locations for the linearization:

$x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



$x_{eq} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$



Let's verify our guess.

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system $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -10\sin x_1 - x_2 \end{cases}$

linearization at $x_{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Jacobian? $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix}$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\begin{bmatrix} 0 \\ 0 \end{bmatrix}} = A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}$$

eigenvalues? $|sI-A| = \begin{vmatrix} s & -1 \\ 10 & s+1 \end{vmatrix} = s^2+s+10 = (s+\frac{1}{2})^2 + \frac{39}{4}$

$\Rightarrow \lambda_{1,2} = -\frac{1}{2} \mp j\frac{\sqrt{39}}{2}$ (as expected)

linearization at $x_{eq} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\begin{bmatrix} \pi \\ 0 \end{bmatrix}} = A = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix} \Rightarrow |sI-A| = s^2+s-10$$

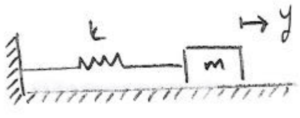
$\Rightarrow \lambda_{1,2} = -3.7,\ 2.7$ (as expected)

Oscillations

Oscillation is the phenomenon displayed by systems that possess a nontrivial (i.e. nonconstant) <u>periodic solution</u> $x(t+T) = x(t) \quad \forall t \geq 0 \quad (T>0 : \text{period})$

The image of a periodic solution in the phase portrait is a closed trajectory which is called a <u>periodic orbit</u> or <u>closed orbit</u>.

<u>Example</u> (Harmonic oscillator)



linear, frictionless mass-spring

$m\ddot{y} + ky = 0$

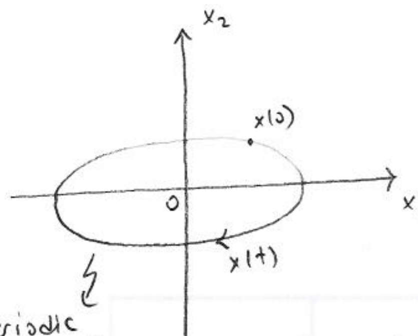
$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m} x_1 \end{aligned}$$

let E = pot. ener. + kin. ener.

$$\Rightarrow E(x) = \frac{1}{2} k x_1^2 + \frac{1}{2} m x_2^2$$

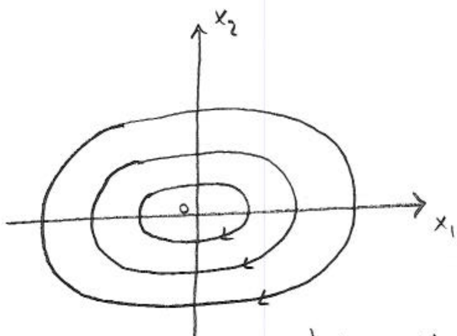
$$\dot{E} = \frac{d}{dt}\left\{ \frac{1}{2} k x_1(t)^2 + \frac{1}{2} m\, x_2(t)^2 \right\} = k x_1 \dot{x}_1 + m x_2 \dot{x}_2 = k x_1 \{x_2\} + m x_2 \left\{ -\frac{k}{m} x_1 \right\} = 0$$

$\Rightarrow E(x(t)) = \text{constant}$, in particular $E(x(t)) = \underbrace{E(x(0))}_{\text{initial energy}}$ $\Rightarrow$ energy is conserved.



periodic orbit $\left\{ \text{an ellipsis defined by } \frac{1}{2} k x_1^2 + \frac{1}{2} m x_2^2 = E(x(0)) \right\}$

$\Rightarrow$



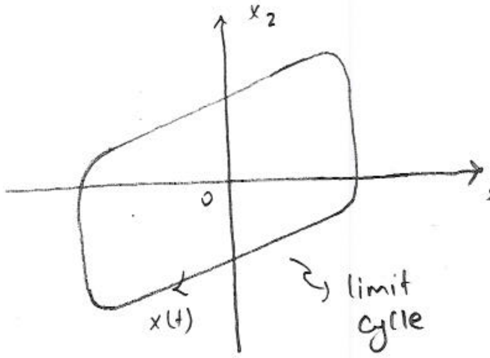
phase portrait

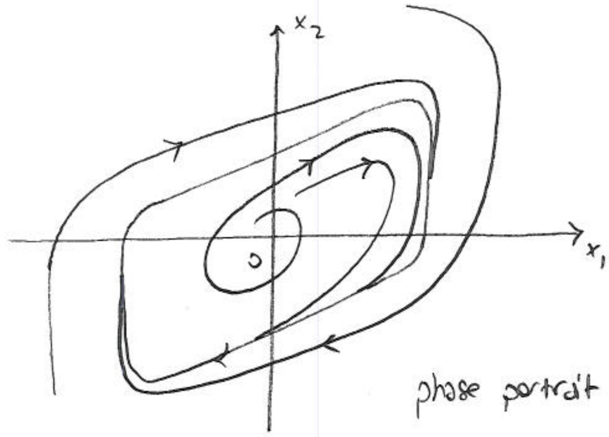
<u>Remark</u>: In general, a linear system cannot have an isolated periodic orbit. (WHY?)

An isolated periodic orbit is called a <u>limit cycle</u>

Example (VDP oscillator, revisited)

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 \qquad (\varepsilon > 0)$$





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For second-order linear systems $\dot{x} = Ax$ $(x \in \mathbb{R}^2)$ periodic orbits exist if and only if the eigenvalues of A are purely imaginary, i.e., $\lambda_{1,2} = \mp j\omega$ $(\omega > 0)$ (WHY?) For nonlinear systems we have:

Poincare - Bendixson Thm Consider the system

(1) $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ & $f: \mathbb{R}^2 \to \mathbb{R}^2$ (assume $\frac{\partial f}{\partial x}$ exists)

Let $M \subset \mathbb{R}^2$ be a closed-bounded subset of the plane and satisfies:

→ M is free of equilibrium points or contains a single equilibrium point x_{eq} where the Jacobian $\left[\frac{\partial f}{\partial x}\right]_{x=x_{eq}}$ has both of its eigenvalues with strictly positive real parts

→ Every trajectory starting in M stays in M for all future time. I.e. $x(0) \in M \Rightarrow x(t) \in M$ $\forall t$ (Then, M is said to be "forward invariant" w.r.t. the system (1))

Then, M contains (at least) one periodic orbit of the system (1).

Example : $\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$
$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$ } system

a) Show that there exists $r > 0$ such that the disk $D = \{x : x_1^2 + x_2^2 \le r^2\}$ is forward invariant. That is, $x(0) \in D \Rightarrow x(t) \in D \quad \forall t \ge 0$.

b) Discuss existence of periodic orbits using PB Thm.

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Sol'n a) Let $V(x) := x_1^2 + x_2^2$. Now let us compute the evolution of $V(x)$ along the trajectories of the system. That is, evaluate $\dot{V} = \frac{d}{dt}\{V(x(t))\}$.

$$\dot{V} = \frac{d}{dt}\{V(x_1(t), x_2(t))\} = \frac{\partial V}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\cdot\frac{dx_2}{dt} = \underbrace{\begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2}\end{bmatrix}}_{(\nabla V)^T} \underbrace{\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}}_{f(x)}$$

$$\Rightarrow \dot{V}(x) = \langle \nabla V(x), f(x) \rangle$$

$$= \left\langle \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \begin{bmatrix} x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ -2x_1 + x_2 - x_2(x_1^2+x_2^2)\end{bmatrix} \right\rangle$$

$$= 2x_1^2 + 2x_1x_2 - 2x_1^2 V(x) - 4x_1x_2 + 2x_2^2 - 2x_2^2 V(x)$$

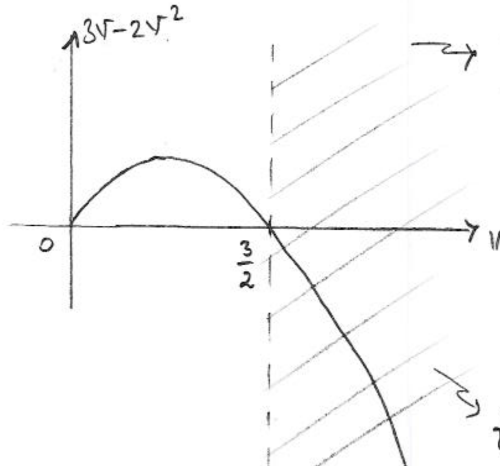
$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2V(x)^2$$

$|2x_1x_2| \le x_1^2 + x_2^2$

$$\le 3x_1^2 + 3x_2^2 - 2V(x)^2$$

$$= 3V(x) - 2V(x)^2$$

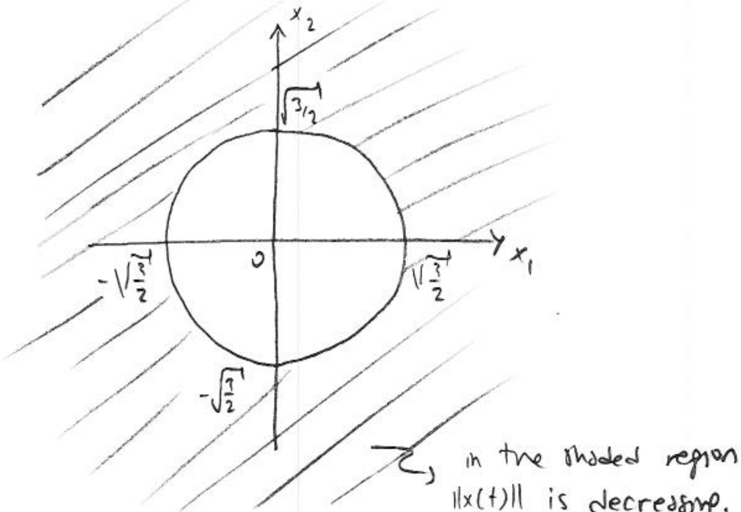
$\Rightarrow \boxed{\dot{V} \le 3V - 2V^2}$ → meaning of this inequality?



→ $\dot{V} \le 0$ in this region

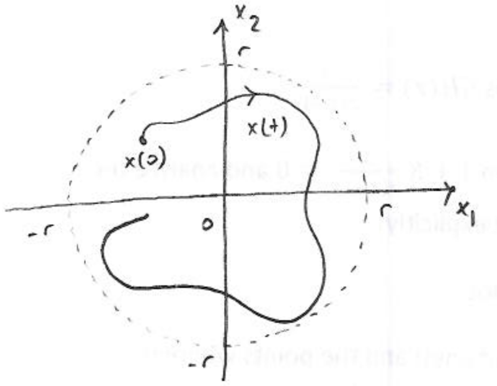
$\{V \ge \frac{3}{2}\}$

$(V(x) = \|x\|^2)$ ⇒



→ in the shaded region $\|x(t)\|$ is decreasing.

Hence, for any $r \geq \sqrt{3/2}$, $x(0) \in \{V(x) \leq r^2\} = D \Rightarrow x(t) \in D \quad \forall t \geq 0$.



b) Equilibria? $f(x) = 0 \Rightarrow x_1 + x_2 - x_1 \cdot \|x\|^2 = 0$ & $-2x_1 + x_2 - x_2 \cdot \|x\|^2 = 0$ $\Big\}$ $\underbrace{\begin{bmatrix} 1-\|x\|^2 & 1 \\ -2 & 1-\|x\|^2 \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only equilibrium.

Linearization at $x_{eq} = 0$?

$$\left[\frac{\partial f}{\partial x}\right]_{x=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow |sI - A| = (s-1)^2 + 2 \Rightarrow \lambda_{1,2} = 1 \mp j\sqrt{2}$$

Now, we've gathered:

$\rightarrow$ $D = \{x_1^2 + x_2^2 \leq r^2\}$ is forward invariant (choose any $r \geq \sqrt{3/2}$)

$\rightarrow$ D contains a single equilibrium $x = 0$

$\rightarrow$ The eigenvalues of the linearization at $x=0$ satisfy $\mathrm{Re}\{\lambda_i\} > 0$.

Hence, D contains a periodic orbit by PB Thm. ▨

Example (2.18) Consider the second-order system

$$\dot{x}_1 = x_2 \quad \& \quad \dot{x}_2 = -g(x_1)$$

where g is continuously differentiable & $zg(z) > 0$ for $z \neq 0$. Let

$$V(x) := \frac{1}{2}x_2^2 + G(x_1) \quad \text{where} \quad G(y) = \int_0^y g(z)dz$$

a) Show that $V(x)$ remains constant along the solutions of the system.

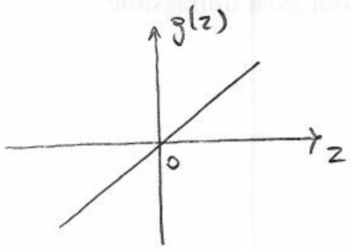
b) Show that, for sufficiently small $\|x(0)\|$, every solution is periodic.

Sol'n a) $\dot{V} = x_2\dot{x}_2 + g(x_1)\dot{x}_1 = -x_2 g(x_1) + g(x_1)x_2 = 0$

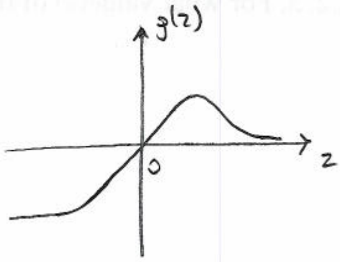
$\Rightarrow V(x(t)) = V(x(0))$ for all $t \geq 0$

b) Note that $zg(z) > 0$ means g visits only the first & third quadrants.

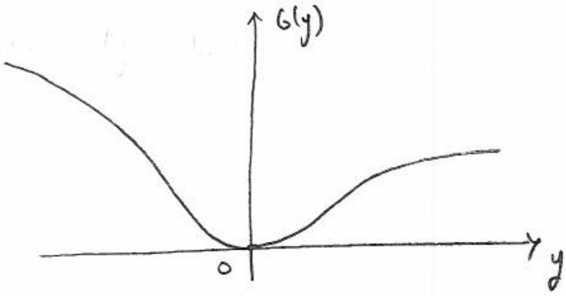
For instance



or



Then $G(y)$ looks like



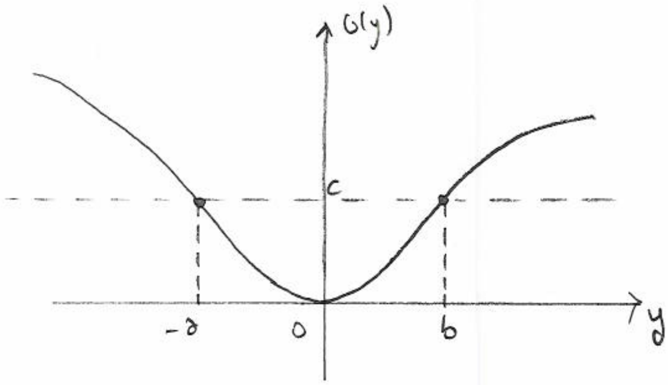
That is, $G(y) > 0$ for all $y \neq 0$

$G(0) = 0$

$G(y)$ strictly decreasing on $(-\infty, 0)$

$G(y)$ strictly increasing on $(0, \infty)$

Hence, we can find $\bar{c} > 0$ such that for each $c \in (0, \bar{c}]$ there exists a unique pair $(a, b > 0)$ such that $G(-a) = G(b) = c$



Claim: There exists $\varepsilon > 0$ such that $\|x\| \le \varepsilon \Rightarrow V(x) \le \bar{c}$.

proof. Let $a_1, b_1 > 0$ be such that $G(-a_1) = G(b_1) = \frac{\bar{c}}{2}$.

Let $\varepsilon_1 := \min\{a_1, b_1\}$. Note that $|y| \le \varepsilon_1 \Rightarrow -a_1 \le y \le b_1 \Rightarrow G(y) \le \frac{\bar{c}}{2}$

Finally, choose $\varepsilon := \min\{\varepsilon_1, \sqrt{\bar{c}}\}$

Now we have $\|x\| \le \varepsilon \Rightarrow V(x) \le \bar{c}$

because $\max\{|x_1|, |x_2|\} \le \|x\| = \sqrt{x_1^2 + x_2^2} \le \varepsilon$

$$\left.\begin{aligned} &\Rightarrow |x_1| \le \varepsilon \le \varepsilon_1 \Rightarrow G(x_1) \le \frac{\bar{c}}{2} \\ &\text{also, } |x_2| \le \varepsilon \le \sqrt{\bar{c}} \Rightarrow \frac{1}{2}x_2^2 \le \frac{\bar{c}}{2} \end{aligned}\right\} \quad V(x) = \frac{1}{2}x_2^2 + G(x_1) \le \bar{c} \quad \blacksquare$$

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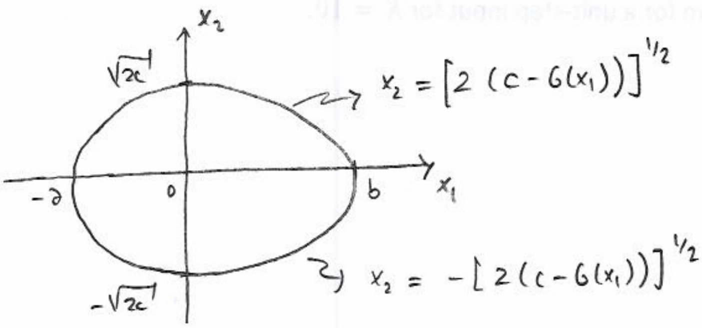
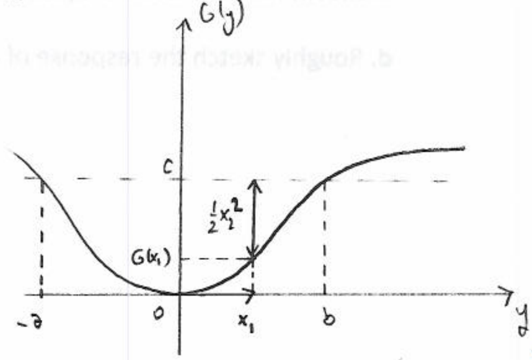
Now, note that for each $c \in (0, \bar{c}]$ the set of points $\{x : V(x) = c\}$ defines a closed curve, symmetric w.r.t. the horizontal axis



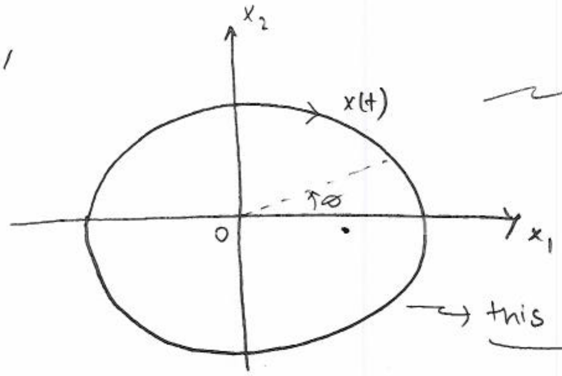


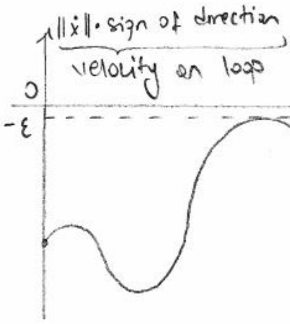
We establish periodicity for small initial conditions ($\|x(0)\| \le \varepsilon$) as follows:

$$\left.\begin{aligned} &\|x(0)\| \le \varepsilon \Rightarrow V(x(0)) \le \bar{c} \\ &\text{by part a, } V(x(t)) = V(x(0)) \end{aligned}\right\} \quad x(t) \in \{x : V(x) = V(x(0))\}$$

→ this set is a closed curve since $V(x(0)) \le \bar{c}$

Hence,





$T \le \frac{\text{perimeter}}{\varepsilon}$

→ $\underbrace{\text{this loop contains no equilibrium point} \Rightarrow x(t) \text{ is periodic!}}_{\text{WHY?}}$

Question: Why cannot we guarantee periodicity for arbitrary initial conditions?

$\boxed{\text{Ch. III}}$ On Existence & Uniqueness of Solutions

System : $\dot{x} = f(t,x)$ $\quad f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ $\quad$ (t: time)

Initial cond. $x(t_0) = x_0$

Solution : A continuous function $x : [t_0, t_1] \to \mathbb{R}^n$ is said to be a solution of the system $\dot{x} = f(t,x)$ if $\dot{x}(t)$ exists and satisfies $\frac{d}{dt} x(t) = f(t, x(t))$ for all $t \in [t_0, t_1]$

Existence : when does a solution exist ?

A sufficient condition : If $f(t,x)$ is continuous w.r.t. its arguments (t,x) then a solution $x : [t_0, t_1] \to \mathbb{R}^n$ exists for some $t_1 > t_0$.

Uniqueness : Can $\dot{x} = f(t,x)$ have multiple solutions ?

Yes, consider $\dot{x} = x^{1/3}$ with $x(0) = 0$

Solution 1 : $x(t) \equiv 0$

solution 2 : $x(t) = \left(\frac{2t}{3}\right)^{3/2} \Rightarrow \frac{d}{dt} x(t) = \frac{3}{2}\left(\frac{2t}{3}\right)^{1/2} \cdot \frac{2}{3} = \left[\left(\frac{2t}{3}\right)^{3/2}\right]^{1/3} = [x(t)]^{\frac{1}{3}}$

Solution 3 : $x(t) = -\left(\frac{2t}{3}\right)^{3/2}$

A sufficient condition for uniqueness is Lipschitz continuity:

Theorem : Let $f(t,x)$ be piecewise continuous in t and satisfy the Lipshitz condition

$$\| f(t,x) - f(t,y) \| \leq L \, \|x-y\| \qquad (\text{for some } L > 0)$$

for all $x, y \in \{ z \in \mathbb{R}^n : \|z - x_0\| \leq r \}$ and for all $t \in [t_0, t_1]$. Then we can find $\delta > 0$ such that the equation $\dot{x} = f(t,x)$ with init. cond. $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$

Exercise : Show that the system $\dot{x} = x^{1/3}$ fails to satisfy the Lipschitz cond.

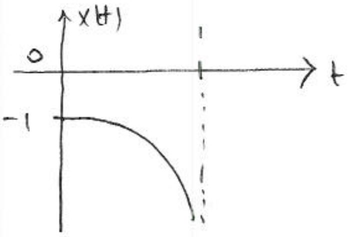
<u>Exercise</u> : Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. A solution $x(\cdot)$ is known to satisfy $x(0) \neq 0$ & $x(T) = 0$ for some finite $T > 0$. Show that f does not satisfy the Lipschitz condition. [Hint: consider sol'n of $\dot{x} = -f(x)$]

<u>Example</u> (Finite escape time) Consider

$$\dot{x} = -x^2 \quad \text{with} \quad x(0) = -1$$

The solution is unique : $x : [0, 1) \to \mathbb{R}$ with

$$x(t) = \frac{1}{t-1} \quad \Rightarrow$$



solution is undefined for $t \geq 1$

Note that as $t \to 1$ $|x(t)| \to \infty$. This phenomenon is called "finite escape time". One way to rule out finite escape times is the global Lipschitz condition:

<u>Theorem</u> : let $f(t, x)$ be piecewise continuous in t and satisfy

$$\|f(t,x) - f(t,y)\| \leq L \|x-y\| \qquad (\text{for some } L > 0)$$

for all $x, y \in \mathbb{R}^n$, for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$.

<u>Exercise</u> : Consider the linear system $\dot{x} = A(t)x$ where A is a continuous function of time. Show that for all x_0, t_0 a unique solution $x : [t_0, \infty) \to \mathbb{R}^n$, with $x(t_0) = x_0$, exists.

Ch. IV Lyapunov Stability

The (autonomous) system

$$\dot{x} = f(x) \qquad (1)$$

with $f: \mathbb{R}^n \to \mathbb{R}^n$

[unless otherwise stated the solution $x(t)$ of (1) uniquely exists for all initial conditions $x(0) = x_0 \in \mathbb{R}^n$ and for all times $t \in [0,\infty)$]

Goal study and characterize the stability of an equilibrium $\bar{x}$ of the system (1), i.e., $f(\bar{x}) = 0$. Without loss of generality we will let $\bar{x} = 0$.

Definition The equilibrium $x = 0$ of the system (1) is said to be

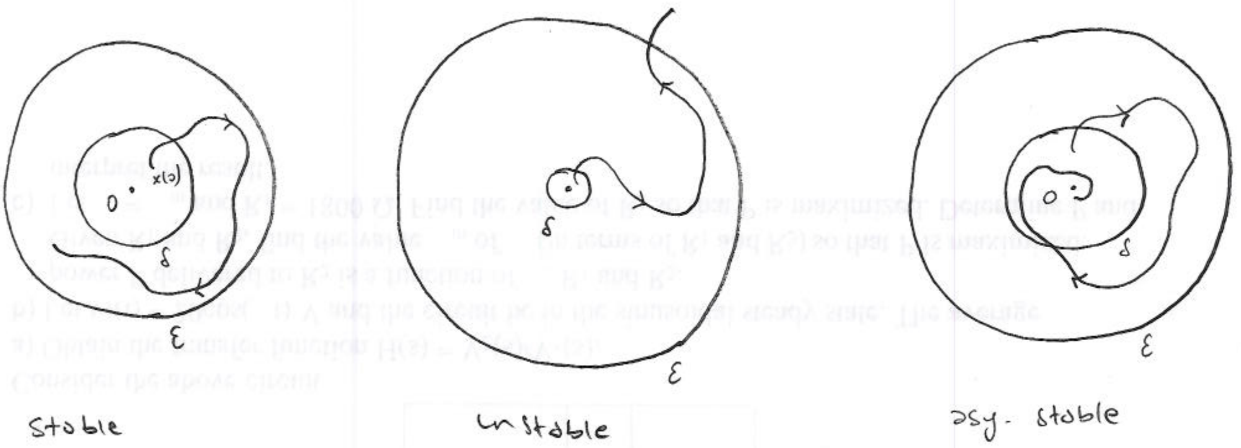
1) Stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \text{for all} \quad t \geq 0.$$

2) Unstable if not stable.

3) Asymptotically stable if stable and δ can be chosen so that

$$\|x(0)\| < \delta \implies \|x(t)\| \to 0 \quad \text{as} \quad t \to \infty \qquad \text{(attractivity)}$$

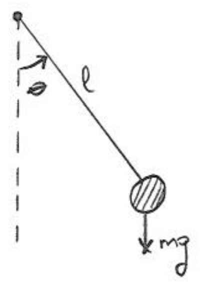


Question : Can an attractive equilibrium be unstable?

Answer: YES! ex: $\dot{x}_1 = \dfrac{x_1^2(x_2 - x_1) + x_2^5}{r^2(1+r^4)}$ & $\dot{x}_2 = \dfrac{x_2^2(x_2 - 2x_1)}{r^2(1+r^4)}$ (Vinograd 1957)

($r^2 = x_1^2 + x_2^2$) The origin is unstable, yet every solution converges to it.

Example (pendulum, revisited)



$$x_1 = \theta, \quad x_2 = \dot\theta \qquad \dot x_1 = x_2, \quad \dot x_2 = -\frac{g}{\ell}\sin x_1 - \frac{k}{m}x_2$$

Equilibria? $x = \begin{bmatrix}0\\0\end{bmatrix}$ downright equilibrium, stable

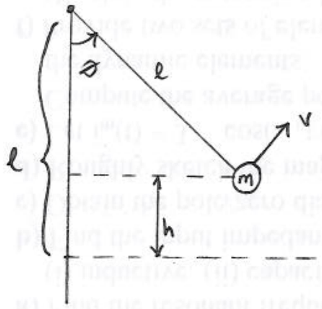
$x = \begin{bmatrix}\pi\\0\end{bmatrix}$ upright equilibrium, unstable

Let us establish the stability of $x=0$. [For simplicity ignore friction, i.e., take $k=0$.]

$$\dot x_1 = x_2$$
$$\dot x_2 = -\frac{g}{\ell}\sin x_1$$

Total energy of the system?

$$E = \underbrace{\text{kin. energy}}_{\frac{1}{2}mv^2} + \underbrace{\text{pot. energy}}_{mgh}$$



$$v = \dot\theta \cdot \ell = \ell x_2$$

$$h = \ell - \ell\cos\theta = \ell(1-\cos x_1)$$

$$\Rightarrow E(x) = \frac{1}{2}m\ell^2 x_2^2 + mg\ell(1-\cos x_1)$$

Let's check the evolution of energy along the solutions of the system.

That is, $E(x(t)) \to ?$ as $t: 0 \to \infty$

Let us compute $\frac{d}{dt}E(x(t))$.

$$\frac{d}{dt}E(x(t)) = \frac{d}{dt}E(x_1(t), x_2(t)) = \frac{\partial E}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial E}{\partial x_2}\cdot\frac{dx_2}{dt}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial E}{\partial x_1} & \frac{\partial E}{\partial x_2} \end{bmatrix}}_{\frac{\partial E}{\partial x} = (\nabla E)^T} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x} = f(x)} = \langle \nabla E, f(x) \rangle = \dot{E}$$

In general, given a function $V: \mathbb{R}^n \to \mathbb{R}$ and a system $\dot{x} = f(x)$ $(f: \mathbb{R}^n \to \mathbb{R}^n)$ we have $\frac{d}{dt}V(x(t)) = \langle \nabla V(x), f(x) \rangle =: \dot{V}$

Now, $$\dot{E} = \left\langle \begin{bmatrix} mgl\sin x_1 \\ ml^2 x_2 \end{bmatrix}, \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} \right\rangle = mgl\sin x_1 \cdot x_2 - ml^2 x_2 \cdot \frac{g}{l}\sin x_1 = 0$$

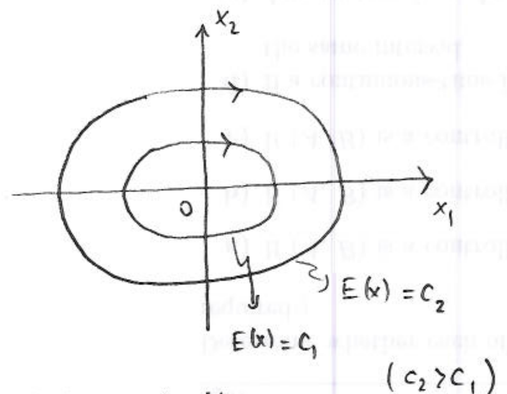
Meaning of $\dot{E} = 0$: the trajectories will always move along the $\{E(x) = \text{constant}\}$ curves, where constant = initial energy $E(x(0))$

Shape of $E(x) = c$?

$$E(x) = \frac{1}{2}ml^2 x_2^2 + mgl(1 - \cos x_1)$$

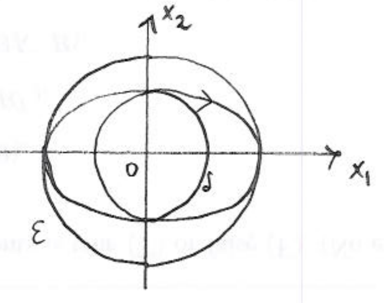
& $\cos x_1 = 1 - \frac{x_1^2}{2} + \frac{x_1^4}{4!} - \frac{x_1^6}{6!} + \cdots \Rightarrow \cos x_1 \cong 1 - \frac{x_1^2}{2}$ for small $|x_1|$

$\Rightarrow$ for small $\|x\|$ we can write $E(x) = \frac{1}{2}ml^2 x_2^2 + \frac{1}{2}mgl x_1^2$



phase portrait for small $\|x(0)\|$

Now, recall definition of stability



$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon$ for all $t \geq 0$

$\Rightarrow$ the origin is stable!

Observe

→ Without explicitly computing the solution $x(t)$, we have determined the stability of $x=0$ through an "energy" function $E(x)$.

→ $E(x)$ (locally) satisfies : $E(x) > 0$ for all $x \neq 0$ & $E(0)=0$. (pos. definiteness)

→ $\dot{E}(x)$ satisfies $\dot{E}(x) \leq 0$. (neg. semi definiteness)

Generalization is due to A. Lyapunov:

locally Lipschitz

Lyapunov's Stability Theorem: Let $x=0$ be an equilibrium of $\underbrace{\dot{x}=f(x)}$ and $D \subset \mathbb{R}^n$ be an open subset containing $x=0$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function satisfying

→ $V(0)=0$ & $V(x) > 0$ in $D-\{0\}$ (1)

→ $\dot{V}(x) \leq 0$ in D (2) $[\dot{V}(x) = \langle \nabla V(x), f(x) \rangle]$

Then $x=0$ is stable. Moreover, if

→ $\dot{V}(x) < 0$ in $D - \{0\}$ (3)

Then $x=0$ is asymptotically stable.

stability under (1)&(2)

Proof: Given $\varepsilon > 0$ choose $r \in (0, \varepsilon)$ s.t. $\overset{r\text{-ball}}{B_r} = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ is contained in D.

Let $\alpha = \min_{\|x\|=r} V(x)$. Then $\alpha > 0$ by (1).

choose $\beta \in (0,\alpha)$ and define $\Omega_\beta := B_r \cap \{V(x) \leq \beta\}$ (note that $\Omega_\beta \subset B_r$)

Claim $x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \quad \forall t \geq 0$

Because Suppose not. Then for some $x(0) \in \Omega_\beta$ we can find some $t_1 > 0$ such that $x(t_1) \notin \Omega_\beta$. Now, $x(0) \in \Omega_\beta \Rightarrow V(x(0)) \leq \beta \Rightarrow V(x(t)) \leq \beta \quad \forall t \geq 0$ by (2).

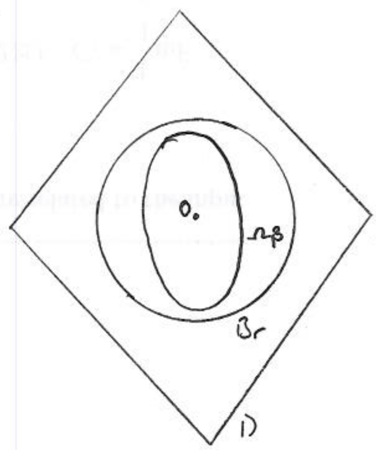
Then $x(t_1) \notin \Omega_\beta$ & $V(x(t_1)) \leq \beta \Rightarrow x(t_1) \notin B_r$, i.e., $\|x(t_1)\| > r$.

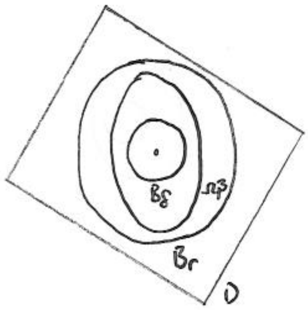
But $\|x(0)\| \leq r$. Hence $\exists\, t_2 \in [0, t_1)$ such that $\|x(t_2)\| = r$. (*)

Moreover, $V(x(t_2)) \leq \beta$ (**)

(*) & (**) $\Rightarrow \beta \geq \min_{\|x\|=r} V(x) = \alpha$

However, we had chosen $\beta < \alpha$. Contradiction! ▨





$$\left.\begin{array}{l} V(x) \text{ continuous} \\ V(0) = 0 \end{array}\right\} \Rightarrow \exists \delta > 0 \text{ such that } \|x\| \le \delta \Rightarrow V(x) < \beta$$

<u>Then</u>: $B_\delta \subset \Omega_\beta$

<u>Because</u>: Note that $\delta < r$, for otherwise ($\delta \ge r$) we would have

$$\beta > \max_{\|x\| \le \delta} V(x) > \min_{\|x\| = r} V(x) = \alpha \Rightarrow \beta > \alpha \Rightarrow \text{contradiction!}$$

Hence $x \in B_\delta \Rightarrow x \in B_r$ & $V(x) \le \beta \Rightarrow x \in \Omega_\beta$. ▨

Now, we can write

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r \quad (\text{recall } r < \varepsilon)$$

That is, $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \ge 0$. Hence, $x = 0$ is <u>stable</u>.

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<u>Asy. stability under (1) & (3)</u> we need to establish $x(t) \to 0$ as $t \to \infty$

<u>Claim</u>: for each $b \in (0, \beta)$ there exists $T > 0$ such that

$$\|x(0)\| < \delta \Rightarrow V(x(t)) \le b \quad \text{for } t \ge T.$$

<u>because</u> Define $-\gamma := \max_{\{x \in \Omega_\beta : V(x) \ge b\}} \dot{V}(x)$. By (3), $\gamma > 0$.

Define $T := \frac{\beta - b}{\gamma}$. Then $V(x(T)) \le b$, for otherwise $V(x(t)) > b$ for $t \in [0, T]$

and $\dot{V}(x(t)) \le -\gamma$ for $t \in [0, T]$

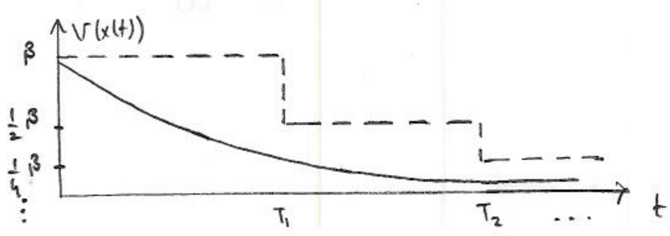
We can write $V(x(T)) = \underbrace{V(x(0))}_{\le \beta} + \int_0^T \underbrace{\dot{V}(x(\tau))}_{\le -\gamma} d\tau \le \beta - \gamma T = b \Rightarrow$ contradiction!

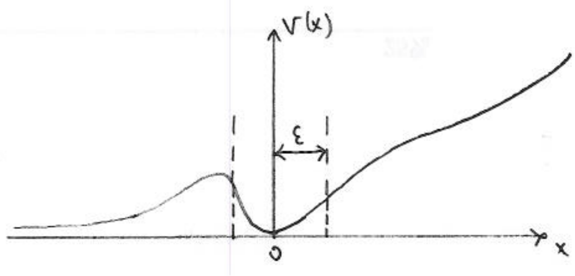
$\dot{V} < 0 \Rightarrow V(x(t)) \le b$ for $t \ge T$. ▨

The claim implies that $\|x(0)\| < \delta \Rightarrow V(x(t)) \to 0$. Since the solution is bounded

($\|x(t)\| < \varepsilon \quad \forall t \ge 0$), V is continuous, and pos. def; $V(x(t)) \to 0 \Rightarrow x(t) \to 0$.

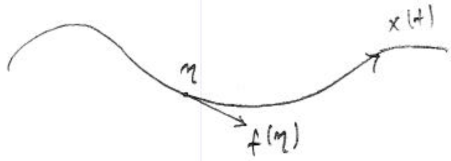
Hence, $x = 0$ is <u>asy. stable</u>.

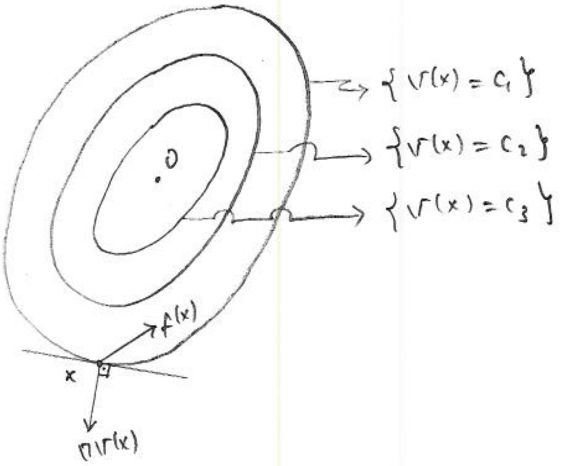




Geometric meaning of $\dot{V} < 0$

$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle$ where $f(x) = \dot{x}$





$c_1 > c_2 > c_3$

level surfaces of a Lyapunov func. $V(x)$

$\langle \nabla V, f(x) \rangle < 0$ means that the angle between the vectors ∇V & $f(x)$ is greater than $90°$. That implies $f(x)$ is pointing to the interior of the surface $\{V(x) = c\}$. Therefore once the trajectory enters the region $\{V(x) \le c\}$, it cannot leave it.

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A Lyapunov function candidate should satisfy $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$. Such functions are called positive definite.

Definition A scalar function $V: D \to \mathbb{R}$ $(D \subset \mathbb{R}^n)$ with $V(0) = 0$ is said to be

→ positive definite if $V(x) > 0$ for $x \neq 0$

→ positive semidefinite if $V(x) \geq 0$

→ negative (semi)definite if $-V(x)$ is positive (semi) definite

→ indefinite if neither of the above.

checking pos. definiteness of a function may not be easy unless it is quadratic:

Definition A quadratic function $V: \mathbb{R}^n \to \mathbb{R}$ is such that it can be written as $V(x) = x^T P x$ where $P \in \mathbb{R}^{n \times n}$ is symmetric, i.e., $P = P^T$. We say the matrix P is positive (semi) definite and write $P > 0$ $(P \geq 0)$ when $x \mapsto x^T P x$ is pos. (semi) def. P is said to be negative (semi) definite $P < 0$ $(P \leq 0)$ when $-P$ is pos. (semi) def.

<u>Fact</u> Given $P^T = P \in \mathbb{R}^{n\times n}$, $P > 0$ $(P \geq 0)$ if and only if all the eigenvalues λ_i of P satisfy $\lambda_i > 0$ $(\lambda_i \geq 0)$ for $i = 1, 2, \dots, n$. Equivalently, $P > 0$ $(P \geq 0)$ iff all the leading principal minors of P are positive (nonnegative).

<u>leading principal minors?</u>

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $\Rightarrow$ LPM are : a, $\det\begin{bmatrix} a & b \\ d & e \end{bmatrix}$, $\det(A)$

<u>Example</u>: Consider $V(x) = [x_1\ x_2\ x_3] \underbrace{\begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x$

lead. principal minors of P: a, a^2, $a^3 - 5a$

For $P > 0$ we need $a > 0$, $a^2 > 0$, and $a(a^2-5) > 0$

Hence, for $a > \sqrt{5}$ $V(x)$ is a positive def. function.

<u>Example</u>: Pendulum with friction

$$\dot{x}_1 = x_2 \qquad (a, b > 0)$$
$$\dot{x}_2 = -a\sin x_1 - b x_2$$

Lyapunov function candidate : $V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$

Note that V is pos. def. in a neighborhood of the origin $x = 0$.

$\dot{V}(x) = a\sin x_1 \cdot \dot{x}_1 + x_2 \dot{x}_2 = -b x_2^2 = -x^T \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}}_{Q} x \Rightarrow \dot{V}(x) \leq 0$

Q is pos. semi def. $\Rightarrow$ $\dot{V}(x)$ is neg. semi def. $\Rightarrow$ the origin is stable

[Note that $\dot{V}$ is not negative definite because we can find points $\eta \neq 0$ that are arbitrarily close to the origin for which $\dot{V}(\eta) = 0$, e.g. $\eta = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}$.]

<u>Question</u>: How about asy. stability?

<u>Answer</u>: Search for a new Lyapunov function.

New $V(x) = \frac{1}{2}x^T P x + a(1-\cos x_1)$ with $P>0$ to be determined

Let $P = \begin{bmatrix} c & d \\ d & e \end{bmatrix}$. For $P>0$ we need $c>0$ & $ce > d^2$

$$\dot{V}(x) = x^T P \dot{x} + a\dot{x}_1 \sin x_1$$

$$= [x_1 \ x_2]\begin{bmatrix} c & d \\ d & e \end{bmatrix}\begin{bmatrix} x_2 \\ -a\sin x_1 - b x_2 \end{bmatrix} + a x_2 \sin x_1$$

$$= [c x_1 + d x_2 \quad d x_1 + e x_2]\begin{bmatrix} x_2 \\ -a\sin x_1 - b x_2 \end{bmatrix} + a x_2 \sin x_1$$

$$= c x_1 x_2 + d x_2^2 - a d x_1 \sin x_1 - b d x_1 x_2 - a e x_2 \sin x_1 - e b x_2^2 + a x_2 \sin x_1$$

$$= a(1-e) x_2 \sin x_1 - a d x_1 \sin x_1 + (c - bd) x_1 x_2 + (d - eb) x_2^2$$

let $e = 1$ & $c = bd$

$$= -a d x_1 \sin x_1 - (b-d) x_2^2$$

let $d = \frac{b}{2}$

$$= -\frac{ab}{2} x_1 \sin x_1 - \frac{b}{2} x_2^2$$

$\sin x_1 = x_1 - \frac{x_1^3}{3!} + \frac{x_1^5}{5!} - \cdots$

$$\cong -\frac{ab}{2} x_1^2 - \frac{b}{2} x_2^2 \quad (\text{for } \|x\| < \varepsilon)$$

Hence $\dot{V}(x)$ is (locally) negative definite

Is $V(x)$ pos. def.? $P = \begin{bmatrix} b^2/2 & b/2 \\ b/2 & 1 \end{bmatrix} \Rightarrow$ LPM: $\frac{b^2}{2} > 0$ & $\frac{b^2}{2} - \left(\frac{b}{2}\right)^2 > 0$

$\Rightarrow P > 0$

By Lyapunov Thm $x=0$ therefore is asy. stable. ▨

— ∘ —

The origin of the pendulum $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a\sin x_1 - b x_2 \end{cases}$ is asy. stable, but NOT every solution satisfies the convergence property $x(t) \to 0$ as $t \to \infty$

Take for instance the solution starting from the other equil. point $x = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$. This observation motivates the following definition.

Region of attraction

Let $x=0$ be an asymptotically stable equilibrium of the system $\dot{x}=f(x)$ ($f:\mathbb{R}^n \to \mathbb{R}^n$). The region of attraction is the set of all points $y \in \mathbb{R}^n$ with the property $x(0)=y \Rightarrow x(t) \to 0$ as $t \to \infty$. When the region of attraction is the entire space $\mathbb{R}^n$, the origin is said to be "globally asymptotically stable" (GAS)

Remark If $x=0$ is GAS then it is the unique equilibrium of $\dot{x}=f(x)$.

Lyapunov Thm for GAS Let $x=0$ be an equilibrium of $\dot{x}=f(x)$. Let $V:\mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and satisfy

1) $V(0)=0$ & $V(x)>0$ for all $x \neq 0$ (V pos. def.)
2) $\|x\| \to \infty \Rightarrow V(x) \to \infty$ (V radially unbounded)
3) $\dot{V}(x)<0$ for all $x \neq 0$ ($\dot{V}$ neg. def.)

Then $x=0$ is GAS.

Reading assignment Read the discussion in the text about the importance of radial unboundedness of V for GAS. [p122-123]

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Lyapunov function is a tool to establish stability. To establish instability we need another tool:

Chetaev's (Instability) Thm Consider the system $\dot{x}=f(x)$ with $f(0)=0$. Suppose there exists $V:\mathbb{R}^n \to \mathbb{R}$, continuously differentiable, $V(0)=0$.

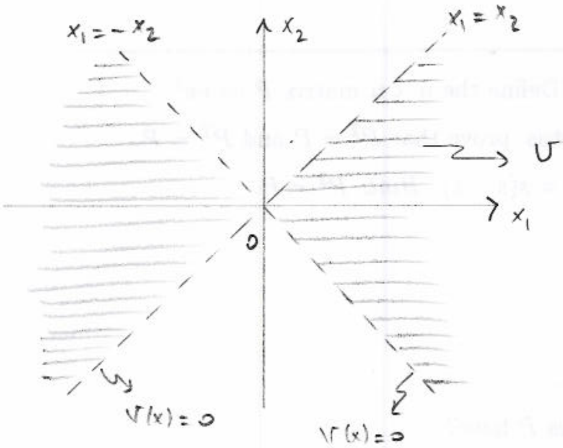
Let $U=\{x \in \mathbb{R}^n : V(x)>0\}$

Suppose the following hold:

1) There exists $r>0$ such that $\dot{V}(x)>0$ for all $x \in U \cap \{x \in \mathbb{R}^n : \|x\| \leq r\}$
2) U has points arbitrarily close to the origin. [That is, for each $\varepsilon>0$ there exists $x \in U$ with $\|x\|<\varepsilon$]

Then the origin is unstable.

Example: $v(x) = \frac{1}{2}(x_1^2 - x_2^2)$ (Note that v is indefinite!)



Consider the system $\begin{cases} \dot{x}_1 = x_1 + g_1(x) \\ \dot{x}_2 = -x_2 + g_2(x) \end{cases}$ with $|g_i(x)| \le k\|x\|^2$ $i = 1, 2$ in some neighborhood D of the origin.

(Note that $g_i(0) = 0$ & $x = 0$ is an equilibrium!)

$$\dot{v}(x) = \langle \nabla v(x), f(x) \rangle = [x_1 \quad -x_2] \begin{bmatrix} x_1 + g_1(x) \\ -x_2 + g_2(x) \end{bmatrix} = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

For $x \in D$ we can write:

$$\begin{aligned} \dot{v}(x) &= x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x) \\ &\ge x_1^2 + x_2^2 - |x_1| \cdot |g_1(x)| - |x_2| \cdot |g_2(x)| \\ &\ge \|x\|^2 - \|x\| \cdot |g_1(x)| - \|x\| \cdot |g_2(x)| \\ &= \|x\|^2 - \|x\| \left(|g_1(x)| + |g_2(x)| \right) \\ &\ge \|x\|^2 - \|x\| \cdot \left(2k\|x\|^2 \right) \\ &= \|x\|^2 \left(1 - 2k\|x\| \right) \end{aligned}$$

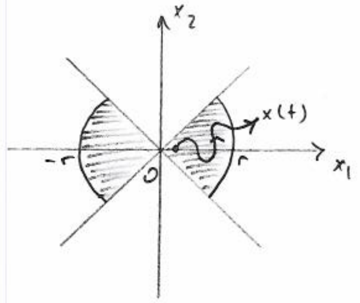
Now, choose $r \in (0, \frac{1}{4k}]$ such that $B_r = \{\|x\| \le r\} \subset D$

Then for $x \in U \cap B_r$ we have

$$\dot{v}(x) \ge \frac{1}{2}\|x\|^2 > 0$$

Hence by Chetaev's Thm the origin is unstable

That is, we can never find $\delta > 0$ small enough such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < r$ for all $t \ge 0$



Example : Euler equations for a rotating rigid spacecraft.

$$J_1 \dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u_1$$
$$J_2 \dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u_2$$
$$J_3 \dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 + u_3$$

$\omega_1, \omega_2, \omega_3$: components of the angular velocity $\omega \in \mathbb{R}^3$ along principal axes

J_1, J_2, J_3 : principal moments of inertia $(J_i > 0)$

u_1, u_2, u_3 : torque inputs

a) Consider the torque free case $(u_i = 0)$. Show that the origin $\omega = 0$ is stable.
Is it asymptotically stable?

b) Let $u_i = -k_i \omega_i$ with $k_1, k_2, k_3 > 0$. Show that the origin is GAS.

Sol'n Energy?

$$E(\omega) = \frac{1}{2}\omega^T J \omega = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 \qquad (\operatorname{diag}(J_1, J_2, J_3) = J \in \mathbb{R}^{3\times 3})$$

$$\Rightarrow \dot{E}(\omega) = J_1\omega_1\dot{\omega}_1 + J_2\omega_2\dot{\omega}_2 + J_3\omega_3\dot{\omega}_3$$
$$= \omega_1\{(J_2-J_3)\omega_2\omega_3\} + \omega_2\{(J_3-J_1)\omega_3\omega_1\} + \omega_3\{(J_1-J_2)\omega_1\omega_2\}$$
$$= \{(J_2-J_3) + (J_3-J_1) + (J_1-J_2)\}\omega_1\omega_2\omega_3 = 0 \qquad \text{(Energy is conserved)}$$

Conclusion: $\dot{E} \le 0 \Rightarrow \omega = 0$ stable

$\dot{E} = 0 \Rightarrow E(\omega(t)) = E(\omega(0)) \Rightarrow \omega(t) \not\to 0 \Rightarrow$ NOT asy. stable

How about the angular momentum M?

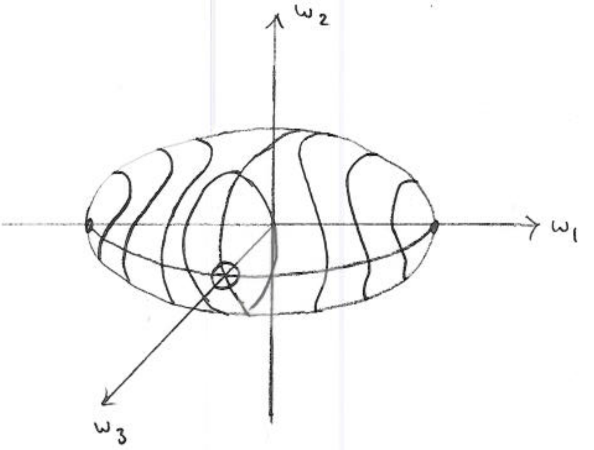
$$M(\omega) = \|J\omega\| \Rightarrow M^2(\omega) = \omega^T J^2 \omega = J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2$$

$$\Rightarrow \frac{d}{dt}\{M^2(\omega(t))\} = 2J_1^2\omega_1\dot{\omega}_1 + 2J_2^2\omega_2\dot{\omega}_2 + 2J_3^2\omega_3\dot{\omega}_3$$
$$= 2J_1\omega_1\{(J_2-J_3)\omega_2\omega_3\} + 2J_2\omega_2\{(J_3-J_1)\omega_3\omega_1\} + 2J_3\omega_3\{(J_1-J_2)\omega_1\omega_2\}$$
$$= 2\{J_1(J_2-J_3) + J_2(J_3-J_1) + J_3(J_1-J_2)\}\omega_1\omega_2\omega_3 = 0$$

(angular momentum is conserved)

Conclusion: Both the energy and the angular momentum are conserved. Hence each trajectory $\omega(t)$ must traverse the intersection of two Lyapunov surfaces; i.e.

$$\omega(t) \in \{\Omega \in \mathbb{R}^3 : E(\Omega) = E(\omega(0))\} \cap \{\Omega \in \mathbb{R}^3 : M^2(\Omega) = M^2(\omega(0))\}$$



b) $[u_i = -k_i \omega_i]$ This time we have $\dot{E}(\omega) = -k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2$

$\Rightarrow \dot{E}(\omega)$ is neg. definite throughout $\mathbb{R}^3$ $\Rightarrow$ $\omega = 0$ is GAS.

LaSalle's Invariance Principle

Pendulum with friction : $\dot{x}_1 = x_2$

$\dot{x}_2 = -a\sin x_1 - b x_2 \qquad (a, b > 0)$

Lyapunov function : $V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$

$\Rightarrow \dot{V}(x) = -b x_2^2 \quad \Rightarrow \dot{V}$ neg. semi definite & $x = 0$ is stable.

How about asy. stability ?

$\dot{V} \leq 0 \quad \Rightarrow \quad V(x(t)) \to c$, some constant $(c \geq 0)$

"By continuity" there exists a solution $\eta(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}$ that satisfies

$V(\eta(t)) \equiv c \quad \Rightarrow \quad \dot{V}(\eta(t)) \equiv 0 \quad \Rightarrow \quad -b\eta_2(t) \equiv 0 \quad \Rightarrow \eta_2(t) \equiv 0$

$\eta_2(t) \equiv 0 \quad \Rightarrow \quad \dot{\eta}_2(t) \equiv 0 \quad \Rightarrow \quad -a\sin\eta_1(t) - b\eta_2(t) \equiv 0 \quad \Rightarrow \quad \eta_1(t) \equiv 0$

Hence $\eta(t) \equiv 0 \quad \Rightarrow \quad c = V(\eta(t)) = V(0) = 0$

$V(x(t)) \to 0 \quad \Rightarrow$ the origin is asy. stable (without a neg. def. $\dot{V}$)

We've made use of the following observation

solutions $x(t)$ must converge to the set $\{ \eta : \underbrace{x(0) = \eta \Rightarrow \dot{V}(x(t)) = 0 \text{ for all } t \geq 0 \}}_{}$

(for pendulum system, this set is the origin!)

Here is the generalization:

Theorem 4.4 [LaSalle] Let $\Omega \subset \mathbb{R}^n$ be a compact set that is positively invariant with respect to the system $\dot{x} = f(x)$, i.e., $x(0) \in \Omega \Rightarrow x(t) \in \Omega \quad \forall t \geq 0$. Also let

→ $V : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable function with $\dot{V}(x) \leq 0$ for all $x \in \Omega$

→ $E := \{ x \in \Omega : \dot{V}(x) = 0 \}$

→ $M := \{ \eta \in E : x(0) = \eta \Rightarrow x(t) \in E \quad \forall t \in \mathbb{R} \}$

M is called "the largest invariant set in E".

Then $x(0) \in \Omega \Rightarrow x(t) \to M$ as $t \to \infty$.

Corollary 4.1 Consider $\dot{x} = f(x)$ with $f(0) = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable, positive definite function on the open set $D \subset \mathbb{R}^n$ containing the origin $x = 0$ such that $\dot{V}(x) \leq 0$ for $x \in D$. Let $S = \{ x \in D : \dot{V}(x) = 0 \}$ and suppose that no solution can stay identically in S other than $x(t) \equiv 0$. Then the origin is asymptotically stable.

Corollary 4.2 Let the conditions in Corollary 4.1 hold with $D = \mathbb{R}^n$ and V radially unbounded. Then the origin is GAS.

Example 4.23 Consider $\dot{x} = [A - BR^{-1}B^T P] x$ where $P = P^T > 0$ satisfies the Riccati eqn. $PA + A^T P + Q - PBR^{-1}B^T P = 0$ with $R = R^T > 0$ and $Q = Q^T \geq 0$. Let $V(x) = x^T P x$ and show that the origin is GAS when

a) $Q > 0$

b) $Q = C^T C$ and the pair (C, A) is observable.

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Sol'n $\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x$

$$= x^T \{ P[A - BR^{-1}B^T P] + [A - BR^{-1}B^T P]^T P \} x$$

$$= x^T \{ PA + A^T P - 2PBR^{-1}B^T P \} x$$

$$= -x^T [Q + PBR^{-1}B^T P] x \qquad (1)$$

a) $Q > 0$ $\quad$ (1) $\Rightarrow$ $\dot{V}(x) \le -x^T Q x$

$\Rightarrow$ $\dot{V}$ is neg. def. (& V rad. unbounded)

$\Rightarrow$ the origin is GAS

b) $Q = C^T C$ $\quad$ (1) $\Rightarrow$ $\dot{V}(x) = -x^T C^T C x - x^T PBR^{-1}B^T P x \le 0$

Let $S = \{ x \in \mathbb{R}^n : \dot{V}(x) = 0 \}$

Then $x \in S \Rightarrow Cx = 0$ & $B^T P x = 0$ (WHY?)

Claim No solution can stay identically in S other than $x(t) \equiv 0$.

Because suppose not. Then there exists a solution $x(t) \not\equiv 0$ satisfying $x(t) \in S$ for all $t \ge 0$.

Then $\dot{x}(t) = Ax(t) - \underbrace{BR^{-1}B^T P x(t)}_{=0} \Rightarrow \dot{x}(t) = Ax(t)$ & $Cx(t) = 0$ for all $t \ge 0$.

$\Rightarrow Ce^{At}x(0) = 0 \quad \forall t \ge 0 \Rightarrow x(0) = 0$ by observability (WHY?)

$x(0) = 0 \Rightarrow x(t) \equiv 0 \Rightarrow$ contradiction.

Therefore $x(t) \equiv 0$ is the only solution that stays in S identically.

The origin is GAS by Corollary 4.2. ∎

Assignment Read Section 4.3 ("Linear Systems & Linearization")

<u>Theorem 4.7</u> [Stability check by linearization] Consider $\dot{x} = f(x)$ with $f(0) = 0$.

Let $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$. Then

1) The origin is asy. stable if $Re\{\lambda_i\} < 0$ for all eigenvalues λ_i of A.

2) The origin is unstable if $Re\{\lambda_i\} > 0$ for at least one eigenvalue λ_i.

<u>Remark</u> The conditions of Thm 4.7 are sufficient only. When $Re\{\lambda_i\} \leq 0$ for all i and there exist at least one eigenvalue on the imaginary axis $Re\{\lambda_i\} = 0$ Thm 4.7 says nothing. In such case the origin of $\dot{x} = f(x)$ can display any behaviour (asy. stable, stable, or unstable)

<u>Example #1</u> System: $\dot{x} = x^3$

Linearization: $\dot{\eta} = [0]\eta \Rightarrow \eta = 0$ is stable (but not asy. stable)

<u>How about $x=0$</u> ?

Let $V(x) = \frac{1}{2}x^2$, $U = \{x : V(x) > 0\}$, U has points arbitrarily close to $x = 0$.

$\dot{V}(x) = x\dot{x} = x^4 \Rightarrow \dot{V} > 0$ on $B_r \cap U$ for any $r > 0$ $\quad (B_r = \{x : \|x\| \leq r\})$

Hence $x = 0$ is unstable by Chetaev's Thm.

In fact $x(t) = \dfrac{x(0)}{\sqrt{1 - 2x(0)^2 t}} \Rightarrow$ the system suffers finite escape times.

<u>Example #2</u> System: $\dot{x} = -x^3$

Linearization: $\dot{\eta} = [0]\eta \Rightarrow \eta = 0$ is stable (but not asy. stable)

Let $V(x) = \frac{1}{2}x^2 \Rightarrow \dot{V}(x) = -x^4 \Rightarrow \dot{V} < 0 \Rightarrow x = 0$ is asy. stable.

Example #3 system : $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 \end{cases}$

linearization: $\dot{\eta} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \eta \Rightarrow \eta = 0$ is unstable

Let $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \Rightarrow \dot{V}(x) = x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = x_1^3 x_2 + x_2(-x_1^3) = 0$

$\Rightarrow x = 0$ is stable (but not asy. stable because $V(x(t)) = \text{constant}$)

Example #4 system $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases}$

linearization : $\dot{\eta} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \eta \Rightarrow \eta = 0$ is unstable

Let $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \Rightarrow \dot{V}(x) = -x_2^4 \Rightarrow \dot{V} \leq 0 \Rightarrow x = 0$ is stable

$\dot{V}(x(t)) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow -x_1(t)^3 - x_2(t)^3 \equiv 0 \Rightarrow x_1(t) \equiv 0$

Hence $\dot{V} \equiv 0 \Rightarrow x(t) \equiv 0 \Rightarrow x = 0$ is asy. stable by LaSalle's invariance principle.

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Summary : All of the below cases are possible.

($x=0$: the origin of the actual system ; $\eta = 0$: the origin of the linearization)

	$x=0$	$\eta=0$
Ex #1	unstable	stable
Ex #2	asy. stable	stable
Ex #3	stable	unstable
Ex #4	asy. stable	unstable

Nonautonomous Systems

System: $\dot{x} = f(t,x)$, $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, t: time

Equilibrium: $x_{eq} \in \mathbb{R}^n$ is an equil. point if $f(t, x_{eq}) = 0 \quad \forall t \geq 0$.

Stability concept becomes subtler for time-varying case:

Definition 4.4 Let $x=0$ be an equilibrium of $\dot{x} = f(t,x)$.
Then the origin is said to be

→ stable if for each $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta > 0$ such that

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \text{for all} \quad t \geq t_0$$

→ unstable if not stable

→ uniformly stable if for each $\varepsilon > 0$ there exists $\delta > 0$ (independent of t_0) such that

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon \quad \text{for all } t_0 \geq 0 \text{ and all } t \geq t_0$$

→ asymptotically stable if stable and for each $t_0 \geq 0$ there exists $c > 0$ such that

$$\|x(t_0)\| < c \implies x(t) \to 0 \quad \text{as} \quad t \to \infty$$

→ uniformly asymptotically stable if uniformly stable and there exists $c > 0$ and for each $d > 0$ there exists $T > 0$ such that

$$\|x(t_0)\| < c \;\;\&\;\; t \geq t_0 + T \implies \|x(t)\| < d \qquad (\text{for all } t_0 \geq 0)$$

→ globally uniformly asymptotically stable if uniformly stable AND for each $\delta > 0$ there exists $\varepsilon < \infty$ such that $\|x(t_0)\| < \delta \implies \|x(t)\| < \varepsilon$ for all $t \geq t_0$ AND for each pair (c,d) of positive numbers there exists $T > 0$ such that

$$\|x(t_0)\| < c \;\;\&\;\; t \geq t_0 + T \implies \|x(t)\| < d \qquad (\text{for all } t_0 \geq 0)$$

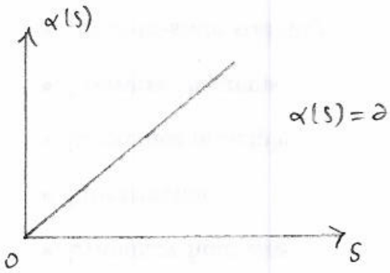
Comparison Functions

class-K A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is said to belong to class-K ($\alpha \in K$) if it is strictly increasing and $\alpha(0) = 0$.

class-K_∞ Let $\alpha \in K$. If $\alpha(s) \to \infty$ as $s \to \infty$ then α is said to belong to class K_∞ ($\alpha \in K_\infty$). Note that $K_\infty \subset K$

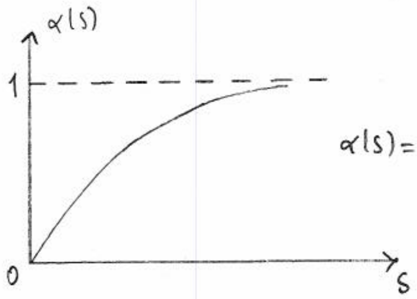
class-KL A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to belong to class KL ($\beta \in KL$) if for each fixed $\bar{t}$ the function $\beta(\cdot, \bar{t})$ is class K and for each fixed $\bar{s}$ the function $\beta(\bar{s}, \cdot)$ is nonincreasing and $\lim_{t \to \infty} \beta(\bar{s}, t) = 0$.

Example :



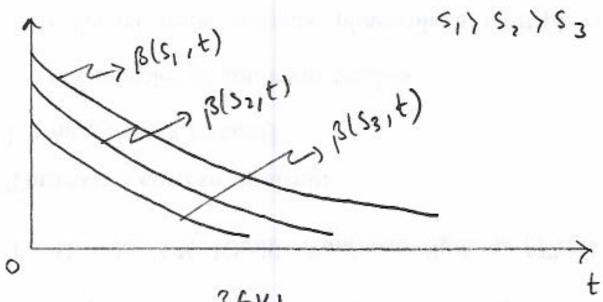
$\alpha(s) = 2s$

$\alpha \in K_\infty$



$\alpha(s) = 1 - e^{-s}$

$\alpha \in K$ but $\alpha \notin K_\infty$



$\beta(s,t) = s^2 e^{-t}$

$\beta \in KL$

Comparison functions are useful tools in characterization of uniform stability :

Lemma 4.5 The equilibrium $x=0$ of $\dot{x}=f(t,x)$ is

→ US if and only if there exist $\alpha \in K$ & $c>0$ (both independent of t_0)

such that $\|x(t_0)\| < c \Rightarrow \|x(t)\| \le \alpha(\|x(t_0)\|)$ for all $t \ge t_0$

→ UAS if and only if there exist $\beta \in KL$ & $c>0$ (both independent of t_0)

such that $\|x(t_0)\| < c \Rightarrow \|x(t)\| \le \beta(\|x(t_0)\|, t-t_0)$ for all $t \ge t_0$

→ GUAS if and only if there exists $\beta \in KL$ (independent of t_0) such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0) \quad \text{for all } t \ge t_0$$

Definition [Exp. Stability] The equilibrium $x=0$ of $\dot{x}=f(t,x)$ is exponentially stable if there exist positive constants c, k, λ such that

$$\|x(t_0)\| < c \Rightarrow \|x(t)\| \le k\|x(t_0)\| e^{-\lambda(t-t_0)} \quad \text{for all } t \ge t_0$$

and globally exp. stable if we can let $c=\infty$.

Example 4.17 system: $\dot{x} = [6t\sin t - 2t]x$ first-order LTV

solution: $x(t) = x(t_0)\exp\underbrace{\{6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2\}}_{g(t,t_0)}$

Let $\bar{g}(t_0) := \max_{t \ge t_0} g(t,t_0)$. Then for each t_0, $\bar{g}(t_0) < \infty$. (WHY?)

Hence, given $\varepsilon > 0$ we can choose $\delta(t_0) = \dfrac{\varepsilon}{\bar{g}(t_0)}$ and write

$$\|x(t_0)\| < \delta(t_0) \Rightarrow \|x(t)\| \le \bar{g}(t_0)\|x(t_0)\| < \bar{g}(t_0)\frac{\varepsilon}{\bar{g}(t_0)} < \varepsilon \quad \text{for all } t \ge t_0$$

The origin therefore is stable.

Uniform stability?

For $t_0 = 2n\pi$ & $t = (2n+1)\pi$ we have $\underbrace{x(t)}_{<\varepsilon} = \underbrace{x(t_0)}_{<\delta}\exp[(4n+1)(6-\pi)\pi]$

⇒ Given $\varepsilon > 0$ there is no $\delta > 0$ that works for all t_0 ⇒ the origin is not unif. stable!

Remark For autonomous case $\dot{x} = f(x)$ (asy.) stability $\equiv$ unif. (asy.) stability.

Example 4.18 system : $\dot{x} = -\frac{1}{1+t} x$ $(t \geq 0)$

solution : $x(t) = x(t_0) \frac{1+t_0}{1+t}$

Unif. stable ? YES, $\|x(t)\| \leq \|x(t_0)\|$ for all $t \geq t_0$ $(\delta = \varepsilon)$

Asy. stable ? YES, $\lim_{t \to \infty} \|x(t)\| = 0$

Unif. asy. stable? NO. Suppose otherwise. Then there exist $c > 0$ and $T > 0$ (independent of t_0) such that

$$\|x(t_0)\| < c \implies \|x(t)\| < \frac{c}{4} \quad \text{for all } t \geq t_0 + T \qquad (*)$$

Take $\|x(t_0)\| = \frac{c}{2}$ and $t_0 = 2T$. Then

$$\|x(t)\|_{t = t_0 + T} = \left\| x(t_0) \frac{1+t_0}{1+t_0+T} \right\| = \left\| x(t_0) \frac{1+2T}{1+3T} \right\| > \|x(t_0)\| \cdot \frac{2}{3} = \frac{c}{3}$$

This contradicts with $(*)$.

— o —

Lyapunov Characterization of uniform stability

Theorem 4.9 [GUAS] Let $x = 0$ be an equilibrium point for $\dot{x} = f(t,x)$ and let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$

and $\dot{V}(t,x) \leq -\alpha_3(\|x\|)$ [Note that $\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$]

for all $t \geq 0$ and all $x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ & $\alpha_3 \in \mathcal{K}$. Then $x = 0$ is GUAS.

Theorem 4.10 [GES] Suppose the conditions in Thm 4.9 hold. Suppose further that we can let $\alpha_1(s) = k_1 s^r$, $\alpha_2(s) = k_2 s^r$, and $\alpha_3(s) = k_3 s^r$ for some $k_1, k_2, k_3, r > 0$. Then the origin is globally exp. stable.

Remark When the conditions of the above theorems hold only for $x \in D$ where D is an open set containing the origin, we have local results : UAS & ES (instead of GUAS & GES)

Proof of Thm 4.10

given $\begin{cases} k_1\|x\|^r \le V(t,x) \le k_2\|x\|^r \\ \dot V(t,x) \le -k_3\|x\|^r \end{cases}$, establish $\|x(t)\| \le c\, e^{-\lambda(t-t_0)}\|x(t_0)\|$

$\dot V \le -k_3\|x\|^r \le -\frac{k_3}{k_2} V$ (let $\mu := \frac{k_3}{k_2}$)

$\Rightarrow \dot V \le -\mu V$ comparison lemma

$\Rightarrow V(t,x(t)) \le e^{-\mu(t-t_0)} V(t_0,x(t_0))$

$\Rightarrow k_1\|x(t)\|^r \le e^{-\mu(t-t_0)} k_2\|x(t_0)\|^r$

$\Rightarrow \|x(t)\| \le \left(\frac{k_2}{k_1}\right)^{1/r} e^{-\frac{\mu}{r}(t-t_0)}\|x(t_0)\|$

The result follows with $c = \left[\frac{k_2}{k_1}\right]^{\frac{1}{r}}$ & $\lambda = \frac{k_3}{k_2 r}$. ▨

Comparison lemma $\dot v(t) \le \gamma v(t) \Rightarrow v(t) \le e^{\gamma(t-t_0)} v(t_0)$

proof Define $u(t) := e^{-\gamma(t-t_0)} v(t)$. Note that $u(t_0) = v(t_0)$

Then $\dot u = -\gamma e^{-\gamma(t-t_0)} v + e^{-\gamma(t-t_0)} \dot v$

$\le -\gamma e^{-\gamma(t-t_0)} v + e^{-\gamma(t-t_0)} \cdot \gamma v$

$= 0$

Now, $\dot u \le 0 \Rightarrow u(t) \le u(t_0)$

$\Rightarrow e^{-\gamma(t-t_0)} v(t) \le v(t_0)$

$\Rightarrow v(t) \le e^{\gamma(t-t_0)} v(t_0)$ ▨

Example (4.20) system: $\begin{cases} \dot{x}_1 = -x_1 - g(t)x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$, assume: $\begin{cases} 0 \le g(t) \le k \\ \dot{g}(t) \le g(t) \quad \text{for all } t \end{cases}$

Show that $x=0$ is GES using $V(t,x) = x_1^2 + [1+g(t)]x_2^2$

Sol'n We can write

$$\|x\|^2 = x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2 \le (1+k)\|x\|^2$$

$$\Rightarrow \quad \|x\|^2 \le V(t,x) \le (1+k)\|x\|^2 \qquad (1)$$

As for $\dot{V}$ we have

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x)$$

$$= \dot{g}(t)x_2^2 + \begin{bmatrix} 2x_1 & 2[1+g(t)]x_2 \end{bmatrix} \begin{bmatrix} -x_1 - g(t)x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$= \dot{g}(t)x_2^2 - 2x_1^2 - 2g(t)x_1x_2 + 2[1+g(t)]x_1x_2 - 2[1+g(t)]x_2^2$$

$$= -2x_1^2 - 2x_2^2 + 2x_1x_2 - x_2^2\underbrace{(2g(t) - \dot{g}(t))}_{\ge 0}$$

$$\le -2x_1^2 - 2x_2^2 + 2x_1x_2$$

$$= -x_1^2 - x_2^2 - (x_1 - x_2)^2$$

$$\le -\|x\|^2$$

That is, $\dot{V}(t,x) \le -\|x\|^2 \quad (2)$.

By (1), (2), and Thm 4.10 the origin is GES. ▨

Exercise Let $x=0$ be UAS for LTV system $\dot{x} = A(t)x$. Show that $x=0$ is GES.

Exercise Let $x=0$ be an equilibrium of $\dot{x}=f(x)$. Suppose $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$ exists and $g(x) := f(x) - Ax$ satisfies $\|g(x)\| \le L\|x\|^2$ for some $L>0$ in some neighborhood of the origin. Show that if A is Hurwitz then the origin of $\dot{x}=f(x)$ is exp. stable. [Recall: A Hurwitz $\Leftrightarrow \exists P=P^T>0$ s.t. $A^TP + PA + I = 0$.]

A Converse Lyapunov Theorem

We know : Existence of a Lyapunov function $\Rightarrow$ stability

We wonder : stability $\overset{?}{\Rightarrow}$ existence of a Lyap function

First, study the linear case :

system : $\dot{x} = Ax$ Suppose: the origin is asy. stable.

Solution $\phi_x(t) = e^{At}x$ (Here, we treat $x \in \mathbb{R}^n$ as our initial cond. $\phi_x(0) = x$)

Asy. stability $\Rightarrow$ $\|\phi_x(t)\| \leq ke^{-\lambda t}\|x\|$ (1)

Define $V(x) = \underbrace{\int_0^\infty \|\phi_x(t)\|^2 dt}_{\text{well-defined due to (1)}} = \int_0^\infty x^T e^{A^T t} e^{At} x \, dt = x^T \left\{ \int_0^\infty e^{A^T t} e^{At} dt \right\} x$

$\rightarrow$ Note that V is pos. def. (WHY?)

How about $\dot{V}$?

$$\dot{V}(x) = \dot{x}^T \left\{ \int_0^\infty e^{A^T t} e^{At} dt \right\} x + x^T \left\{ \int_0^\infty e^{A^T t} e^{At} dt \right\} \dot{x}$$

$\dot{x} = Ax$

$$= x^T \left\{ \int_0^\infty \underbrace{\left[A^T e^{A^T t} e^{At} + e^{A^T t} e^{At} A\right] dt}_{d(e^{A^T t} e^{At})} \right\} x$$

$$= x^T \left\{ e^{A^T t} e^{At} \Big|_{t=0}^{t=\infty} \right\} x$$

WHY?

$$= -\|x\|^2$$

$\rightarrow$ Hence $\dot{V}$ is neg. def.

Conclusion $V(x) = \int_0^\infty \|\phi_x(t)\|^2 dt$ works as a Lyap function.

This or similar ideas lie behind the converse Lyapunov theorems. [See, for instance, "A smooth Lyapunov function from a class-KL estimate..." A.R. Teel & L. Praly, ESAIM: COCV, 2000]

Theorem 4.17 Let $x=0$ be the GAS equilibrium of $\dot{x}=f(x)$ where f is locally Lipschitz. Then there exist a Lyapunov function $V:\mathbb{R}^n \to \mathbb{R}$ along with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

and $\langle \nabla V(x), f(x) \rangle \le -\alpha_3(\|x\|)$ for all $x \in \mathbb{R}^n$

Example 4.45 System: $\begin{cases} \dot{x}_1 = h(t)x_2 - g(t)x_1^3 \\ \dot{x}_2 = -h(t)x_1 - g(t)x_2^3 \end{cases}$

$h(t), g(t)$: bounded, differentiable. Moreover, $g(t) \ge k > 0$ for all $t \ge 0$.

a) Is $x=0$ UAS?

b) Is $x=0$ ES? [Use Thm 4.15: the origin of $\dot{x}=f(t,x)$ is ES if and only if the linearization $\dot{x}=A(t)x$, with $A(t) = \frac{\partial}{\partial x} f(t,x)\Big|_{x=0}$, is ES.]

c) Is $x=0$ GUAS?

d) Is $x=0$ GES?

Sol'n a, c) Take $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 = \|x\|^2/2$.

Then $\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$

$$= x_1\{h(t)x_2 - g(t)x_1^3\} + x_2\{-h(t)x_1 - g(t)x_2^3\}$$

$$= -g(t)\{x_1^4 + x_2^4\}$$

$$\le -k(x_1^4 + x_2^4)$$

$$\le -\frac{k}{2}\|x\|^4$$

because: $(x_1^2)^2 + (x_2^2)^2 = \frac{1}{2}(x_1^2+x_2^2)^2 - x_1^2x_2^2 + \frac{1}{2}(x_1^2)^2 + \frac{1}{2}(x_2^2)^2$

$$= \frac{1}{2}\|x\|^4 + \frac{1}{2}[x_1^2 - x_2^2]^2$$

$$\ge \frac{1}{2}\|x\|^4$$

Hence we have $\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$ & $\dot{V}(x) \le -\alpha_3(\|x\|)$ with $\alpha_1(s) = \alpha_2(s) = s^2/2$ and $\alpha_3(s) = s^4/2$. ($\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$) Then Thm 4.9 $\Rightarrow$ GUAS $\Rightarrow$ UAS b,d) $A(t) = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix}$

Is the LTV system $\dot{\eta} = A(t)\eta$ ES? [Note that the eigenvalues of $A(t)$ tell nothing for $\dot{x} = A(t)x$, see Ex 4.22.]

Try $V(\eta) = \frac{1}{2}\eta_1^2 + \frac{1}{2}\eta_2^2$.

$\dot{V} = \langle \nabla V, A(t)\eta \rangle = [\eta_1 \quad \eta_2] \begin{bmatrix} 0 & h(t) \\ -h(t) & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0$

$\Rightarrow V(\eta(t)) = \text{constant}$

$\Rightarrow V(\eta(t)) \not\to 0$

$\Rightarrow \eta(t) \not\to 0$

$\Rightarrow \eta = 0$ is NOT AS

$\Rightarrow \eta = 0$ is NOT ES

Thm 4.15

$\Rightarrow x = 0$ is NOT ES

$\Rightarrow x = 0$ is NOT GES

Ch. XIV Nonlinear Design Tools

Backstepping

system : $\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi \\ \dot{\xi} = u \end{cases}$ $\eta \in \mathbb{R}^n, \ \xi \in \mathbb{R}$

u : control input

Assume : there is a known feedback law $\phi : \mathbb{R}^n \to \mathbb{R}$ (with $\phi(0) = 0$) under which the origin $\eta = 0$ of the subsystem

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

is asy. stable. Moreover, we know a (smooth, pos.def.) Lyap function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies

$$\langle \nabla V(\eta), f(\eta) + g(\eta)\phi(\eta) \rangle \leq -W(\eta)$$

with some pos. def. $W : \mathbb{R}^n \to \mathbb{R}$.

Goal : find a suitable control input u to stabilize the origin $\begin{bmatrix} \eta \\ \xi \end{bmatrix} = 0$.

Sol'n : First, rewrite the system as

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)[\xi - \phi(\eta)]$$
$$\dot{\xi} = u$$

Under change of variables $z = \xi - \phi(\eta)$ we have

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta) z$$
$$\dot{z} = u - \dot{\phi}(\eta) = \underbrace{u - \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]}$$

let's call this term v
treat it as the "control input"

$\left(\frac{\partial \phi}{\partial \eta} := \nabla \phi(\eta)^T \right.$

Finally we have

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta) z$$
$$\dot{z} = v$$

To design a stabilizing v (and eventually a stabilizing u) let's use the composite Lyap function

$$V_c(\eta,\xi) = V(\eta) + \frac{1}{2}z^2$$

Then,

$$\dot{V}_c = \frac{\partial V}{\partial \eta}\left[f(\eta) + g(\eta)\phi(\eta)\right] + \frac{\partial V}{\partial \eta} g(\eta) z + zv$$

$$\leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + zv$$

Choose $v = -\frac{\partial V}{\partial \eta} g(\eta) - kz \qquad (k>0)$

$$\leq -W(\eta) - kz^2$$

To summarize:

$$V_c(\eta,\xi) = V(\eta) + \frac{1}{2}(\xi - \phi(\eta))^2 \qquad V_c \text{ is pos. det.}$$

$$\dot{V}_c \leq -W(\eta) - k(\xi - \phi(\eta))^2 \qquad \dot{V}_c \text{ is neg. det.}$$

Hence, the origin $\begin{bmatrix}\eta\\ \xi\end{bmatrix} = 0$ is asy. stable under the feedback

$$u(\eta,\xi) = \frac{\partial \phi}{\partial \eta}\left[f(\eta) + g(\eta)\xi\right] + v$$

$$= \frac{\partial \phi}{\partial \eta}\left[f(\eta) + g(\eta)\xi\right] - \frac{\partial V}{\partial \eta} g(\eta) - k(\xi - \phi(\eta))$$

Moreover, if

→ all assumptions hold globally

→ $V(\eta)$ radially unbounded

then the origin is GAS.

<u>Example 14.8</u> Stabilize the origin of

$$\left.\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u \end{aligned}\right\} \text{ Note that } \eta = x_1 \ \& \ \xi = x_2 \text{ here!}$$

Step 1 Find a feedback law $\phi: \mathbb{R} \to \mathbb{R}$ such that the origin of

$$\left.\dot{x}_1 = \underbrace{x_1^2 - x_1^3}_{\text{potentially harmful term}} + \underbrace{x_2}_{\text{useful term}}\right|_{x_2 = \phi(x_1)} \quad (1) \quad \text{is asy. stable}$$

Let $\phi(x_1) = -x_1^2 - x_1$. Then (1) becomes

$$\dot{x}_1 = -x_1 - x_1^3 \quad (2)$$

For $V(x_1) = \frac{1}{2}x_1^2$, subsystem (2) yields

$$\dot{V}(x_1) = -x_1^2 - x_1^4 =: -W(x_1)$$

$\dot{V} < 0 \Rightarrow x_1 = 0$ is asy. stable. (Indeed, $x_1 = 0$ is GES. Why?)

Step 2 Apply backstepping. Consider the original system.

Let $z = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$

Dynamics of the (x_1, z) system?

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 + \phi(x_1) - \phi(x_1) = \underbrace{x_1^2 - x_1^3 + \phi(x_1)}_{-x_1 - x_1^3} + \underbrace{x_2 - \phi(x_1)}_{z}$$

$$\dot{z} = \dot{x}_2 + (1 + 2x_1)\dot{x}_1 = u + (1 + 2x_1)[-x_1 - x_1^3 + z]$$

Hence,
$$\dot{x}_1 = -x_1 - x_1^3 + z$$
$$\dot{z} = u + (1 + 2x_1)[-x_1 - x_1^3 + z]$$

Take $V_c(x_1 z) = \frac{1}{2}x_1^2 + \frac{1}{2}z^2$ i.e. $\boxed{V_c = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2}$

$$\Rightarrow \dot{V}_c = x_1\dot{x}_1 + z\dot{z} = -x_1^2 - x_1^4 + x_1 z + zu + z(1 + 2x_1)[-x_1 - x_1^3 + z]$$
$$= -x_1^2 - x_1^4 + \underbrace{z[u + x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z)]}_{\text{choose a proper } u \text{ to make this term negative}}$$

Let $u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z) - z$ i.e. $\boxed{u = -x_1 - (1 + 2x_1)(x_1^2 - x_1^3 + x_2) - (x_2 + x_1 + x_1^2)}$

Then $\dot{V}_c = -x_1^2 - x_1^4 - z^2$

i.e. $\boxed{\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2}$

Finally, $\left.\begin{array}{l} V_c > 0 \text{ \& rad unbounded} \\ \dot{V}_c < 0 \end{array}\right\} x = 0$ is GAS

Example 14.9 $\dot{x}_1 = x_1^2 - x_1^3 + x_2$ Stabilize the origin. [Recursive design]

$\dot{x}_2 = x_3$

$\dot{x}_3 = u$

Sol'n From the previous example we know that the subsystem

$$\left\{ \begin{array}{l} \dot{x}_1 = x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 = x_3 \end{array} \right\}_{x_3 = \phi(x_1, x_2)}$$ for $\phi(x_1, x_2) := -x_1 - (1+2x_1)(x_1^2 - x_1^3 + x_2) - (x_2 + x_1 + x_1^2)$ has an asy. stable origin, where

$V(x_1, x_2) = \frac{1}{2} x_1^2 + \frac{1}{2}(x_2 + x_1 + x_1^2)^2$ works as a Lyapunov function.

This time we let $z = x_3 - \phi(x_1, x_2)$. Then

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$\dot{x}_2 = \phi(x_1, x_2) + z$$

$$\dot{z} = u - \overbrace{\left[\frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(\phi(x_1, x_2) + z) \right]}^{\square} = u - \square$$

Letting $V_c(x_1, x_2, z) = V(x_1, x_2) + \frac{1}{2} z^2$ we have

$$\dot{V}_c = \underbrace{\frac{\partial V}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2}\phi(x_1, x_2)}_{\substack{\text{negative def. w.r.t. } (x_1, x_2) \\ \text{from the previous example}}} + \frac{\partial V}{\partial x_2} z + zu - z\square$$

choosing $u = \square - \frac{\partial V}{\partial x_2} z - z$ we obtain

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 - z^2$$

We have $\left. \begin{array}{l} V_c > 0 \text{ \& rad unbounded} \\ \dot{V}_c < 0 \end{array} \right\} \Rightarrow x = 0$ is GAS. ∎

Recursive backstepping can be adapted to the systems in "strict feedback" form:

$$\dot{\eta} = f(\eta) + g(\eta)\,\xi_1$$

$$\dot{\xi}_1 = h_1(\eta, \xi_1) + l_1(\eta, \xi_1)\,\xi_2$$

$$\dot{\xi}_2 = h_2(\eta, \xi_1, \xi_2) + l_2(\eta, \xi_1, \xi_2)\,\xi_3$$

$$\vdots$$

$$\dot{\xi}_{k-1} = h_{k-1}(\eta, \xi_1, \dots, \xi_{k-1}) + l_{k-1}(\eta, \xi_1, \dots, \xi_{k-1})\,\xi_k$$

$$\dot{\xi}_k = h_k(\eta, \xi_1, \dots, \xi_k) + l_k(\eta, \xi_1, \dots, \xi_k)\,u$$

Conditions:

→ $\eta \in \mathbb{R}^n$, $\xi_i \in \mathbb{R}$ $\quad i = 1, 2, \dots, k$

→ $l_i(\cdot) \neq 0 \quad i = 1, 2, \dots, k$

→ There exist $\phi : \mathbb{R}^n \to \mathbb{R}$ with $\phi(0) = 0$

and $V : \mathbb{R}^n \to \mathbb{R}$, $W : \mathbb{R}^n \to \mathbb{R}$ both pos. def. such that

$$\langle \nabla V(\eta), f(\eta) + g(\eta)\phi(\eta) \rangle = -W(\eta)$$

Exercise Develop the procedure (or read the text)

Remark: For real-world engineering applications of backstepping see the book "Nonlinear & Adaptive Control" by Krstic et al., 1995. Examples include active suspension problem in cars & jet engine stabilization in planes.

Input-to-State Stability

Consider the LTI system

$$\dot{x} = Ax + Bu$$

where → $A \in \mathbb{R}^{n\times n}$ is Hurwitz [i.e. all the eigenvalues of A satisfy $\mathrm{Re}\{\lambda_i\} < 0$]

→ the input $u \in \mathbb{R}^m$ represents the undesired disturbance (as opposed to control input)

Question: the effect of input on the input-free behaviour $\|x(t)\| \to 0$?

solution: $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$

A Hurwitz $\Rightarrow$ $\|e^{At}\| \le k e^{-\lambda t}$ for some $k, \lambda > 0$

Therefore,

$$\|x(t)\| \le k e^{-\lambda t}\|x(0)\| + \int_0^t k e^{-\lambda(t-\tau)} \|B\| \cdot \|u(\tau)\|\, d\tau$$

$$\le k e^{-\lambda t}\|x(0)\| + k\|B\| \sup_{\tau\in[0,t]} \|u(\tau)\| \cdot e^{-\lambda t}\overbrace{\int_0^t e^{\lambda\tau}\, d\tau}^{\frac{e^{\lambda t}-1}{\lambda}}$$

$$\le k e^{-\lambda t}\|x(0)\| + \frac{k\|B\|}{\lambda} \sup_{\tau\in[0,t]} \|u(\tau)\| \qquad (1)$$

Inequality (1) implies:

→ For zero init. state $x(0) = 0$, small input $\Rightarrow$ small state

→ For nonzero init. state, small input $\Rightarrow$ eventually small state

This behaviour of linear systems motivated the below definition

<u>Definition 4.7</u> The system $\dot{x} = f(t,x,u)$ is said to be input-to-state stable (ISS) if there exist $\beta \in KL$ and $\gamma \in K$ such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup_{\tau\in[t_0,t]} \|u(\tau)\|\right) \quad \text{for all } t \ge t_0$$

<u>Remark</u> If $\dot{x} = f(t,x,u)$ is ISS, then the origin of the system $\dot{x} = f(t,x,0)$ is GUAS. [The reverse is not true in general.]

Example : Consider $\dot{x} = -x + ux$. For $u(t) \equiv 0$, the system becomes $\dot{x} = -x$, for which the origin is GUAS. This system however is not ISS because bounded input can result in unbounded state. Take for instance $u(t) \equiv 2$. Then $\dot{x} = x$ and $\|x(t)\| \to \infty$ if $x(0) \neq 0$.

Example 4.58 Let $\dot{x} = f(t, x, u)$ be ISS. Show that $\lim_{t\to\infty} u(t) = 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$.

Sol'n

Let $x(t_0)$, $u(t)$ be given.

$$\text{ISS} \Rightarrow \|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{\tau\in[t_0,t]} \|u(\tau)\|\right) \qquad (1)$$

$u(t) \to 0 \iff$ for each $k \in \{1, 2, \dots\}$ we can find $T_k > t_0$ such that

$$\|u(t)\| < \frac{1}{2^k} \quad \text{for all } t \ge T_k \qquad (2).$$

$x(t) \to 0 \iff$ for each $\varepsilon > 0$ we can find $T_\varepsilon > t_0$ such that

$$\|x(t)\| < \varepsilon \quad \text{for all } t \ge T_\varepsilon \qquad (3)$$

Given $\varepsilon > 0$, first choose k such that

$$\gamma\left(\frac{1}{2^k}\right) < \frac{\varepsilon}{2} \qquad (4)$$

Then, choose M such that

$$M \ge \beta(\|x(t_0)\|, T_k - t_0) + \gamma\left(\sup_{\tau\in[t_0,T_k]} \|u(\tau)\|\right) \qquad (5)$$

Finally, choose Δ such that

$$\beta(M, \Delta) < \frac{\varepsilon}{2} \qquad (6)$$

$(1) \& (5) \Rightarrow \|x(T_k)\| \le M \qquad (7)$

$$(1) \& (7) \Rightarrow \|x(t)\| \le \beta(M, t - T_k) + \gamma\left(\sup_{\tau\in[T_k,t]} \|u(\tau)\|\right) \quad \text{for } t \ge T_k$$

by (2)

$$\le \beta(M, t - T_k) + \gamma\left(\frac{1}{2^k}\right)$$

by (4)

$$< \beta(M, t - T_k) + \frac{\varepsilon}{2} \qquad (8)$$

Let $T_\varepsilon := T_k + \Delta$. Then $(6) \& (8) \Rightarrow (3)$ because for $t \ge T_\varepsilon$ we have

$$\|x(t)\| < \sup_{t \ge T_\varepsilon} \beta(M, t - T_k) + \frac{\varepsilon}{2} = \beta(M, \Delta) + \frac{\varepsilon}{2} < \varepsilon. \quad \blacksquare$$

<u>Theorem 4.19</u> Let $V: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a cont. diff. func. such that

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \le -W_3(x) \quad \forall \ \|x\| \ge \rho(\|u\|) > 0$$

for all $(t,x,u) \in [0,\infty) \times \mathbb{R}^n \times \mathbb{R}^m$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and W_3 is a continuous pos. def. function. Then the system is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

<u>Example 4.27</u> Consider $\begin{cases} \dot{x}_1 = -x_1 + x_2^2 \\ \dot{x}_2 = -x_2 + u \end{cases}$

Establish ISS by the Lyapunov function candidate $V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$

<u>Sol'n</u> Note that V is pos. def. & radially unbounded. Therefore (by Lemma 4.3) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ satisfying $\underbrace{\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \quad \text{for all } x \in \mathbb{R}^2}_{(1)}$

Moreover,
$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2^3\dot{x}_2 \\
&= -x_1^2 + x_1x_2^2 - x_2^4 + x_2^3 u \\
&= -\frac{1}{2}(x_1^2 + x_2^4) - \frac{1}{2}(x_1^2 - 2x_1x_2^2 + x_2^4) + x_2^3 u \\
&= -\frac{1}{2}(x_1^2 + x_2^4) - \frac{1}{2}(x_1 - x_2^2)^2 + x_2^3 u \\
&\le -\frac{1}{2}(x_1^2 + x_2^4) + x_2^3 u \\
&\le \underbrace{-\frac{1}{4}(x_1^2 + x_2^4)}_{W_3(x)} - \underbrace{\left\{\frac{1}{4}(x_1^2 + x_2^4) - |x_2|^3 \cdot |u|\right\}}_{g(x,u)} = -W_3(x) - g(x,u)
\end{aligned}$$

(W_3 is pos. def.)

<u>Claim</u> $\underbrace{\max\{|x_1|, |x_2|\}}_{\|x\|_\infty} \ge \underbrace{\max\{4|u|, 16u^2\}}_{\rho(\|u\|)} \Rightarrow g(x,u) \ge 0$

($\rho(s) := \max\{4s, 16s^2\} \in \text{class-}\mathcal{K}$)

<u>proof</u> if $\max\{|x_1|, |x_2|\} \ge \max\{4|u|, 16u^2\}$ then

- either "$|x_2| \ge 4|u|$" $\Rightarrow \frac{1}{4}(x_1^2 + x_2^4) - |x_2|^3 \cdot |u| \ge \frac{1}{4}x_2^4 - |x_2|^3|u| = \frac{1}{4}|x_2|^3(|x_2| - 4|u|) \ge 0$
- or "$|x_2| < 4|u|$" $\Rightarrow |x_1| \ge 16u^2 \Rightarrow \frac{1}{4}(x_1^2 + x_2^4) - |x_2|^3|u| \ge \frac{1}{4}x_1^2 - |x_2|^3|u| > \frac{1}{4}x_1^2 - 64u^4 \ge \frac{1}{4}(16u^2)^2 - 64u^4 = 0$

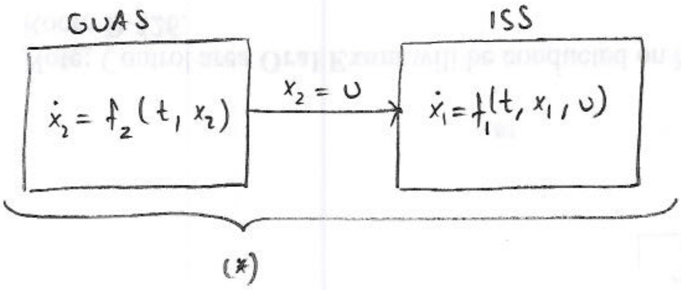
Hence we can write

$$\dot{V}(x) \leq -W_3(x) \quad \text{for all} \quad \|x\| \geq \rho(\|u\|) \qquad (2)$$

ISS follows by (1), (2), and Thm 4.19. ☒

— o —

ISS plays an important role in the stability analysis of cascade systems. In particular:



$\Rightarrow$ The origin of the overall system (*) is GUAS.

Lemma 4.7 Consider $\dot{x} = f(t,x)$ where

$$\dot{x} = \left\{ \begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2) \\ \dot{x}_2 &= f_2(t, x_2) \end{aligned} \right\} = f(t,x) \qquad x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}$$

If $\left\{ \begin{array}{l} \bullet\ \dot{x}_1 = f_1(t, x_1, u) \text{ is ISS } \& \\ \bullet\ \text{The origin of } \dot{x}_2 = f_2(t, x_2) \text{ is GUAS} \end{array} \right\}$ then the origin of $\dot{x} = f(t,x)$ is GUAS.

Example 4.56 Show that the origin is GAS for $\left\{ \begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -x_2^3 \end{aligned} \right.$

Sol'n First, consider $\dot{x}_1 = f_1(x_1, u) := -x_1^3 + u$. Take $V_1(x_1) = \frac{1}{2} x_1^2$.

$$\dot{V}_1 = x_1 \dot{x}_1 = -x_1^4 + x_1 u = -\frac{1}{2} x_1^4 - \frac{1}{2}(x_1^4 - 2x_1 u)$$

Note that $|x_1| \geq 2^{1/3} |u|^{1/3} \Rightarrow x_1^4 - 2x_1 u \geq 0$. Therefore $\boxed{\dot{x}_1 = f_1(x_1, u) \text{ is ISS}}$ since

$$\dot{V}_1(x_1) \leq -\frac{1}{2} x_1^4 \quad \text{for } |x_1| \geq 2^{1/3} |u|^{1/3} =: \rho(|u|)$$

Now consider $\dot{x}_2 = -x_2^3$ with $V_2(x_2) = \frac{1}{2} x_2^2$. Then $\dot{V}_2 = -x_2^4$ (neg. def.) Therefore

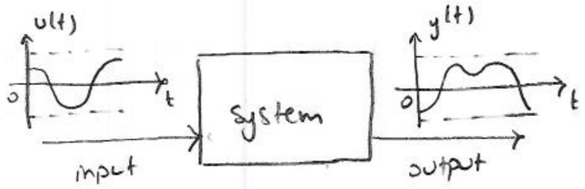
$\boxed{\text{the origin of } x_2\text{-system is GUAS}}$

Hence (by Lemma 4.7) the origin of the cascade system $\left\{ \begin{aligned} \dot{x}_1 &= -x_1^3 + x_2 \\ \dot{x}_2 &= -x_2^3 \end{aligned} \right.$ is GAS. ☒

Ch. II Input - Output Stability

Input-output stability is concerned with the following question :

→ (When) is the output "well-behaved" under "well-behaved" input ?



In particular, we desire : bounded input ⇒ bounded output.

Recall BIBO stability for LTI systems:

$$(1) \begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \\ y = Cx + Du & y \in \mathbb{R}^q \end{cases}$$

System (1) is BIBO stable if there exists $g > 0$ such that for $x(0) = 0$

we can write $\sup_{t \geq 0} \|y(t)\| \leq g \sup_{t \geq 0} \|u(t)\|$ for all input functions $u : [0, \infty) \to \mathbb{R}^m$

Recall For system (1):

→ BIBO stability ⟺ All the poles of the TF matrix $[C(sI - A)^{-1} B + D]$ are with negative real parts

→ A Hurwitz ⇒ BIBO stability

→ If system (1) is both controllable & observable then

A Hurwitz ⟺ BIBO stability

Generalization of BIBO stability : "$\mathcal{L}$ - stability"

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Signal Norms

Signal $u:[0,\infty) \to \mathbb{R}^m$

L_2 - norm : $\|u\|_{\mathcal{L}_2} := \left[\int_0^\infty u(t)^T u(t) dt \right]^{1/2}$

L_∞ - norm : $\|u\|_{\mathcal{L}_\infty} := \sup_{t \geq 0} \|u(t)\|$ → vector norm

L_p - norm : $\|u\|_{\mathcal{L}_p} := \left[\int_0^\infty \|u(t)\|^p dt \right]^{1/p}$ $(p \geq 1)$

" $\mathcal{L}_p^m$ " denotes the space of all signals $u:[0,\infty) \to \mathbb{R}^m$ with finite $\mathcal{L}_p$-norms.

$\mathcal{L}$ - Stability

Definition: Consider the system

$$(1) \begin{cases} \dot{x} = f(t,x,u), & x(0) = x_0 \\ y = h(t,x,u) \end{cases}$$

The system (1) is said to be $\mathcal{L}_p$ - stable if there exists $\alpha \in K$ and for each $x_0 \in \mathbb{R}^n$, there exists $\beta \geq 0$ such that

$$\|y\|_{\mathcal{L}_p} \leq \alpha(\|u\|_{\mathcal{L}_p}) + \beta$$

where $y:[0,\infty) \to \mathbb{R}^q$ is the output produced by the initial condition $x(0) = x_0$ and the forcing signal $u:[0,\infty) \to \mathbb{R}^m$. The system is said to be finite-gain $\mathcal{L}_p$ - stable if we can find $\gamma \geq 0$ such that

$$\|y\|_{\mathcal{L}_p} \leq \gamma \|u\|_{\mathcal{L}_p} + \beta$$

The smallest possible γ (if well-defined) is called the $\mathcal{L}_p$ gain of the system.

<u>Theorem 5.4</u> Consider the LTI system

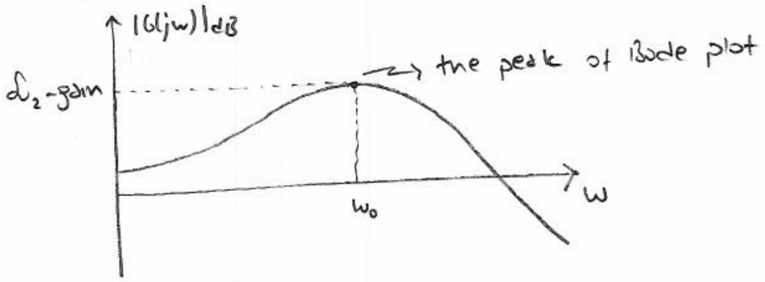
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \text{where } A \text{ is Hurwitz.}$$

Let $G(s) = C[sI - A]^{-1}B + D$. The system is $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain given as

$$\gamma = \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2$$

⇒ induced 2-norm, i.e., the largest singular value of $G(j\omega)$

<u>Remark</u> For a single-input single-output LTI system with Hurwitz A matrix, the $\mathcal{L}_2$-gain can be read from the Bode plot:



<u>Theorem 5.3</u> Consider the system

$$\left.\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}\right\} (1) \qquad f: \text{locally Lipschitz}, \quad h: \text{continuous}$$

Assume: → The system (1) is ISS

→ There exist $\alpha_1, \alpha_2 \in K$, $\eta \geq 0$ such that

$$\|h(x,u)\| \leq \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta \quad \text{for all } x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

Then the system (1) is $\mathcal{L}_\infty$-stable.

<u>Proof</u> ISS ⇒ $\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right)$ for some $\beta \in KL$, $\gamma \in K$

$$\Rightarrow \|y(t)\| \leq \alpha_1\{\beta(\cdot) + \gamma(\cdot)\} + \alpha_2(\|u(t)\|) + \eta$$

$\alpha_1(a+b) \leq \alpha_1(2a) + \alpha_1(2b)$ (WHY?)

$$\leq \alpha_1\{2\beta(\cdot)\} + \alpha_1\{2\gamma(\cdot)\} + \alpha_2(\|u(t)\|) + \eta$$

$$\leq \underbrace{\alpha_1\left(2\gamma\left(\sup_{\tau \in [0,t]} \|u(\tau)\|\right)\right) + \alpha_2\left(\sup_{\tau\in[0,t]} \|u(\tau)\|\right)}_{=: \gamma_0\left(\sup_{\tau \in [0,t]} \|u(\tau)\|\right)} + \underbrace{\alpha_1\left(2\beta(\|x(0)\|, 0)\right) + \eta}_{=: \beta_0}$$

$$\Rightarrow \|y\|_{\mathcal{L}_\infty} \leq \gamma_0\left(\|u\|_{\mathcal{L}_\infty}\right) + \beta_0 \quad \blacksquare$$

[Note: $\gamma_0(s) = \alpha_1(2\gamma(s)) + \alpha_2(s)$. Hence $\gamma_0 \in K$.]

Example :
$$\dot{x}_1 = -x_1^3 + x_2 \qquad y = x_1 + x_2$$
$$\dot{x}_2 = -x_1 - x_2^3 + u$$

Lyapunov function $V(x) = x_1^2 + x_2^2$

$$\Rightarrow \dot{V}(x) = -2x_1^4 + \cancel{2x_1x_2} - \cancel{2x_1x_2} - 2x_2^4 + 2x_2u$$
$$= -(x_1^4 + 2x_1^2x_2^2 + x_2^4) - (x_1^4 - 2x_1^2x_2^2 + x_2^4) + 2x_2u$$
$$= -(x_1^2+x_2^2)^2 - (x_1^2-x_2^2)^2 + 2x_2u$$
$$\leq -\|x\|^4 + 2\|x\| \cdot \|u\|$$
$$= -\frac{1}{2}\|x\|^4 - \frac{1}{2}\|x\|\left(\|x\|^3 - 4\|u\|\right)$$

Therefore $\dot{V}(x) \leq -\frac{1}{2}V(x)^2$ when $\|x\| \geq 4^{1/3}\|u\|^{1/3}$ $\Rightarrow$ system ISS

Also, $\|h(x,u)\| = \|x_1 + x_2\| \leq |x_1| + |x_2| \leq \sqrt{2}\,\|x\|$

By Thm. 5.3 the system is $\mathcal{L}_\infty$-stable.

Lemma Consider the system

$$\left.\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}\right\} \qquad f: \text{locally Lipschitz}, \quad h: \text{continuous}$$

Let $V: \mathbb{R}^n \to \mathbb{R}$ be a positive semi definite function such that

$$\langle \nabla V, f(x,u) \rangle \leq a\left(\gamma^2\|u\|^2 - \|y\|^2\right) \quad \text{where } a, \gamma > 0$$

Then the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$-gain no larger than γ. In particular,

$$\|y\|_{\mathcal{L}_2} \leq \gamma\|u\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{a}}$$

<u>proof</u> $\dot{V} \le a\left(\gamma^2 \|u\|^2 - \|y\|^2\right)$

$$\Rightarrow \quad V(x(t)) - V(x(0)) \le a\gamma^2 \int_0^t \|u(\tau)\|^2 d\tau - a\int_0^t \|y(\tau)\|^2 d\tau \quad (1)$$

$$a\int_0^t \|y(\tau)\|^2 d\tau \le a\int_0^t \|y(\tau)\|^2 d\tau + V(x(t))$$

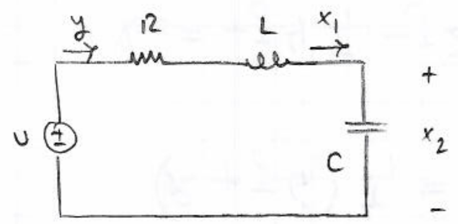
$$\le a\gamma^2 \int_0^t \|u(\tau)\|^2 d\tau + V(x(0)) \qquad \text{by } (1)$$

$$\Rightarrow \quad \int_0^t \|y(\tau)\|^2 d\tau \le \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau + \frac{V(x(0))}{a}$$

$$\Rightarrow \quad \left(\int_0^t \|y(\tau)\|^2 d\tau\right)^{1/2} \le \gamma \left(\int_0^t \|u(\tau)\|^2 d\tau\right)^{1/2} + \left(\frac{V(x(0))}{a}\right)^{1/2}$$

$$\Rightarrow \quad \|y\|_{\mathcal{L}_2} \le \gamma \|u\|_{\mathcal{L}_2} + \sqrt{V(x(0))/a} \qquad \blacksquare$$

<u>Example</u> [RLC circuit]



$$L\dot{x}_1 = V_L = u - Rx_1 - x_2$$

$$C\dot{x}_2 = x_1$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Total stored energy $\quad V(x) = \frac{1}{2} L x_1^2 + \frac{1}{2} C x_2^2 = x^T \begin{bmatrix} L/2 & 0 \\ 0 & C/2 \end{bmatrix} x$

Conservation of power $\Rightarrow \quad \underset{\substack{\Downarrow \\ uy}}{P_S} = \underset{\substack{\Downarrow \\ Ry^2}}{P_R} + \underbrace{P_L + P_C}_{\dot{V}} \quad \Rightarrow \quad uy = \dot{V} + Ry^2 \quad (1)$

<u>Claim</u> : Eq. (1) $\Rightarrow$ $\mathcal{L}_2$ gain $= \frac{1}{R}$. $\left(G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Z(s)} \right.$. Thm 5.4 $\Rightarrow$ $\mathcal{L}_2$ gain $= \max_\omega |G(j\omega)|$

$$|G(j\omega)|_{max} = |G(j\omega)|_{\omega = \frac{1}{\sqrt{LC}}} = \frac{1}{R}$$

Because : $\dot{V} = uy - Ry^2 = -\frac{1}{2R}(u - Ry)^2 + \frac{1}{2R}u^2 - \frac{R}{2}y^2 \leq \frac{R}{2}\left(\frac{1}{R^2}u^2 - y^2\right)$

$$\left.\begin{array}{l} \dot{V} \leq \frac{R}{2}\left(\frac{1}{R^2}u^2 - y^2\right) \\ \text{\& previous lemma} \end{array}\right\} \Rightarrow \mathcal{L}_2 \text{ gain} = \frac{1}{R}$$

In general :

Lemma 6.5 If the system

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \qquad x \in \mathbb{R}^n \,;\ u, y \in \mathbb{R}^m$$

(input & output are of same dimension)

is "output strictly passive" with

$$u^T y \geq \dot{V} + \delta y^T y$$

where $\delta > 0$ and $V: \mathbb{R}^n \to \mathbb{R}$ is pos. semidefinite, then it is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$-gain no larger than $1/\delta$.

Example :

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a x_1^3 - k x_2 + u \\ y &= x_2 \end{aligned} \qquad \Bigg| \quad a, k > 0$$

let $V(x) = \frac{a}{4}x_1^4 + \frac{1}{2}x_2^2$

$$\begin{aligned} \Rightarrow \dot{V} &= a x_1^3 x_2 - a x_2 x_1^3 - k x_2^2 + x_2 u \\ &= -k x_2^2 + x_2 u \\ &= -ky^2 + yu \end{aligned}$$

$$\Rightarrow uy = \dot{V} + ky^2$$

Therefore by Lem 6.5 the system is $\mathcal{L}_2$ stable with $\mathcal{L}_2$-gain $\leq \frac{1}{k}$.

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Example (LTI system)

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Suppose there exists $P = P^T \geq 0$ satisfying the Riccati equation

$$A^T P + PA + \frac{1}{\gamma^2} PBB^T P + C^T C = 0$$

for some $\gamma > 0$. Then the system is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\leq \gamma$.

Because : Let $V(x) = x^T P x$

$$\Rightarrow \dot{V}(x) = x^T PAx + x^T A^T P x + 2x^T PBu$$

& $$2x^T PBu = -\gamma^2 \| u - \frac{1}{\gamma^2} B^T P x \|^2 + \gamma^2 \|u\|^2 + \frac{1}{\gamma^2} x^T PBB^T P x$$

$$\Rightarrow \dot{V}(x) = x^T \underbrace{\left(A^T P + PA + \frac{1}{\gamma^2} PBB^T P \right)}_{-C^T C} x + \gamma^2 \|u\|^2 - \gamma^2 \| u - \frac{1}{\gamma^2} B^T P x \|^2$$

$$\leq -x^T C^T C x + \gamma^2 \|u\|^2$$

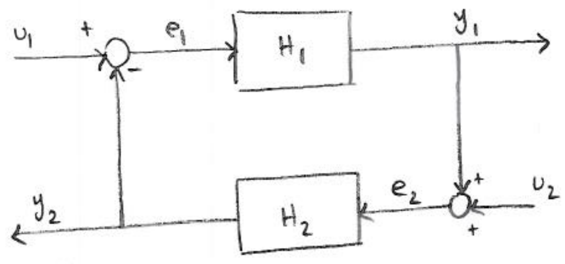
$$= \gamma^2 \|u\|^2 - \|y\|^2 .$$

The result follows by THE lemma.

The Small Gain Theorem

"$\mathcal{L}$-stability tools are useful in studying the stability of feedback connections".

Consider



Assume

(A1) The feedback connection is well-defined. That is, for each pair of input signals (u_1, u_2) the signals e_1, e_2, y_1, y_2 uniquely exist.

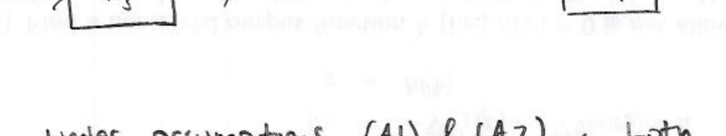
(A2) Systems H_1 & H_2 are both finite-gain $\mathcal{L}$-stable. That is, there exist $\gamma_1, \gamma_2 > 0$ and

$$\|y_1\|_{\mathcal{L}} \leq \gamma_1 \|e_1\|_{\mathcal{L}} + \beta_1$$

$$\|y_2\|_{\mathcal{L}} \leq \gamma_2 \|e_2\|_{\mathcal{L}} + \beta_2$$

where β_1, β_2 are determined by the initial conditions.

Now, we can consider the overall feedback connection as a single system whose input is $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and output is either $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ or $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. Let H_3 & H_4 denote the corresponding systems:



<u>Theorem 5.6</u> Under assumptions (A1) & (A2), both systems H_3 & H_4 are finite gain $\mathcal{L}$-stable if $\gamma_1 \gamma_2 < 1$.

proof: $e_1 = u_1 - y_2$

$$\Rightarrow \|e_1\|_{\mathcal{L}} \le \|u_1\|_{\mathcal{L}} + \|y_2\|_{\mathcal{L}}$$
$$\le \|u_1\|_{\mathcal{L}} + \gamma_2 \|e_2\|_{\mathcal{L}} + \beta_2 \qquad (1)$$

$e_2 = u_2 + y_1$

$$\Rightarrow \|e_2\|_{\mathcal{L}} \le \|u_2\|_{\mathcal{L}} + \|y_1\|_{\mathcal{L}}$$
$$\le \|u_2\|_{\mathcal{L}} + \gamma_1 \|e_1\|_{\mathcal{L}} + \beta_1 \qquad (2)$$

$$(1) \,\&\, (2) \Rightarrow \|e_1\|_{\mathcal{L}} \le \|u_1\|_{\mathcal{L}} + \gamma_2 \left\{ \|u_2\|_{\mathcal{L}} + \gamma_1 \|e_1\|_{\mathcal{L}} + \beta_1 \right\} + \beta_2$$
$$= \|u_1\|_{\mathcal{L}} + \gamma_2 \|u_2\|_{\mathcal{L}} + \gamma_1\gamma_2 \|e_1\|_{\mathcal{L}} + \gamma_2\beta_1 + \beta_2$$

$$\Rightarrow (1-\gamma_1\gamma_2)\|e_1\|_{\mathcal{L}} \le \|u_1\|_{\mathcal{L}} + \gamma_2 \|u_2\|_{\mathcal{L}} + \gamma_2\beta_1 + \beta_2$$

$$\Rightarrow \|e_1\|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2}\left(\|u_1\|_{\mathcal{L}} + \gamma_2 \|u_2\|_{\mathcal{L}} \right) + \frac{\gamma_2\beta_1 + \beta_2}{1-\gamma_1\gamma_2} \qquad (3)$$

Likewise, $$\|e_2\|_{\mathcal{L}} \le \frac{1}{1-\gamma_1\gamma_2}\left(\|u_2\|_{\mathcal{L}} + \gamma_1 \|u_1\|_{\mathcal{L}} \right) + \frac{\gamma_1\beta_2 + \beta_1}{1-\gamma_1\gamma_2} \qquad (4)$$

Note that $$\|e\|_{\mathcal{L}} = \left\| \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right\|_{\mathcal{L}} \le \left\| \begin{bmatrix} e_1 \\ 0 \end{bmatrix} \right\|_{\mathcal{L}} + \left\| \begin{bmatrix} 0 \\ e_2 \end{bmatrix} \right\|_{\mathcal{L}} = \|e_1\|_{\mathcal{L}} + \|e_2\|_{\mathcal{L}}$$

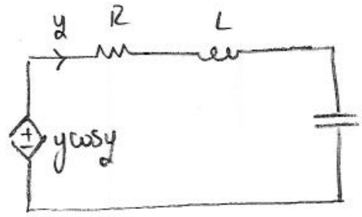
Also, $$\max\left\{ \|u_1\|_{\mathcal{L}}, \|u_2\|_{\mathcal{L}} \right\} \le \|u\|_{\mathcal{L}}$$

Hence, $$(3)\,\&\,(4) \Rightarrow \|e\|_{\mathcal{L}} \le \underbrace{\frac{2+\gamma_1+\gamma_2}{1-\gamma_1\gamma_2}}_{=:\gamma_0} \|u\|_{\mathcal{L}} + \underbrace{\frac{\gamma_2\beta_1 + \gamma_1\beta_2 + \beta_1 + \beta_2}{1-\gamma_1\gamma_2}}_{=:\beta_0}$$

Therefore, system H_3 is finite-gain $\mathcal{L}$-stable. ∎

(Proving H_4 is finite-gain $\mathcal{L}$ stable is similar.)

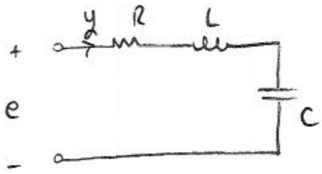
Example



Find conditions on $R, L, C > 0$ so that the heat dissipated on the resistor is finite. That is, $\int_0^\infty R\,y(t)^2\,dt < \infty$.

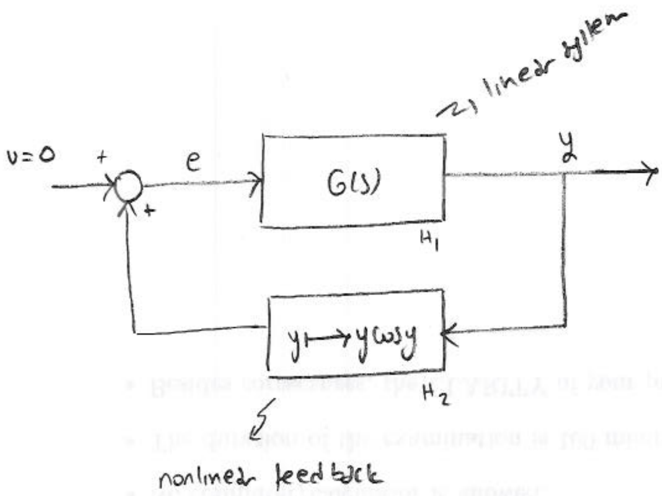
Sol'n

Consider



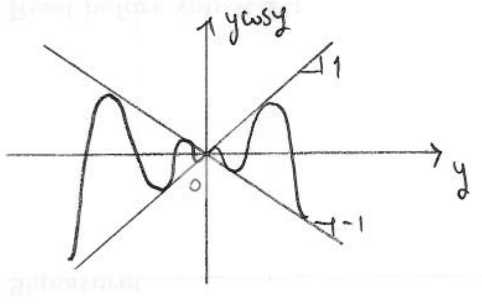
$$\Rightarrow G(s) = \frac{Y(s)}{E(s)} = \frac{1}{R + Ls + \frac{1}{Cs}} = \frac{\frac{1}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Now, the original circuit admits the following block diagram



$$\gamma_1 = \sup_\omega |G(j\omega)| = \frac{1}{R} \quad \Rightarrow \quad H_1 \text{ is finite-gain } \mathcal{L}_2\text{-stable with } \gamma_1 = \frac{1}{R}$$

$\gamma_2 = ?$



$|y\cos y| \le 1 \cdot |y|$ for all $y \in \mathbb{R}$

$\Rightarrow \gamma_2 = 1$

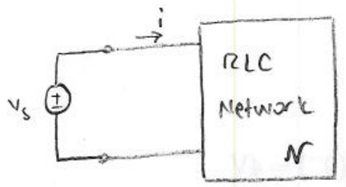
$\Rightarrow H_2$ is finite-gain $\mathcal{L}_2$-stable with $\gamma_2 = 1$

Now, for $\boxed{R > 1}$ we have $\gamma_1\gamma_2 < 1$ and the closed-loop system is $\mathcal{L}_2$-stable

$$\Rightarrow \|y\|_{\mathcal{L}_2} \le \underbrace{\gamma_0 \|u\|_{\mathcal{L}_2}}_{0} + \beta_0 \quad \Rightarrow \quad \|y\|_{\mathcal{L}_2} \le \beta_0 \quad \Rightarrow \quad \int_0^\infty R y^2\,dt = R\beta^2 < \infty.$$

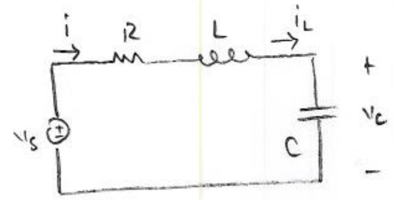
Ch. VI Passivity

Motivation: electrical networks



The circuit N is passive if the energy absorbed by the network during any time interval $[0,T]$ is no less than the energy stored (by the energy storing components) in the network over the same interval.

Example : [Series RLC circuit]



$$\text{Energy absorbed} = \int_0^T v_s(t)\, i(t)\, dt$$

$$\text{Energy stored} = \{E_c(T) + E_L(T)\} - \{E_c(0) + E_L(0)\}$$

where $E_c = \frac{1}{2} C v_c^2$ & $E_L = \frac{1}{2} L i_L^2$

Let input $u = v_s$, output $y = i$ | states $x_1 = i_L$, $x_2 = v_c$

Model: $Rx_1 + L\dot{x}_1 + x_2 = u$ & $C\dot{x}_2 = x_1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u \\ \frac{1}{C} x_1 \end{bmatrix} =: f(x,u)$$

$$y = x_1 =: h(x,u)$$

Define "the storage function" $V(x) = E_L + E_c = \frac{1}{2} L x_1^2 + \frac{1}{2} C x_2^2$

$$\Rightarrow \dot{V} = L x_1 \dot{x}_1 + C x_2 \dot{x}_2 = L x_1 \left\{ -\frac{R}{L} x_1 - \cancel{\frac{1}{L} x_2} + \frac{1}{L} u \right\} + \cancel{C x_2 \left\{ \frac{1}{C} x_1 \right\}}$$

$$= -R x_1^2 + x_1 u$$

$$= uy - Ry^2$$

Hence, $uy = \dot{V} + Ry^2 \Rightarrow \boxed{uy \geq \dot{V}} \Rightarrow \underbrace{\int_0^T u(t) y(t)\, dt}_{\text{Energy absorbed}} \geq \underbrace{V(x(T)) - V(x(0))}_{\text{Energy stored}}$

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Definition 6.3 The system (assume $f(0,0)=0$ & $h(0,0)=0$)

$$\begin{cases} \dot{x} = f(x,u) & x \in \mathbb{R}^n \\ y = h(x,u) & u, y \in \mathbb{R}^m \end{cases}$$

(Note: input & output are of same dim.)

is said to be passive if there exists a continuously differentiable pos. semidef. function $V: \mathbb{R}^n \to \mathbb{R}$ (called the storage function) such that

$$\underbrace{\langle u, y \rangle}_{u^T y} \geq \underbrace{\langle \nabla V, f \rangle}_{\dot{V}} \quad \text{for all } x \in \mathbb{R}^n \text{ \& } u \in \mathbb{R}^m$$

In particular, a passive system is called:

→ lossless if $u^T y = \dot{V}$;

→ output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some $\rho: \mathbb{R}^m \to \mathbb{R}^m$ satisfying $y^T \rho(y) > 0$ for all $y \neq 0$;

→ strictly passive if $u^T y \geq \dot{V} + \Psi(x)$ for some pos. def. $\Psi: \mathbb{R}^n \to \mathbb{R}$.

— o —

For memoryless systems $u \longrightarrow \boxed{h(\cdot)} \longrightarrow y = h(u)$ we have the following def:

Definition 6.1 The system $y = h(u)$ with $h: \mathbb{R}^m \to \mathbb{R}^m$ is said to be

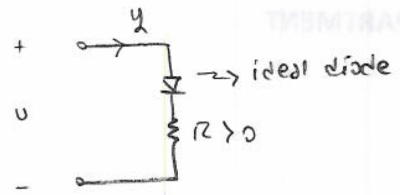
→ passive if $u^T y \geq 0$;

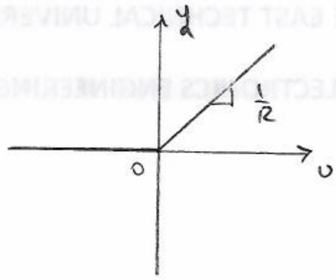
→ lossless if $u^T y = 0$;

→ output strictly passive if $u^T y \geq y^T \rho(y)$ where $y^T \rho(y) > 0$ for all $y \neq 0$.

Examples

piecewise linear resistor





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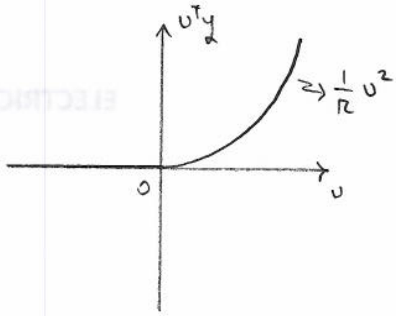
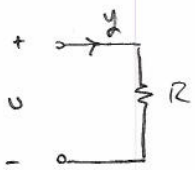


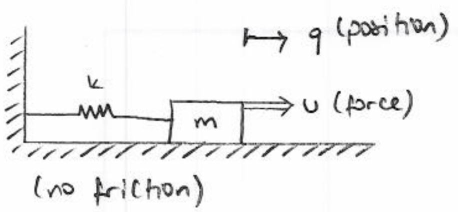
$u^T y \geq 0 \Rightarrow$ passive

LTI resistor



$u^T y = R y^2 \Rightarrow$ output strictly passive

mass-spring



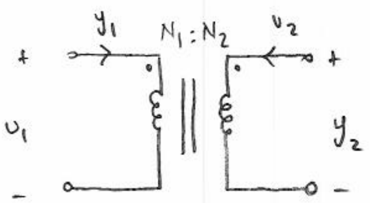
$y = \dot{q}$ (velocity)

model: $m\ddot{q} + kq = u$

$$\left.\begin{aligned} V &= \text{pot. energy} + \text{kin. energy} \\ &= \tfrac{1}{2}kq^2 + \tfrac{1}{2}m\dot{q}^2 \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{V} &= kq\dot{q} + m\dot{q}\ddot{q} \\ &= kq\dot{q} + \dot{q}\{-kq + u\} = \dot{q}u = yu \end{aligned}$$

$\Rightarrow uy = \dot{V} \Rightarrow$ lossless

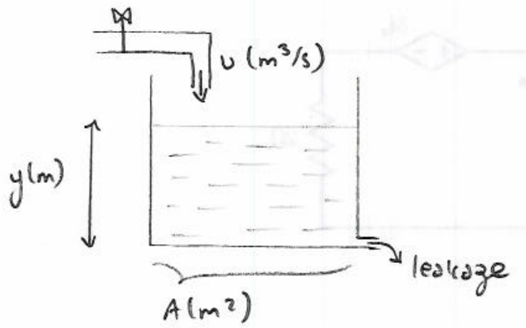
ideal transformer



$$\left.\begin{aligned} \frac{u_1}{N_1} &= \frac{y_2}{N_2} \\ N_1 y_1 + N_2 u_2 &= 0 \end{aligned}\right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{N_2}{N_1} \\ \frac{N_2}{N_1} & 0 \end{bmatrix}}_{H \text{ (skew-symmetric, } H^T = -H)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow u^T y = u^T H u = \tfrac{1}{2} u^T H u + \tfrac{1}{2} u^T H^T u = u^T \left[\frac{H + H^T}{2} \right] u = 0 \Rightarrow \text{lossless.}$$

water reservoir



model: $A\dot{y} = u - \alpha y$ → leakage term $(\alpha > 0)$

storage function (pot. energy of the water) $V = \underbrace{(Ay)}_{\text{total mass}} \cdot \underbrace{\frac{y}{2}}_{\text{average height}}$

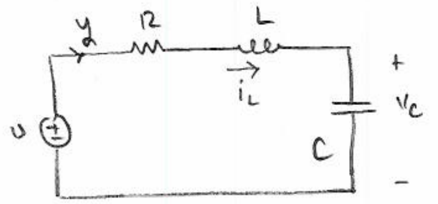
$\Rightarrow \dot{V} = yA\dot{y} = y(u - \alpha y) \Rightarrow uy = \dot{V} + \alpha y^2$

$\Rightarrow$ output strictly passive (also, strictly passive since $x = y$)

— o —

Lemma 6.5 (revisited) If the system $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ is output strictly passive with $u^T y \geq \dot{V} + \delta y^T y$ $(\delta > 0)$, then it is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain $\leq \frac{1}{\delta}$.

Example (revisited)



for $V = \frac{1}{2} L i_L^2 + \frac{1}{2} C v_C^2$

we've obtained $uy = \dot{V} + Ry^2$ (1)

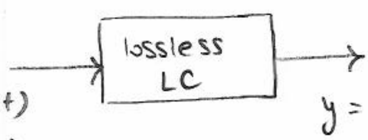
(1) => OSP & finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$-gain $\leq \frac{1}{R}$.

How about $R = 0$ case?

$R = 0 \Rightarrow uy = \dot{V} \Rightarrow$ lossless

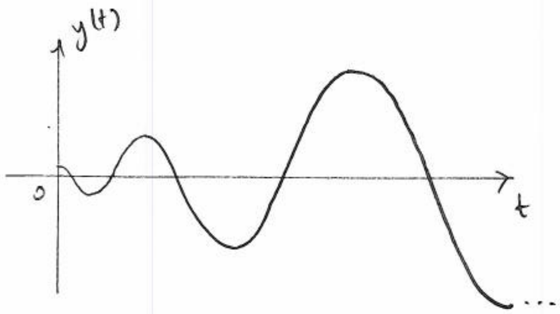
— o —

Note that the lossless case is not $\mathcal{L}_2$ stable because:



$u = \cos(wt)$

$\left(w = \frac{1}{\sqrt{LC}}\right)$



Passivity & Stability

Lemma 6.6 If the system $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ is passive with a positive definite storage function V, then the origin of the unforced system $\dot{x} = f(x,0)$ is stable.

proof passivity $\Rightarrow$ $\dot{V} \le u^T y$

$u = 0 \Rightarrow \dot{V} \le 0$

$\left.\begin{matrix} V \text{ pos. def} \\ \& \ \dot{V} \le 0 \end{matrix}\right\} \Rightarrow$ origin stable. ∎

Lemma 6.7 (a) If the system $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ is strictly passive then the origin of $\dot{x} = f(x,0)$ is asy. stable. Furthermore, if the storage function is radially unbounded then the origin is GAS.

proof strict passivity $\Rightarrow$ $u^T y \ge \langle \nabla V, f(x,u) \rangle + \Psi(x)$ with pos. def. Ψ

$u = 0 \Rightarrow \dot{V}(x) \le -\Psi(x) \quad (1)$

Claim : (1) $\Rightarrow$ V is pos. def.

Because : let $\phi(t,\eta)$ denote the solution of $\dot{x} = f(x,0)$ starting from η, that is, $\phi(0,\eta) = \eta$. Let us integrate (1):

$$V(\phi(T,\eta)) - \underbrace{V(\phi(0,\eta))}_{V(\eta)} \le -\int_0^T \Psi(\phi(t,\eta))\,dt$$

since $V \ge 0$ we have $V(\eta) \ge \int_0^T \Psi(\phi(t,\eta))\,dt \quad (2)$

Suppose V is not pos. def. Then there should exist $\eta \neq 0$ such that $V(\eta) = 0$. Then

(2) $\Rightarrow \int_0^T \Psi(\phi(t,\eta))\,dt = 0 \Rightarrow \phi(t,\eta) = 0$ for all $t \in [0,T]$ because Ψ pos. def.

$\Rightarrow 0 = \phi(t,\eta)\big|_{t=0} = \eta$. But $\eta \neq 0$. Contradiction. Hence V is pos. def.

Finally, V pos. def. & (1) $\Rightarrow$ AS ;

V pos. def & (1) & V rad. unbounded $\Rightarrow$ GAS.

∎

Question : output strict passivity $\overset{?}{\Longrightarrow}$ the origin of $\dot{x} = f(x,0)$ AS

Answer : Not necessarily. We further need "some sort of" observability property.

Recall : LTI system $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$ observable $\Longleftrightarrow$ $\left\{ y(t) = 0 \;\; \forall t \in [0,\delta] \;\Rightarrow\; x(t) = 0 \;\; \forall t \in [0,\delta] \right\}$

where $\delta > 0$ is arbitrary. This motivates:

Definition 6.5 [ZSO] The system $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ is said to be zero-state observable

if for all $\delta > 0$ and all solutions $x(t)$ of $\dot{x} = f(x,0)$ we can write

$$x(t) \in \left\{ z \in \mathbb{R}^n : h(z,0) = 0 \right\} \;\; \forall t \in [0,\delta] \iff x(t) = 0 \;\; \forall t \in [0,\delta]$$

Lemma 6.7(b) If the system $\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$ is output strictly passive and zero state observable then the origin of $\dot{x} = f(x,0)$ is asy. stable. Furthermore, if the storage function is radially unbounded then the origin is GAS.

Proof OSP $\Rightarrow$ $u^T y \geq \langle \nabla V, f(x,u) \rangle + y^T \rho(y)$ with $y^T \rho(y) > 0 \;\; \forall y \neq 0$

$u = 0 \;\Rightarrow\; \dot{V}(x) \leq -y^T \rho(y) \quad (1)$

Claim : (1) + ZSO $\Rightarrow$ V is pos. def.

Because : let $\phi(t,\eta)$ be the solution of $\dot{x} = f(x,0)$ starting from η. Integrate (1) :

$$V(\phi(T,\eta)) - V(\phi(0,\eta)) \leq -\int_0^T h^T(\phi(t,\eta),0)\, \rho\big(h(\phi(t,\eta),0)\big)\, dt$$

$$\Rightarrow V(\eta) \geq \int_0^T h^T(\phi(t,\eta),0)\, \rho\big(h(\phi(t,\eta),0)\big)\, dt \quad (2)$$

Suppose V is not pos. def. Then we can find $\eta \neq 0$ such that $V(\eta) = 0$. Then

(2) $\Rightarrow$ $h(\phi(t,\eta),0) = 0$ for all $t \in [0,T]$

by ZSO

$\Rightarrow$ $\phi(t,\eta) = 0$ for all $t \in [0,T]$

$\Rightarrow$ $\eta = 0$. Contradiction to $\eta \neq 0$. Hence V is pos. def.

Return to eqn. (1) : $\dot{V} \le -y^T \rho(y)$

$\dot{V} \equiv 0 \Rightarrow y(t) \equiv 0 \Rightarrow x(t) \equiv 0$ by ZSO. Hence, we've obtained :

(a) No solution except $x(t) \equiv 0$ can identically stay in $\{\dot{V} = 0\}$

(b) V pos. def

(1) + (a) + (b) $\Rightarrow$ origin is AS (by Corollary 4.1)

(1) + (a) + (b) + V rad. unb. $\Rightarrow$ the origin is GAS (Corollary 4.2) ▨

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_1^3 - k x_2 + u \qquad a, k > 0$$
$$y = x_2$$

storage function : $V(x) = \frac{1}{4} a x_1^4 + \frac{1}{2} x_2^2$ (pos. def. & radially unbounded)

$$\dot{V} = a x_1^3 \dot{x}_1 + x_2 \dot{x}_2 = \cancel{a x_1^3 x_2} + x_2 \{ \cancel{-a x_1^3} - k x_2 + u \} = -k x_2^2 + x_2 u$$

$\Rightarrow \dot{V} = -ky^2 + uy$ (output strictly passive)

when $u \equiv 0$

$y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0 \Rightarrow x(t) \equiv 0$

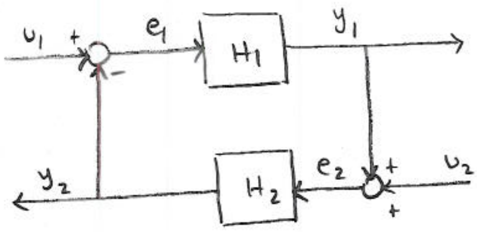
Therefore the system is zero-state observable.

stability of the origin of $\dot{x} = f(x, 0)$

OSP + ZSO + V rad. unb. $\Rightarrow$ GAS .

Analysis of Feedback systems via Passivity

Feedback connection of two systems:



Each system H_1 & H_2 is represented

either by $\begin{cases} \dot{x}_i = f_i(x_i, e_i) \\ y_i = h_i(x_i, e_i) \end{cases}$ (dynamic)

or by $y_i = h_i(e_i)$ (memoryless)

Assume: The feedback connection is well-defined. That is, for all inputs u_1 & u_2 the solution uniquely exist. Also, $f_i(0,0) = 0$ & $h_i(0,0)$ $(h_i(0) = 0)$ $i = 1,2$.

Theorem 6.1 The feedback connection of two passive systems is passive.

Proof Let V_1 & V_2 be the storage functions for H_1 & H_2. (If the system i is memoryless, take $V_i = 0$.) Then

$$e_i^T y_i \geq \dot{V}_i$$

Define the overall storage function $V(x) := V_1(x_1) + V_2(x_2)$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then
$$\begin{aligned} \dot{V} = \dot{V}_1 + \dot{V}_2 &\leq e_1^T y_1 + e_2^T y_2 \\ &= (u_1 - y_2)^T y_1 + (y_1 + u_2)^T y_2 \\ &= u_1^T y_1 + u_2^T y_2 \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ \& } y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= u^T y \end{aligned}$$

$\square$

How about $\mathcal{L}_2$-stability?

<u>Lemma 6.8</u> The feedback connection of two output strictly passive systems with $e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i$, $\delta_i > 0$ is finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain no larger than $\dfrac{1}{\min\{\delta_1, \delta_2\}}$.

<u>proof</u> Let $V := V_1 + V_2$ & $\delta = \min\{\delta_1, \delta_2\}$. Then

$$
\begin{aligned}
u^T y &= e_1^T y_1 + e_2^T y_2 \\
&\geq \dot{V}_1 + \dot{V}_2 + \delta_1 y_1^T y_1 + \delta_2 y_2^T y_2 \\
&\geq \dot{V} + \delta y^T y
\end{aligned}
$$

The result then follows from Lemma 6.5.

<u>Example</u> Consider the feedback connection of

$$
H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_1^3 - b x_2 + e_1 \\ y_1 = x_2 \end{cases} \qquad \& \qquad H_2 : \; y_2 = k e_2
$$

with $a, b, k > 0$. Let $V_1(x) = \frac{a}{4} x_1^4 + \frac{1}{2} x_2^2$

$$
\Rightarrow \dot{V}_1 = a x_1^3 x_2 - a x_1^3 x_2 - b x_2^2 + x_2 e_1 = -b y_1^2 + e_1 y_1 \qquad (1)
$$

Also, $e_2 y_2 = \frac{1}{k} y_2^2$ (2)

Then (1) & (2) $\Rightarrow$

$$
\begin{aligned}
\dot{V}_1 &= -b y_1^2 - \frac{1}{k} y_2^2 + e_1 y_1 + e_2 y_2 \\
&\leq -\min\left\{b, \frac{1}{k}\right\} (y_1^2 + y_2^2) + u_1 y_1 + u_2 y_2
\end{aligned}
$$

$$
\Rightarrow \dot{V}_1 + \min\left\{b, \frac{1}{k}\right\} y^T y \leq u^T y \quad \Rightarrow \quad \mathcal{L}_2 \text{ gain} \leq \frac{1}{\min\{b, \frac{1}{k}\}}
$$

<u>Theorem 6.4</u> Let $\begin{cases} H_1 : \text{strictly passive} \\ H_2 : \text{memoryless, passive} \end{cases}$

Then the origin of the closed-loop system when $u=0$ is asymptotically stable. Furthermore, if the storage function of H_1 is radially unbounded then the origin is GAS.

<u>Proof</u> Let V_1 be the storage function for $H_1 : \begin{cases} \dot{x}_1 = f_1(x_1, e_1) \\ y_1 = h_1(x_1, e_1) \end{cases}$

Then $\dot{V}_1 \le e_1^T y_1 - \psi_1(x_1)$ for some pos. def ψ_1

Note that $e_1^T y_1 + e_2^T y = u^T y = 0$ since $u=0$.

Therefore $\dot{V}_1 \le -e_2^T y_2 - \psi_1(x_1)$

Also, H_2 passive $\Rightarrow e_2^T y_2 \ge 0 \Rightarrow \dot{V}_1(x_1) \le -\psi_1(x_1)$ (1)

From the proof of lemma 6.7 we recall : strict passivity $\Rightarrow V_1$ pos. def.

The result then follows by (1). ∎

<u>Theorem 6.3</u> Consider the feedback connection of two dynamic systems

$$H_i : \begin{cases} \dot{x}_i = f_i(x_i, e_i) \\ y_i = h_i(x_i, e_i) \end{cases} \qquad i = 1,2 \qquad (f_i(0,0) = 0 \ \& \ h_i(0,0) = 0)$$

The origin of the closed-loop system when $u=0$ is asy. stable if one of the following conditions hold

1) both systems (H_1, H_2) are strictly passive

2) both systems are output strictly passive and zero state observable

3) one of the systems is SP and the other is OSP & ZSO.

Moreover, if both storage functions (V_1, V_2) are rad. unbnd. then the origin is GAS.

proof Define $V(x) := V_1(x_1) + V_2(x_2)$.

From the proof of lemma 6.7 we know: V_1, V_2 pos. def. w.r.t. x_1, x_2.

Hence $V(x)$ is pos. def. w.r.t. $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

[case 1] $\dot{V}(x) \le \underbrace{e_1^T y_1 + e_2^T y_2}_{u^T y = 0} - \Psi_1(x_1) - \Psi_2(x_2)$, Ψ_i pos. def. w.r.t. x_i

$\Rightarrow \dot{V}(x) \le -\Psi_1(x_1) - \Psi_2(x_2) =: -\Psi(x)$, Ψ pos. def. w.r.t. x

Hence, $\dot{V}(x) \le -\Psi(x) \Rightarrow$ AS.

[case 3] $\dot{V}(x) \le -\Psi_1(x_1) - y_2^T \rho_2(y_2)$, $y_2^T \rho_2(y_2) > 0$ for all $y_2 \neq 0$

Let us study the solutions that stay identically in the set $\{\dot{V}(x) = 0\}$

$$\dot{V}(x) = 0 \Rightarrow \begin{cases} x_1 = 0 \\ y_2 = 0 \end{cases}$$

Now, $y_2 \equiv 0 \Rightarrow e_1 \equiv 0$

Then $\{e_1 \equiv 0 \ \& \ x_1 \equiv 0\} \Rightarrow y_1 \equiv 0$ (because $h_1(0,0) = 0$)

$\Rightarrow e_2 \equiv 0$

Also, $\{e_2 \equiv 0 \ \& \ y_2 \equiv 0\} \Rightarrow x_2 \equiv 0$ (by ZSO)

Therefore the only solution that can stay identically in $\{\dot{V}(x) = 0\}$ is the trivial solution $x(t) \equiv 0$. The result follows by invariance principle (Cor 4.1).

[case 2] Exercise. ∎

Example 6.9 Let

$$H_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_1^3 - k x_2 + e_1 \\ y_1 = x_2 \end{cases} \quad \& \quad H_2 : \begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = -b x_3 - x_4^3 + e_2 \\ y_2 = x_4 \end{cases}$$

$a, b, k > 0$.

Establish the GAS of the origin $x=0$ of the interconnection $\begin{cases} e_1 = -y_2 \\ e_2 = y_1 \end{cases}$

Sol'n Let $V_1 = \frac{a}{4} x_1^4 + \frac{1}{2} x_2^2$

$$\Rightarrow \dot{V}_1 = a x_1^3 \dot{x}_1 + x_2 \dot{x}_2$$

$$= a x_1^3 x_2 - a x_1^3 x_2 - k x_2^2 + x_2 e_1$$

$$= -k y_1^2 + y_1 e_1 \Rightarrow H_1 \text{ is OSP} \quad (A)$$

When $e_1 \equiv 0$ & $y_1 \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$

Hence, H_1 is ZSO. (B)

Now, let $V_2 = \frac{b}{2} x_3^2 + \frac{1}{2} x_4^2$

$$\Rightarrow \dot{V}_2 = b x_3 \dot{x}_3 + x_4 \dot{x}_4$$

$$= b x_3 x_4 - b x_3 x_4 - x_4^4 + x_4 e_2$$

$$= -y_2^4 + y_2 e_2 \Rightarrow H_2 \text{ is OSP} \quad (C)$$

When $e_2 \equiv 0$ & $y_2 \equiv 0 \Rightarrow x_4 \equiv 0 \Rightarrow \dot{x}_4 \equiv 0 \Rightarrow x_3 \equiv 0$

Hence, H_2 is ZSO. (D)

Finally, note that both V_1 & V_2 are radially unbounded. (E)

Then (A), (B), (C), (D), (E) $\Rightarrow$ the origin $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$ is GAS. ▨

Passivity-Based Control

System $\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases}$ (1)

with $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, $f(0,0) = 0$, $h(0) = 0$, f, h locally Lipschitz.

WANT: stabilize the origin $x=0$ by output feedback $u = -\phi(y)$. That is, the origin of $\dot{x} = f(x, -\phi(h(x)))$ is (asy.) stable.

Theorem 14.4 Suppose the system (1) is

→ passive with a radially unbounded pos. def. storage function, and

→ zero-state observable.

Then the origin $x=0$ can be stabilized by $u = -\phi(y)$, where $\phi : \mathbb{R}^m \to \mathbb{R}^m$ is any locally Lipschitz function satisfying $\phi(0) = 0$ and $y^T \phi(y) > 0$ for all $y \neq 0$. In particular, the origin of $\dot{x} = f(x, -\phi(h(x)))$ is GAS.

proof Let $V: \mathbb{R}^n \to \mathbb{R}$ be the storage function. Set $u = -\phi(y)$. Then

$$\dot{V} \le u^T y = -y^T \phi(y) \le 0$$

$\Rightarrow$ $\dot{V}$ is neg. semidef. & $\dot{V} = 0 \Rightarrow y = 0$

By ZSO we can write

$$y(t) \equiv 0 \Rightarrow u(t) \equiv 0 \Rightarrow x(t) \equiv 0.$$

Hence we have

$\left.\begin{array}{l} \to V \text{ pos. def. \& rad. unbounded} \\ \to \dot{V} \text{ neg. semidef.} \\ \to \dot{V} \equiv 0 \Rightarrow x(t) \equiv 0 \end{array}\right\} \Rightarrow x=0$ is GAS by invariance principle.

$\boxtimes$

Question What if the assumptions of Thm 14.4 are not directly satisfied?

Answer Then we may try to

→ choose (if possible) a different output or

→ redefine the input

· Consider a special case of the system (1)

$$\dot{x} = f(x) + G(x)u$$

Suppose : → there exists radially unb. pos. def. V such that $\langle \nabla V(x), f(x) \rangle \leq 0$

→ we are free to choose the output $y = h(x)$.

Suggestion : Choose $h(x) := G(x)^T \nabla V(x)$

Then : $\dot{V} = \langle \nabla V, f(x) + G(x)u \rangle = \langle \nabla V, f(x) \rangle + \langle \nabla V, G(x)u \rangle$

$$\leq u^T G(x)^T \nabla V(x)$$

$$= u^T h(x)$$

$$= u^T y$$

Hence, $\dot{V} \leq u^T y$ and the system $\left\{ \begin{array}{l} \dot{x} = f(x) + G(x)u \\ y = h(x) \end{array} \right.$ is passive.

If it is also ZSO then the assumptions of Thm 14.4 are satisfied.

Example
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + u$$

Let $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. Then

$$\dot{V} = x_1^3 x_2 - x_2 x_1^3 + x_2 u = x_2 u$$

Choose $y = x_2$. Then $\dot{V} = u^T y$ & the system is passive. How about ZSO?

$u \equiv 0$ & $y \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$

Hence the system is ZSO from the output $y = x_2$.

Choose, for instance, $u = -kx_2$ or $u = -b\,\text{sat}(x_2)$ with $k, b > 0$. Then the origin of the closed-loop is GAS by Thm 14.4.

<u>Question</u> : what if the system

$$\begin{cases} \dot{x} = f(x) + G(x)u \\ y = h(x) \end{cases}$$

is not passive with a pos. def. rad. unbounded storage function?

<u>Answer</u> : We can try "feedback passivation". That is, choose (if possible)

$u = \alpha(x) + \beta(x) v$ (v : the new input)

such that the new system

$$\begin{cases} \dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x) v \\ y = h(x) \end{cases}$$

satisfies the assumptions of Thm 14.4.

<u>Example</u> [m-link robot manipulator]

system: $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$

$q \in \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}^m$, $M, C, D \in \mathbb{R}^{m \times m}$

$M = M^T \geq \varepsilon I$, $\dot{M} - 2C$ skew-symmetric, $D = D^T \geq 0$

WANT: stabilize the point $\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} q_r \\ 0 \end{bmatrix}$ q_r: fixed reference position

Define the error $e = q - q_r \Rightarrow \dot{e} = \dot{q}$. (Now we want $\begin{bmatrix} e \\ \dot{e} \end{bmatrix} \to 0$)

$\Rightarrow M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$ (1)

sys. (1) is not passive (regardless of the output) with pos. def. rad unbounded storage function because $\begin{bmatrix} e \\ \dot{e} \end{bmatrix} = 0$ is not a equilibrium point (for $u=0$).

Let $u = g(q) - K_p e + v$ with $K_p \in \mathbb{R}^{m \times m}$ & $K_p = K_p^T > 0$

Then we have $M\ddot{e} + C\dot{e} + D\dot{e} + K_p e = v$ (2)

Storage function?

Let $V(e,\dot e) = \frac{1}{2}\dot e^T M(q)\dot e + \frac{1}{2} e^T K_p e$

$$\Rightarrow \dot V = \dot e^T M(q)\ddot e + \frac{1}{2}\dot e^T \dot M(q)\dot e + e^T K_p \dot e$$

$$= \dot e^T\{-C\dot e - D\dot e - \cancel{K_p e} + v\} + \frac{1}{2}\dot e^T \dot M \dot e + \cancel{e^T K_p \dot e}$$

$$= \underbrace{\frac{1}{2}\dot e^T(\dot M - 2C)\dot e}_{\substack{=0 \text{ because} \\ \dot M - 2C \text{ is skew-sym}}} - \dot e^T D \dot e + \dot e^T v \le \dot e^T v \quad \text{(recall: } D \text{ is pos. semidef.)}$$

$\Rightarrow \dot V \le \dot e^T v$

choose $y = \dot e$. Then Sys. (2) is passive with pos. def. rad. unb. V.

How about ZSO?

$v \equiv 0$ & $y \equiv 0 \Rightarrow \dot e \equiv 0 \Rightarrow \ddot e \equiv 0 \Rightarrow e \equiv 0 \Rightarrow$ Sys. (2) is ZSO.

Choose $v = -\phi(\dot e)$ with $\phi(0) = 0$, $\dot e^T \phi(\dot e) > 0$ for $\dot e \ne 0$.

Then the origin $\begin{bmatrix} e \\ \dot e \end{bmatrix} = 0$ is GAS by Thm 14.4.

Now, the original input reads: $u = g(q) - K_p e - \phi(\dot e)$

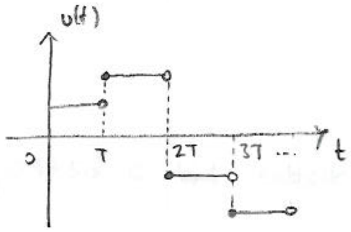
$$= g(q) - K_p(q - q_r) - \phi(\dot q)$$

Special case: $u = g(q) - K_p(q - q_r) - K_d \dot q$ with $K_d \in \mathbb{R}^{m\times m}$, $K_d = K_d^T > 0$.

DISCRETE-TIME SYSTEMS

Continuous-time LTI system: $\dot{x} = \tilde{A}x + \tilde{B}u$

WANT: control by piecewise constant input $u(t)$



$u(t) = u(kT)$ for all $t \in [kT, (k+1)T)$

where $T > 0$ is fixed

Solution $x(t) = ?$

$$x(t) = e^{\tilde{A}(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{\tilde{A}(t-\tau)} B u(\tau) d\tau$$

$$\Rightarrow \quad x[(k+1)T] = e^{\tilde{A}T} x(kT) + \int_{kT}^{(k+1)T} e^{\tilde{A}[(k+1)T-\tau]} \tilde{B} u(kT) d\tau$$

$w = \tau - kT$

$$= e^{\tilde{A}T} x(kT) + \left[e^{\tilde{A}T} \int_0^T e^{-\tilde{A}w} \tilde{B} dw \right] u(kT)$$

Define $A := e^{\tilde{A}T}$ & $B := e^{\tilde{A}T} \int_0^T e^{-\tilde{A}w} \tilde{B} dw$

Then we have the discrete-time linear system:

$$x[(k+1)T] = A x(kT) + B u(kT)$$

For simplicity take $T=1$ then $x(k+1) = Ax(k) + Bu(k)$ $k = 0, 1, 2, \dots$

Shorthand notation: $\boxed{x^+ = Ax + Bu}$ DT LTI system

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

We've obtained $\{\dot{x} = \tilde{A}x + \tilde{B}u\} \to \{x^+ = Ax + Bu\}$

How about $\{x^+ = Ax + Bu\} \xrightarrow{?} \{\dot{x} = \tilde{A}x + \tilde{B}u\}$. That is, can an arbitrary DT system always be obtained from a CT system? NO. (WHY?)

<u>Remark</u> DT linear systems can display peculiar behaviours that are absent for CT linear systems.

<u>Ex 1</u> Solutions of $x^+ = Ax$ may converge to the origin in finite-time. That is, we can have $x(0) \neq 0$ and $x(N) = 0$ for some $N < \infty$. No such thing can happen for $\dot{x} = Ax$.

<u>Ex 2</u> First order linear system $\dot{x} = ax$ does either of the following:

$$\begin{cases} |x(t)| \to \infty & \text{if } a > 0 \\ x(t) \to 0 & \text{if } a < 0 \\ x(t) = x(0) & \text{if } a = 0 \end{cases}$$

, but it never oscillates. However,

take $x^+ = -x$. Then $x(k) = [-1]^k x(0)$. That is, a first order DT linear system can display oscillations.

— o —

DT nonlinear autonomous sys.

$x^+ = f(x)$, $f: \mathbb{R}^n \to \mathbb{R}^n$

Solution:

$$x(k) = f^k(x(0)) = \underbrace{f(f(\cdots f}_{k \text{ times}}(x(0))\cdots))$$

<u>Equilibrium point?</u> x_e is an equilibrium of the system if $x(k) = x_e$ for $k = 0, 1, 2, \ldots$ is a solution. Equivalently, if

$$\boxed{f(x_e) = x_e}$$

Henceforth, without loss of generality, we will let $x = 0$ be an equilibrium point of our system, i.e., $f(0) = 0$. We will also assume f continuous.

<u>Definition</u> [Stability] Consider the system $x^+ = f(x)$. The equilibrium $x=0$ is

→ <u>stable</u> if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(k)\| < \varepsilon \quad \text{for all } k = 0, 1, 2, \dots$$

→ <u>unstable</u> if not stable.

→ <u>asy. stable</u> if stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{k\to\infty} \|x(k)\| = 0$$

→ <u>globally asy. stable</u> if stable and $\lim_{k\to\infty} \|x(k)\| = 0$ for all $x(0)$.

→ <u>globally exp. stable</u> if there exist $c>0$ and $0<\gamma<1$ such that all solutions satisfy $\|x(k)\| \le c\gamma^k \|x(0)\|$ for all $k \ge 0$

<u>Theorem</u> The origin of the linear system $x^+ = Ax$ is stable if and only if all the eigenvalues λ_i of A satisfy $|\lambda_i| \le 1$ and whenever $|\lambda_i| = 1$, the corresponding Jordan block is of size 1.

<u>Theorem</u> For $x^+ = Ax$ the following are equivalent

(1) The origin is asy. stable.

(2) $|\lambda_i| < 1$ for all eigenvalues of A.

(3) For each $Q = Q^T > 0$ there exists $P = P^T > 0$ that uniquely satisfies $A^TPA - P + Q = 0$.

(4) The origin is GES.

Let us prove $(3) \Leftrightarrow (4)$

<u>(4) ⇒ (3)</u> $x=0$ GES $\Rightarrow$ $\|x(k)\| \le c\gamma^k \|x(0)\|$ with $c>0$ & $0<\gamma<1$

Given $Q = Q^T > 0$, define $P_N := \sum_{k=0}^{N} A^{kT} Q A^k$

Claim $(P_N)_{N=1}^{\infty}$ is a Cauchy sequence. That is, for each $\varepsilon > 0$ we can find K such that for all $N, M > K$ we have $\|P_N - P_M\| < \varepsilon$.

Because Since $x(k) = A^k x(0)$, we can write for $x(0) \neq 0$

$$\frac{\|A^k x(0)\|}{\|x(0)\|} = \frac{\|x(k)\|}{\|x(0)\|} \leq c\gamma^k. \qquad (*)$$

Note that $x(0)$ can be chosen arbitrarily. Hence $(*)$ implies

$$\|A^k\| \leq c\gamma^k.$$

Now, given $\varepsilon > 0$ choose K large enough so that

$$\frac{\|Q\| c^2 \gamma^{2K}}{1-\gamma^2} < \varepsilon$$

Let N, M satisfy $N > M > K$. We can write

$$\|P_N - P_M\| = \Big\| \sum_{k=M+1}^{N} A^{kT} Q A^k \Big\|$$

$$\leq \sum_{k=K}^{\infty} \|A^{kT}\| \cdot \|Q\| \cdot \|A^k\| \qquad \|A^{kT}\| = \|A^k\|$$

$$\leq \|Q\| \cdot \sum_{k=K}^{\infty} [c\gamma^k]^2$$

$$= \frac{\|Q\| c^2 \gamma^{2K}}{1-\gamma^2} < \varepsilon \qquad \blacksquare$$

Since $(P_N)_{N=1}^{\infty}$ is a Cauchy sequence, $P := \lim_{N\to\infty} P_N$ exists.

By definition, $P = P^T > 0$. Now,

$$A^T P A - P + Q = A^T \Big(\sum_{k=0}^{\infty} A^{kT} Q A^k \Big) A - \sum_{k=0}^{\infty} A^{kT} Q A^k + Q$$

$$= \sum_{k=1}^{\infty} A^{kT} Q A^k - \Big\{ Q + \sum_{k=1}^{\infty} A^{kT} Q A^k \Big\} + Q$$

$$= 0$$

Uniqueness? Suppose not. Then there would exist $P_1 \neq P_2$ both satisfying

$$A^T P_i A - P_i + Q = 0 \qquad i = 1, 2$$

which implies

$$A^T (P_1 - P_2) A - (P_1 - P_2) = 0$$

yielding

$$A^{kT} (P_1 - P_2) A^k = P_1 - P_2$$

$$\Rightarrow P_1 - P_2 = \lim_{k \to \infty} A^{kT} (P_1 - P_2) A^k$$

$A^k \to 0$ because $\|A^k\| \le c\gamma^k$

$$= 0$$

$\Rightarrow P_1 = P_2$, contradiction.

<u>(3) ⇒ (4)</u> Suppose we have $P = P^T > 0$ & $Q = Q^T > 0$ satisfying $A^T P A - P + Q = 0$

Define $V(x) = x^T P x$. Then $\lambda_{min}(P) \|x\|^2 \le V(x) \le \lambda_{max}(P) \|x\|^2$. Given any solution $x(k) = A^k x(0)$ we can write

$$\begin{aligned} V(x(k+1)) - V(x(k)) &= x(k)^T A^T P A x(k) - x(k)^T P x(k) \\ &= x(k)^T \{A^T P A - P\} x(k) \\ &= -x(k)^T Q x(k) \\ &\le -\lambda_{min}(Q) \|x(k)\|^2 \\ &\le -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(x(k)) \end{aligned}$$

$$\Rightarrow V(x(k+1)) \le \underbrace{\left[1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}\right]}_{=: \beta < 1} V(x(k)) \qquad \Rightarrow V(x(k)) \le \beta^k V(x(0)) \qquad (**)$$

$$(**) \Rightarrow \|x(k)\|^2 \le \frac{1}{\lambda_{min}(P)} V(x(k)) \le \frac{1}{\lambda_{min}(P)} \beta^k V(x(0)) \le \frac{\lambda_{max}(P)}{\lambda_{min}(P)} \beta^k \|x(0)\|^2$$

$$\Rightarrow \|x(k)\| \le \left[\frac{\lambda_{max}(P)}{\lambda_{min}(P)}\right]^{1/2} \left[\left(1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(P)}\right)^{1/2}\right]^k \|x(0)\| \qquad \blacksquare$$

Lyapunov Theory for DT Systems

Theorem Consider the system $x^+ = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ continuous & $f(0) = 0$. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuous pos. def. function. Define $\Delta V(x) := V(f(x)) - V(x)$. Then

1) If ΔV is negative semi def. then the origin is stable.
2) If ΔV is negative def. then the origin is asy. stable.
3) If ΔV is neg. def. and V radially unbounded then the origin is GAS.

Theorem Consider the system $x^+ = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ continuous & $f(0) = 0$. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuous pos. def. function satisfying

$$c_1 \|x\|^2 \le V(x) \le c_2 \|x\|^2$$

$$\& \quad V(f(x)) - V(x) \le -c_3 \|x\|^2$$

for some positive constants c_1, c_2, c_3. Then the origin is globally exp. stable

Theorem [Invariance] Consider $x^+ = f(x)$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ continuous & $f(0) = 0$. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a continuous pos. def. function. Suppose ΔV is neg. semidef. and $\Delta V(x(k)) \equiv 0$ only when $x(k) \equiv 0$. Then the origin is asy. stable. Furthermore, if V is rad. unbounded then the origin is globally asy. stable.

Example [output coupled oscillators]

oscillator 1 : $x_1^+ = Q(x_1 + \lambda_1 C^T(y_2 - y_1))$, $y_1 = Cx_1$

oscillator 2 : $x_2^+ = Q(x_2 + \lambda_2 C^T(y_1 - y_2))$, $y_2 = Cx_2$

$Q \in \mathbb{R}^{n \times n}$, orthogonal : $Q^TQ = I$
$C \in \mathbb{R}^{m \times n}$
$\lambda_1, \lambda_2 \ge 0$, coupling strength

Assume :
- $\to$ (C, Q) observable
- $\to$ $0 < \lambda_1 + \lambda_2 \le 1$
- $\to$ $CC^T = I_{m \times m}$

Claim The oscillators synchronize. That is, $\|x_1(k) - x_2(k)\| \to 0$ as $k \to \infty$.

Define the error : $e = x_1 - x_2$

Then $e^+ = x_1^+ - x_2^+ = Q[x_1 + \lambda_1 C^T C(x_2 - x_1)] - Q[x_2 + \lambda_2 C^T C(x_1 - x_2)]$

$$= Q[x_1 - x_2 + (\lambda_1 + \lambda_2) C^T C(x_2 - x_1)]$$

$$= Q[I - (\lambda_1+\lambda_2) C^T C] e =: Ae$$

Let $V(e) = \|e\|^2 = e^T e$

$$\Rightarrow V(e^+) - V(e) = (e^+)^T e^+ - e^T e = e^T A^T A e - e^T e$$

$$= e^T \{ [I - (\lambda_1+\lambda_2) C^T C] \underbrace{Q^T Q}_{I} [I - (\lambda_1+\lambda_2) C^T C] \} e - e^T e$$

$$= e^T \{ [I - (\lambda_1+\lambda_2) C^T C]^2 - I \} e$$

$$= e^T \{ \cancel{I} - 2(\lambda_1+\lambda_2) C^T C + (\lambda_1+\lambda_2)^2 C^T C - \cancel{I} \} e$$

$$= - e^T \{ \underbrace{[2(\lambda_1+\lambda_2) - (\lambda_1+\lambda_2)^2]}_{=: \gamma > 0} C^T C \} e$$

$\Rightarrow V(e^+) - V(e) = -\gamma \|Ce\|^2$ (1) $\Rightarrow$ the origin $e = 0$ of $e^+ = Ae$ is stable.

<u>How about asy. stability?</u>

Suppose $\Delta V(e(k)) \equiv 0 \Rightarrow Ce(k) \equiv 0$ (2)

Then $e(k+1) = Q[I - (\lambda_1+\lambda_2) C^T C] e(k)$

$$= Qe(k) - (\lambda_1+\lambda_2) QC^T \underbrace{Ce(k)}_{=0}$$

$$= Qe(k) \quad (3)$$

Now, (2) & (3) imply $e(k) \equiv 0$ because (C, Q) observable. Hence:

$$\Delta V(e(k)) \equiv 0 \Rightarrow e(k) \equiv 0 \quad (4)$$

By invariance principle (1) & (4) $\Rightarrow e(k) \to 0$.

That is, $\|x_1(k) - x_2(k)\| \to 0$ & the oscillators (asymptotically) synchronize.

An Easy Way to Design Observers

<u>Observer</u>: Given the system $\begin{cases} x^+ = f(x) \\ y = h(x) \end{cases}$ with $f: \mathbb{R}^n \to \mathbb{R}^n$ & $h: \mathbb{R}^n \to \mathbb{R}^m$

the system $\hat{x}^+ = g(\hat{x}, y)$ with $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be an observer if the solution of the following coupled system in $\mathbb{R}^{2n}$

$$\begin{cases} x^+ = f(x) \\ \hat{x}^+ = g(\hat{x}, h(x)) \end{cases}$$

satisfies $\|x(k) - \hat{x}(k)\| \to 0$ as $k \to \infty$ for all initial conditions $x(0), \hat{x}(0)$.

Recall the <u>Luenberger observer</u> for linear systems

system: $x^+ = Ax$, $y = Cx$

observer: $\hat{x}^+ = A\hat{x} + L(y - C\hat{x})$

error: $e = \hat{x} - x$

error dynamics:
$$\begin{aligned} e^+ &= \hat{x}^+ - x^+ \\ &= A\hat{x} + L(y - C\hat{x}) - Ax \\ &= A(\hat{x} - x) + L(Cx - C\hat{x}) \\ &= [A - LC](\hat{x} - x) \\ &= [A - LC]e \end{aligned}$$

Now, the dynamics $e^+ = [A-LC]e$ imply that if the eigenvalues of $[A-LC]$ satisfy $|\lambda_i| < 1$ then $e(k) \to 0$ as $k \to \infty$. That is, $\|\hat{x}(k) - x(k)\| \to 0$. Hence to design an observer, choose the "observer gain" $L \in \mathbb{R}^{n \times m}$ such that the eigenvalues of $[A-LC]$ satisfy $|\lambda_i| < 1$.

<u>Question</u>: When can we choose such L?

<u>Answer</u>: When (C, A) is detectable.

Glad's Observer

$$\text{System} \begin{cases} x^+ = f(x) \\ y = h(x) \end{cases} \qquad (1)$$

Assumptions:

A1) The output y is scalar

A2) The inverse mapping $f^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ exists. (f is invertible)

A3) For each $\xi \in \mathbb{R}^n$ the following equation

$$\begin{bmatrix} h(x) \\ h(f^{-1}(x)) \\ h(f^{-2}(x)) \\ \vdots \\ h(f^{-(n-1)}(x)) \end{bmatrix} = \xi$$

has a unique solution $x \in \mathbb{R}^n$.

Under these assumptions Glad proposes the following observer structure

Observer: $\hat{x}^+ = f(\eta) \qquad (2)$

where $\eta \in \mathbb{R}^n$ solves the following set of equations

$$\left.\begin{aligned} h(\eta) &= y \\ h(f^{-1}(\eta)) &= h(f^{-1}(\hat{x})) \\ h(f^{-2}(\eta)) &= h(f^{-2}(\hat{x})) \\ &\vdots \\ h(f^{-(n-1)}(\eta)) &= h(f^{-(n-1)}(\hat{x})) \end{aligned}\right\} \quad (*)$$

Remark Note that η is a function of $\hat{x}$ & y, i.e., $\eta = \eta(\hat{x}, y)$.

Theorem System (2) is an observer for system (1). In particular, for all initial conditions, we have $\hat{x}(k) = x(k)$ for all $k \geq n$.

Note: This type of observer, where equality $\hat{x}(k) = x(k)$ is achieved in finite time, is called "deadbeat observer."

Proof of Thm. Let's use shorthand notation: $x_k = x(k)$, $\hat{x}_k = \hat{x}(k)$, $\eta_k = \eta(k)$ and drop the parantheses: $x_{k+1} = f x_k$, $y_k = h x_k$, $\hat{x}_{k+1} = f \eta_k$.

time $k=0$

$$(*) \Rightarrow h\eta_0 = y_0 = hx_0$$

time $k=1$

$$h\eta_1 = hx_1$$
$$hf^{-1}\eta_1 = hf^{-1}\hat{x}_1 = hf^{-1}f\eta_0 = h\eta_0 = hx_0 = hf^{-1}x_1$$

time $k=2$

$$h\eta_2 = hx_2$$
$$hf^{-1}\eta_2 = hf^{-1}x_2$$
$$hf^{-2}\eta_2 = hf^{-2}\hat{x}_2 = hf^{-2}f\eta_1 = hf^{-1}\eta_1 = hf^{-1}x_1 = hf^{-2}x_2$$

$\vdots$

time $k=n-1$

$$\left.\begin{aligned} h\eta_{n-1} &= hx_{n-1} \\ hf^{-1}\eta_{n-1} &= hf^{-1}x_{n-1} \\ &\vdots \\ hf^{-(n-1)}\eta_{n-1} &= hf^{-(n-1)}x_{n-1} \end{aligned}\right\} \Rightarrow \eta_{n-1} = x_{n-1} \quad \text{by } (A3)$$

$$\Rightarrow \hat{x}_n = f\eta_{n-1} = fx_{n-1} = x_n.$$

Once $\hat{x}_n = x_n$, $(*)$ guarantees that $\hat{x}_k = x_k$ for all $k \geq n$. ∎

Example [Chaotic Oscillator]

$$x_1^+ = 1 + x_2 - a x_1^2$$
$$x_2^+ = b x_1 + x_3$$
$$x_3^+ = -b x_1$$

From "Chaos from switched circuits: discrete maps" Chua et al. Proceedings of the IEEE, 1987.

$$\Rightarrow f(x) = \begin{bmatrix} 1 + x_2 - a x_1^2 \\ b x_1 + x_3 \\ -b x_1 \end{bmatrix} \Rightarrow f^{-1}(x) = \begin{bmatrix} -\frac{1}{b} x_3 \\ -1 + \frac{a}{b^2} x_3^2 + x_1 \\ x_2 + x_3 \end{bmatrix} \quad \text{(WHY?)}$$

For this system, suppose we can measure $y = x_3 =: h(x)$. Design an observer $\hat{x}^+ = g(\hat{x}, y)$.

Sol'n Apply Glad's method.

Solve for $\eta = [\eta_1 \ \eta_2 \ \eta_3]^T$ in $[h\eta \ \ hf^{-1}\eta \ \ hf^{-2}\eta]^T = [y \ \ hf^{-1}\hat{x} \ \ hf^{-2}\hat{x}]^T$

$f^{-2}(x) = ?$

$$f^{-2}(x) = \begin{bmatrix} \text{don't care} \\ \text{don't care} \\ -1 + \frac{a}{b^2} x_3^2 + x_1 + x_2 + x_3 \end{bmatrix} \Rightarrow hf^{-2}(x) = -1 + \frac{a}{b^2} x_3^2 + x_1 + x_2 + x_3$$

$$\Rightarrow \begin{bmatrix} h\eta \\ hf^{-1}\eta \\ hf^{-2}\eta \end{bmatrix} = \begin{bmatrix} \eta_3 \\ \eta_2 + \eta_3 \\ -1 + \frac{a}{b^2}\eta_3^2 + \eta_1 + \eta_2 + \eta_3 \end{bmatrix} = \begin{bmatrix} y \\ \hat{x}_2 + \hat{x}_3 \\ -1 + \frac{a}{b^2}\hat{x}_3^2 + \hat{x}_1 + \hat{x}_2 + \hat{x}_3 \end{bmatrix}$$

$$\left.\begin{aligned} &\Rightarrow \eta_3 = y \\ &\Rightarrow \eta_2 = \hat{x}_2 + \hat{x}_3 - y \\ &\Rightarrow \eta_1 = \hat{x}_1 + \frac{a}{b^2}\hat{x}_3^2 - \frac{a}{b^2} y^2 \end{aligned}\right\} \quad \eta(\hat{x}, y) = \begin{bmatrix} \hat{x}_1 + \frac{a}{b^2}(\hat{x}_3^2 - y^2) \\ \hat{x}_2 + \hat{x}_3 - y \\ y \end{bmatrix}$$

Finally, the observer dynamics read :

$$\hat{x}^+ = f(\eta) = \begin{bmatrix} 1+\eta_2 - a\eta_1^2 \\ b\eta_1 + \eta_3 \\ -b\eta_1 \end{bmatrix} = \begin{bmatrix} 1+\hat{x}_2+\hat{x}_3 - y - a\left[\hat{x}_1 + \frac{a}{b^2}(\hat{x}_3^2 - y^2)\right]^2 \\ b\hat{x}_1 + \frac{a}{b}(\hat{x}_3^2-y^2) + y \\ -b\hat{x}_1 - \frac{a}{b}(\hat{x}_3^2 - y^2) \end{bmatrix} =: g(\hat{x}, y)$$

$\blacksquare$

<u>Exercise</u> Verify experimentally (in MATLAB) that this observer works.