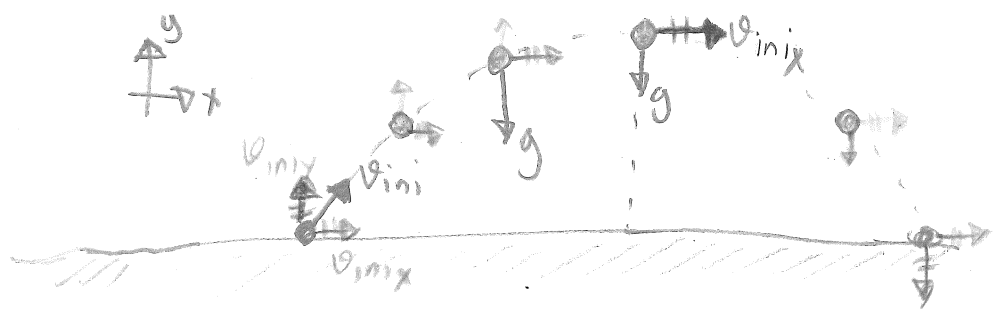
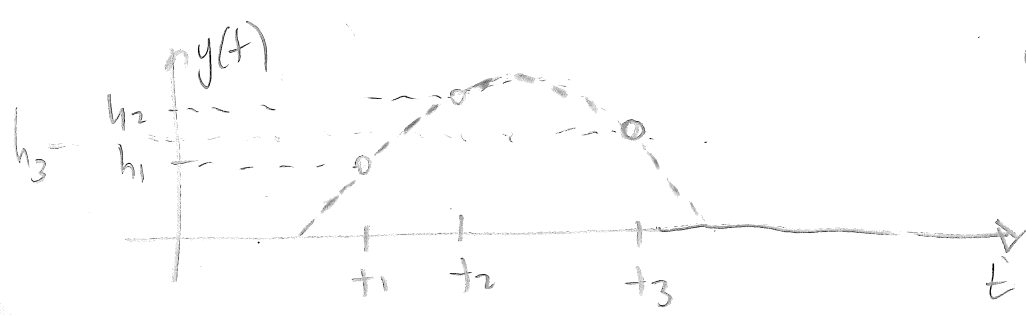


To understand (or get a feeling) about random processes, we start with deterministic signals. In signal processing, a major approach is to use the properties of the signals involved in an application to improve the performance of that application. Let's assume that we are tracking the position of a ball under projectile motion:



Projectile motion

The height of the ball above the ground at time "t", that is $y(t)$, is a quadratic function of time. Assume that, we observe the height by a radar system and get the ^{height} measurements/observations at 3 different time instants.



$$y(t_1) = h_1$$

$$y(t_2) = h_2$$

$$y(t_3) = h_3$$

The question is whether we can find the height at any other instant. For example, can we predict the time that ball will hit the ground?

Kinematics tells us that $y(t)$ is quadratic function of time, that is $y(t) = at^2 + bt + c$ $a, b, c \in \mathbb{R}$.

If we can (a, b, c) then we can identify the quadratic and then evaluate the height at any time t_x $y(t_x) = at_x^2 + bt_x + c$.

Since we have $\left. \begin{matrix} y(t_1) = h_1 \\ y(t_2) = h_2 \\ y(t_3) = h_3 \end{matrix} \right\} \rightarrow$ We can get (a, b, c)

from the measurements and we can do the height prediction as intended.

(Actually

$$y(t) = h_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + h_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + h_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)}$$

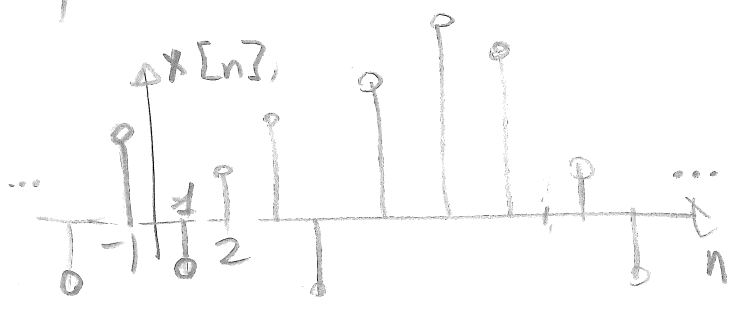


(Why?)

(Lagrange interpolator)

Forgetting the application details, what we have done is fitting the observations to a quadratic function. In this case, we are sure that the signal class is 2nd order polynomials due to basic physics.

Let's assume that we observe the samples of a band-limited function. The question is whether, we can have the value of the function at an unobserved point or not?



The sampling theorem says that if we sample $x(t)$ bandlimited to f_{max} (Hz) at $2f_{max}$ samples/sec. or higher rate (Nyquist rate or higher)

we can reconstruct the continuous-time function $x(t)$ from its samples without any error.

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc} \left(\frac{t}{T_s} - n \right) \quad \left. \vphantom{\sum} \right\} \text{Sampling Theorem}$$

where $f_s = \frac{1}{T_s} \geq 2f_{max}$

Hence, $x(t_x) = \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}\left(\frac{t_x}{T_s} - n\right)$.

(4)

↑
"prediction" for $t=t_x$
(reconstruction)
in this
case

↓
observations

Again, in this case, we can predict an unobserved time value from the observations, since we have assumed at the beginning that the function is an element of band-limited functions and collected sufficiently many samples to uniquely identify it. (In this case, we need infinite of samples collected above Nyquist rate.)

↔
In these two problems, there is a deterministic mapping between observed and unobserved quantities.

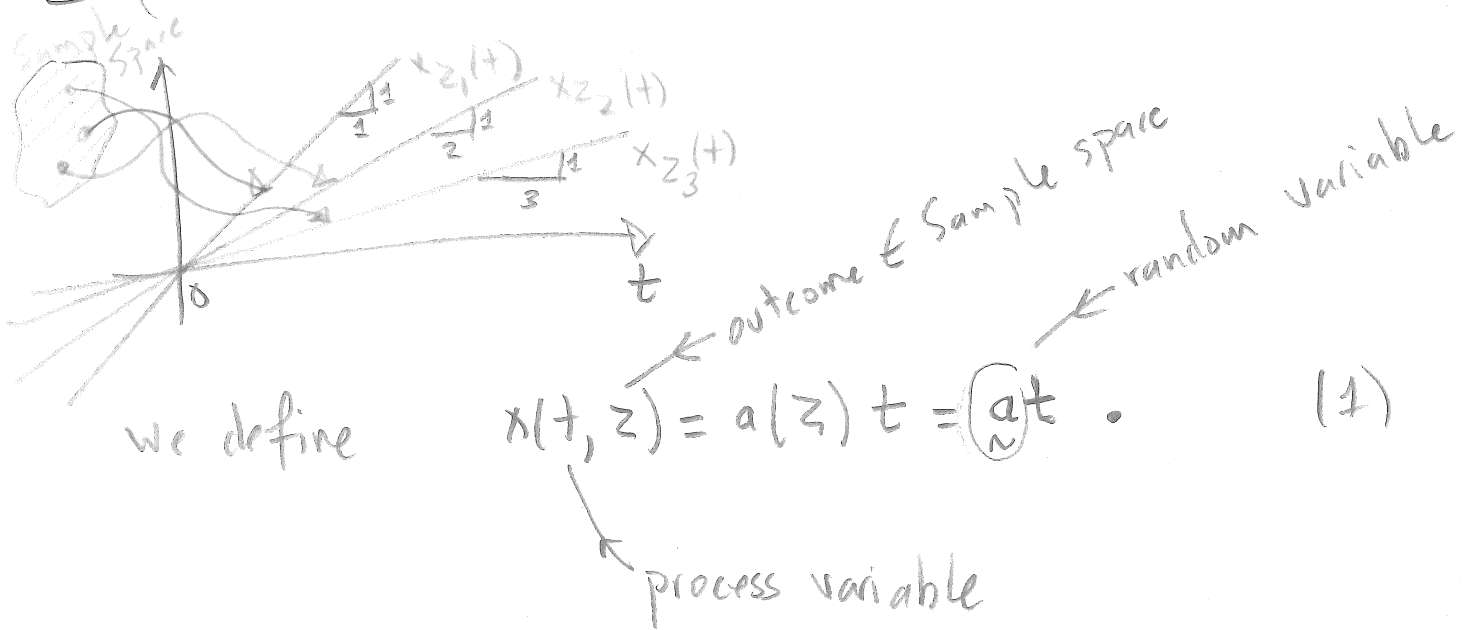
↔
In many problems of interest, the mapping observed and unobserved quantities is not deterministic, but random. For example, the weather prediction for tomorrow given today's and yesterday's temperature is a

problem of this kind. We would like to model (5) such random experiments and use these models in applications as we did for deterministic signals.

Random Processes:

We introduce the random processes with a simple example.

Ex: (Random slope signal)



we define $x(t, z) = a(z)t = \underbrace{a}_N t \quad (1)$

Here, we have functions at as the function class, that is linear functions passing through origin. In (1), "t" is a deterministic quantity (process variable), while slope comes from sample space. (To emphasize random variables further, I may use $\frac{a}{N}$ instead of A)

Let's assume that $f_A(a) \sim \text{Unif}[0,1]$, that ⁽⁶⁾
is slope is uniformly distributed in $[0,1]$. In the figure
above "3 realizations" are shown with slopes $1, \frac{1}{2}, \frac{1}{3}$.

Every realization is formed by an outcome selection
from sample space (fixing Z to Z_1 or Z_2 or Z_3).

You may consider that every realization is constructed
by selecting an element of sample space, i.e. outcome
of random experiment. Hence, when Z is fixed/selected
we have deterministic functions such as

$$\begin{aligned}x(t) &= t \\x(t) &= \frac{1}{2}t \\x(t) &= \frac{1}{3}t\end{aligned}$$

At this point, you may have feeling that since
 $A \sim \text{Unif}[0,1]$, each of these functions are equally likely;
This is correct but we need more accurate modeling.
clearly

Joint Pdf Description of Random processes:

Given r.p. $x(t)$

① 1st Order pdf Description

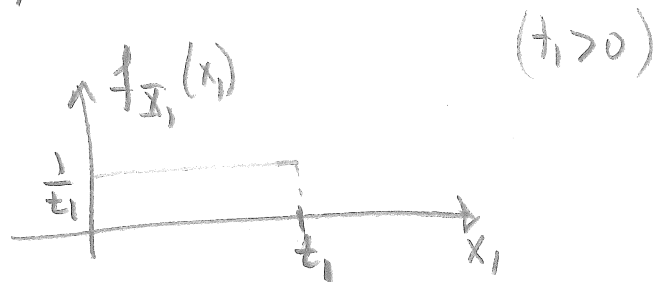
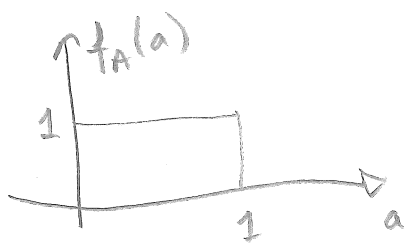
⑦

The density of the sample of the r.p. at $t=t_1$, that is

$\underbrace{X_1 = x(t_1)}_{\text{r.v.}}$ and $f_{X_1(t_1)}(x_1)$ is needed for the description.
← sampling time.

Ex: (Random slope signal)
 $X_1 = x(t_1) = a \cdot t_1$

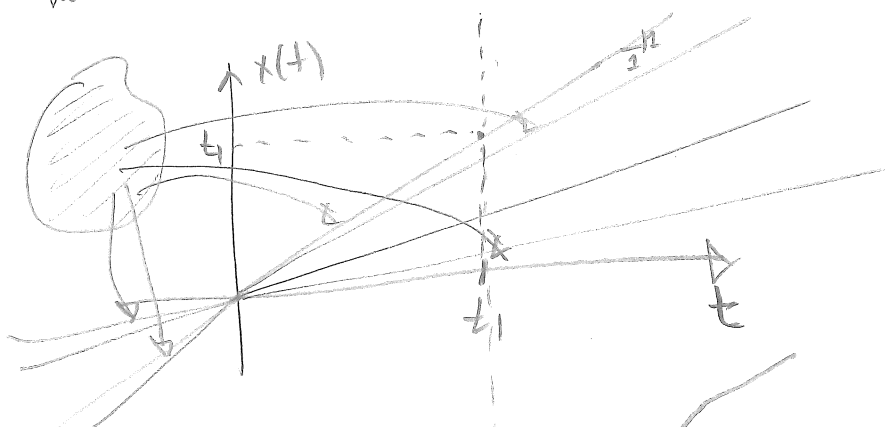
$a \sim \text{Unif}[0, 1]$



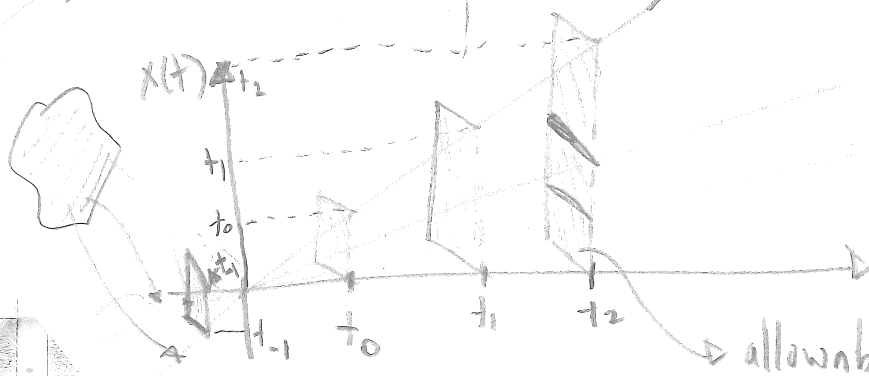
$$f_A(a) = \begin{cases} 1, & 0 \leq a \leq 1 \\ 0, & \text{other} \end{cases}$$

$$f_{X_1}(x_1) = \frac{f_A(x_1/t_1)}{t_1}$$

Let's think about the realizations for this process



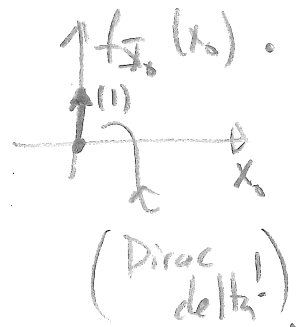
If we focus only on $t=t_1$ instant $x(t_1)$ is limited to $[0, t_1]$ values and $X_1 = x(t_1)$ is Uniformly distributed in this interval.



allowable range and pdf.

The second figure shows the p.d.f.'s for different t instants. The allowable range or support of the density function is $[0, t_x]$ where t_x is the time instance. (Note that the p.d.f for $x(t)$ at $t=0$ is

$$x_0 = x(0) = 0 \rightarrow f_{x_0}(x_0) = \delta(x)$$



Hence,

$x(t) : \text{r.p.}$
 \sim
 x

($x(t, z)$
 \sim
 more
 specifically)

→ When $t = t_1$, $x_1 = x(t_1) : \text{r.v.}$
 (fixed to t_1)

→ When $z = z_k$, $x(t, z_k) : \text{deterministic function}$
 (outcome fixed to z_k)

→ When $z = z_k$, $t = t_1$, $x(t_1, z_k) = \text{scalar}$

↑
 real/complex
 number.

2nd Order Pdf. Description

9

Given r.p. $\underline{x}(t)$, the joint density of r.v.'s

$$\left. \begin{aligned} X_1 &= \underline{x}(t_1) \\ X_2 &= \underline{x}(t_2) \end{aligned} \right\}$$

that is,

$$f_{X_1, X_2}(x_1, x_2) \quad \text{or} \quad f_{\underline{x}(t_1), \underline{x}(t_2)}(x_1, x_2)$$

is ^{the} required description.

Ex: (Random Slope Signal)

$$X_1 = \underline{x}(t_1) = \underline{a} t_1$$

$$X_2 = \underline{x}(t_2) = \underline{a} t_2$$

$$\longrightarrow f_{\underline{x}(t_1), \underline{x}(t_2)}(x_1, x_2) = ?$$

$$\underline{a} \sim \text{Unif}[0, 1]$$

$$\left(\begin{array}{l} t_1 > 0 \\ t_2 > 0 \end{array} \right)$$

Let's write the joint density as

$$f_{\underline{x}(t_1), \underline{x}(t_2)}(x_1, x_2) = \underbrace{f_{\underline{x}(t_1)}(x_1)}_{\text{calculated above (1st order density)}} \cdot \underbrace{f_{\underline{x}(t_2)|\underline{x}(t_1)}(x_2|x_1)}_{(?)}$$

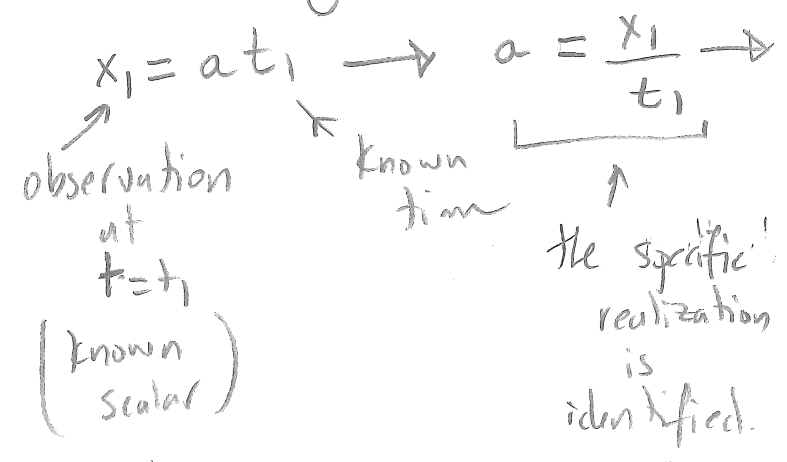
calculated above
(1st order density)

(?)

$f_{X(t_2)|X(t_1)}(x_2|x_1) = ?$

Q: Find the density of $X_2 = at_2$ given $x_1 = at_1$

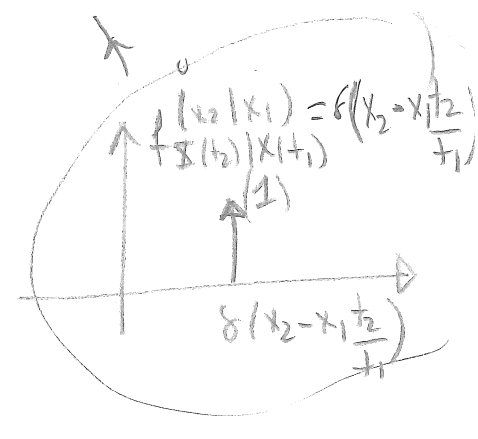
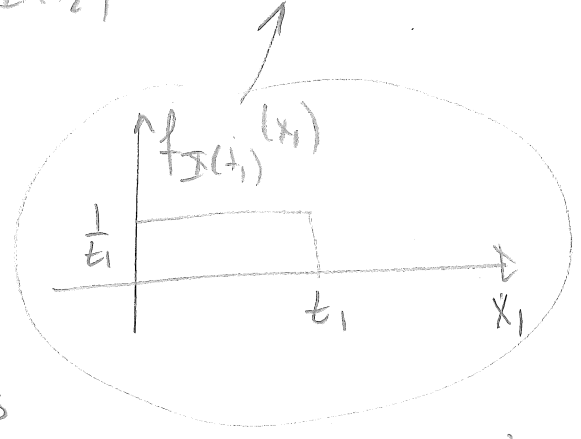
From conditioning event,



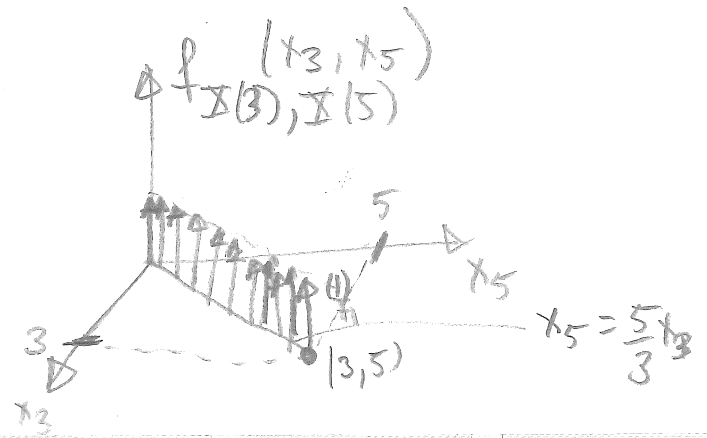
$a = \frac{x_1}{t_1}$ (deterministic) \rightarrow $f_{X(t_2)=x_2 | X(t_1)=x_1}(x_2|x_1) = \delta(x_2 - \frac{x_1 t_2}{t_1})$ (known)

Hence,

$f_{X(t_1), X(t_2)}(x_1, x_2) = f_{X(t_1)}(x_1) \cdot f_{X(t_2)|X(t_1)}(x_2|x_1) \quad \begin{matrix} t_1 > 0 \\ t_2 > 0 \end{matrix}$



Sketch for $t_1=3$
 $t_2=5$



Comments: (1) 2nd Order p.d.f description contains (11)
 the information of 1st order p.d.f description.

Since $f_{X_1(t_1)}(x_1) = \int_{-\infty}^{\infty} f_{X(t_1), X(t_2)}(x_1, x_2) dx_2$

(2) A r.p. is fully characterized with Nth order joint pdf description for $\forall N$ (or as $N \rightarrow \infty$)

$$f_{X(t_1), X(t_2), \dots, X(t_N)}(x_1, x_2, \dots, x_N) \quad \forall(t_1, t_2, \dots, t_N)$$

Ex: Calculate 1st Order pdf from 2nd Order pdf description for random slope signal

Solu $f_{X(t_1), X(t_2)}(x_1, x_2) = f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1) = \frac{f_A\left(\frac{x_1}{t_1}\right)}{t_1} \cdot \delta\left(x_2 - x_1 \frac{t_2}{t_1}\right)$

(1) $f_{X(t_1)}(x_1) = \int_{-\infty}^{\infty} f_{X(t_1), X(t_2)}(x_1, x_2) dx_2 = \frac{f_A\left(\frac{x_1}{t_1}\right)}{t_1} \int_{-\infty}^{\infty} \delta\left(x_2 - x_1 \frac{t_2}{t_1}\right) dx_2$

(2) $f_{X(t_2)}(x_2) = \int_{-\infty}^{\infty} f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 = \int_{-\infty}^{\infty} \frac{f_A\left(\frac{x_1}{t_1}\right)}{t_1} \delta\left(x_2 - x_1 \frac{t_2}{t_1}\right) dx_1$

$\delta(x_2 - x_1 \frac{t_2}{t_1}) = \delta\left(x_1 \frac{t_2}{t_1} - x_2\right)$

$\int_{-\infty}^{\infty} f_A\left(\frac{u+x_2}{t_1}, \frac{t_1}{t_2}\right) \delta(u) \cdot \frac{t_1}{t_2} du$

$= f_A\left(x_2/t_2\right) / t_2$

Note that

For the random slope example,

(12)

if $x(t)$ is observed at $t_1 \neq 0$ time instant, its value at any other time instant, say t_2 , is simply $x(t_2) = x(t_1) \cdot \frac{t_2}{t_1}$. This is expected from this simple process and in general, we can only give $f_{x(t_2)|x(t_1)}(x_2|x_1)$, i.e. the density for the value of $x(t_2)$.

Moment Descriptions for Random Processes (Partial description):

For many random processes, the process description by joint p.d.f is not easy or analytically feasible. For such processes, we use a simpler description called partial description or moment description of r.v.'s.

1st Order Moment description:

$x(t)$: r.p.

$m_x(t) = E\{x(t)\}$: mean function

Note that $m_x(t) = \int_{-\infty}^{\infty} x_1 \underbrace{f_{x(t)}(x_1)}_{\text{1st order density description}} dx_1$

← 1st order density description

2nd Order Moment Description:

$$\left. \begin{array}{l} x_1 = \tilde{x}(t_1) \\ x_2 = \tilde{x}(t_2) \end{array} \right\} \rightarrow R_x(t_1, t_2) = E\{x(t_1)x(t_2)\} : \text{Auto-correlation func.}$$

$$\rightarrow C_x(t_1, t_2) = E\{[x(t_1) - \mu_x(t_1)][x(t_2) - \mu_x(t_2)]\} : \text{Covariance function.}$$

Note that $R_x(t_1, t_2)$ and $C_x(t_1, t_2)$ can be calculated from the joint density of $\tilde{x}(t_1)$ and $\tilde{x}(t_2)$; but, in many problems we can calculate the partial descriptions without calculating the joint densities by calculating expectations.

Ex: (Random slope signal)

$$\tilde{x}(t) = \tilde{a}t, \quad \tilde{a} : \text{Unif. } [0, 1]$$

① mean function: $\mu_x(t_1) = E\{\tilde{a}t_1\} = \frac{t_1}{2}$

② correlation function: $R_x(t_1, t_2) = E\{\tilde{x}(t_1)\tilde{x}(t_2)\}$

$$= E\{\tilde{a}^2 t_1 t_2\}$$

$$= E\{\tilde{a}^2\} t_1 t_2$$

$$= \left[\int_0^1 a^2 f_a(a) da \right] t_1 t_2 = \frac{t_1 t_2}{3}$$

joint density requires calculation

density of \tilde{a} is given! easy!!

Covariance function: $C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$ (Why?)

$$= \frac{1}{3} t_1 t_2 - \frac{t_1 t_2}{4} = \frac{t_1 t_2}{12}$$

Ex: (Random phase cosine)

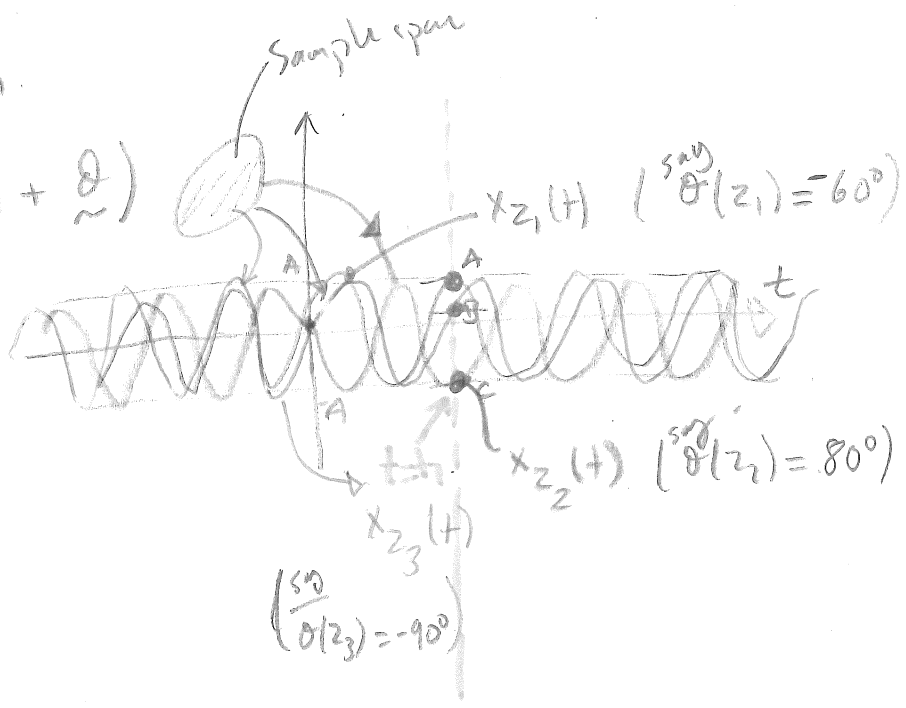
$$x(t) = A \cos(\omega t + \theta)$$

θ : Unif. $[0, 2\pi)$

Find joint p.d.f and moment descriptions of the process

a) Joint p.d.f description.

$$x_1 = x(t_1) = A \cos(\omega t_1 + \theta)$$



In the illustration given above, we observe that 3 realizations of r.p. $x(t)$ (that is $x_{z_1}(t)$, $x_{z_2}(t)$, $x_{z_3}(t)$) end up with the values of A, B, C shown in the figure. These realizations correspond to say θ of $-60^\circ, 80^\circ$ and -90° . Of course, θ is not limited to these values, but uniformly distributed in $[0, 2\pi)$.

1st Order p.d.f:

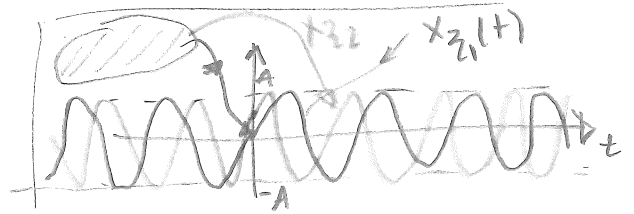
$$x_1 = x_{\sim}(t_1) = A \cos(\omega t_1 + \theta_{\sim}) \stackrel{(d)}{=} A \cos(\alpha_{\sim}) \stackrel{(d)}{=} A \cos(\phi_{\sim})$$

(15)

$\alpha_{\sim} = \omega t_1 + \theta_{\sim} \sim \text{Unif.}[\omega t_1, \omega t_1 + 2\pi)$
 $\theta_{\sim} \sim \text{Unif.}[0, 2\pi)$ $\phi_{\sim} \sim \text{Unif.}[0, 2\pi)$

Notes: (1) $a \stackrel{(d)}{=} b$ means that r.v.'s a and b have the same density/distribution.

(2) In the last equality on RHS of the equation above, we make use of the fact that $\cos(\cdot)$ is a periodic function with 2π period.

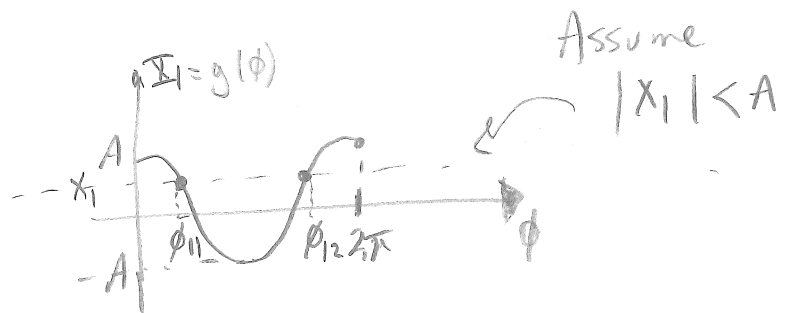


We see that $x_1 = x_{\sim}(t_1) \stackrel{(d)}{=} A \cos(\phi_{\sim})$,
 $\phi_{\sim} \sim \text{Unif.}[0, 2\pi)$

hence the density of $x_{\sim}(t_1)$ does not depend on t_1 .
 Let's find the density.

$$x_1 = A \cos(\phi_{\sim})$$

$g(\phi)$



$$f_{x_1}(x_1) = \frac{f_{\phi}(\phi_{11})}{|g'(\phi_{11})|} + \frac{f_{\phi}(\phi_{12})}{|g'(\phi_{12})|}$$

$$= \frac{1/2\pi}{|-A \sin(\phi_{11})|} + \frac{1/2\pi}{|-A \sin(\phi_{12})|}$$

Assume $|x_1| < A$

$$\phi_{1k} = \pm \cos^{-1}\left(\frac{x_1}{A}\right)$$

or $k = \{1, 2\}$

$$x_1 = A \cos(\phi_{1k})$$

$$= \frac{1}{2\pi} + \frac{1}{2\pi} \frac{1}{A\sqrt{1-(\frac{x_1}{A})^2}}$$

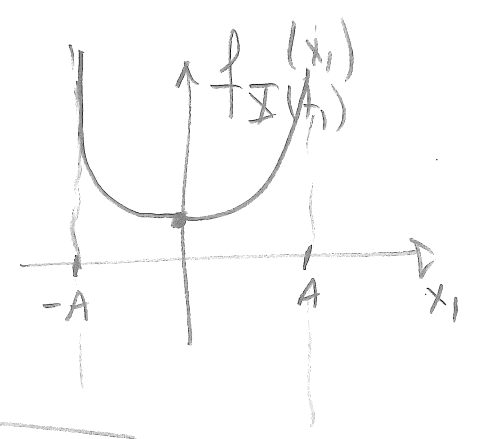
Since

$$\frac{x_1}{A} = \cos(\phi_{1k})$$

and

$$|\sin \phi_{1k}| = \sqrt{1 - \cos^2 \phi_{1k}}$$

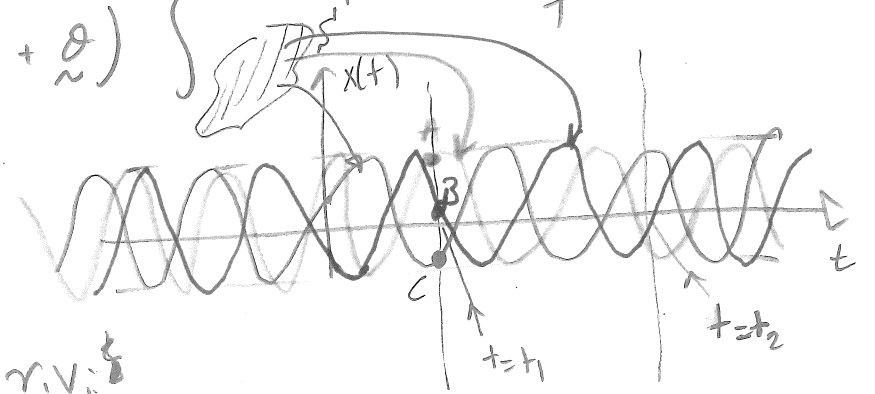
$$f_{X(t_1)}(x_1) = \begin{cases} \frac{1}{\pi\sqrt{A^2-x_1^2}}, & -A < x_1 < A \\ 0, & \text{other} \end{cases}$$



2nd Order Density:

$$\left. \begin{aligned} x_1 &= x(t_1) = A \cos(\omega t_1 + \theta) \\ x_2 &= x(t_2) = A \cos(\omega t_2 + \theta) \end{aligned} \right\} \text{2 function of } 1-\text{rv}$$

$$f_{X(t_1), X(t_2)}(x_1, x_2) = ?$$



Let's use conditional r.v.:

$$f_{X(t_1), X(t_2)}(x_1, x_2) = \underbrace{f_{X(t_1)}(x_1)}_{\text{1st order density}} \cdot \underbrace{f_{X(t_2)|X(t_1)}(x_2|x_1)}_{?}$$

(found before)

Meaning:

Given that (we know) $X(t_1) = x_1$ (say can be A, B, C in the figure for a 3 outcome experiment and we know that it is say A), what is $X(t_2)$ dist.?

$$f_{\tilde{X}(t_2)|\tilde{X}(t_1)}(x_2|x_1) = ?$$

(17)

This calculation should be done by the fact that x_1 value of $x_1 = A \cos(\omega t_1 + \theta)$ is observed. Hence, x_1 is no more a r.v., but a fixed value in $[-A, A]$

Given this value of $x_1 = \tilde{X}(t_1)$, what can you say about $\tilde{X}_2 = A \cos(\omega t_2 + \theta)$ ($t_1 \neq t_2$)?

In the random slope experiment, the observation at time t_1 , $x(t_1) = x_1 = \tilde{a} t$ was sufficient to "observe" the value of realization of a r.v. ($t \neq 0$). Let's see what we have for the random phase cosine.

$$\begin{aligned} X_2 = \tilde{X}(t_2) &= A \cos(\omega t_2 + \theta) && (\Delta t \stackrel{\Delta}{=} t_2 - t_1) \\ &= A \cos(\omega t_1 + \theta + \omega(t_2 - t_1)) \\ &= \underbrace{A \cos(\omega t_1 + \theta)}_{x_1} \cos(\omega \Delta t) - \underbrace{A \sin(\omega t_1 + \theta)}_{\pm \sqrt{A^2 - x_1^2}} \sin(\omega \Delta t) \\ &= x_1 \cos(\omega \Delta t) \pm \sqrt{A^2 - x_1^2} \sin(\omega \Delta t) \end{aligned}$$

two possible values of x_2

$$f_{\tilde{X}(t_2)|\tilde{X}(t_1)}(x_2|x_1) = \frac{1}{2} \delta(x_2 - \hat{x}_{2,1}) + \frac{1}{2} \delta(x_2 - \hat{x}_{2,2})$$

$$\begin{aligned} \hat{x}_{2,1} &= x_1 \cos(\omega \Delta t) + \sqrt{A^2 - x_1^2} \sin(\omega \Delta t) \\ \hat{x}_{2,2} &= x_1 \cos(\omega \Delta t) - \sqrt{A^2 - x_1^2} \sin(\omega \Delta t) \end{aligned}$$

Note that x_2/x_1 r.v. is a binary valued r.v., that is it takes either the value of $\hat{x}_{2,1}$ or $\hat{x}_{2,2}$. We have not calculated the probability for $\hat{x}_{2,1}$ and $\hat{x}_{2,2}$ outcomes; but, since θ r.v. at the beginning of problem is $\text{Unif}[0, 2\pi)$ and we expect them to be equally likely which is the case.

Hence,

$$f_{X(t_2), X(t_1)} = f_{X(t_1)} \cdot f_{X(t_2)|X(t_1)}$$

$$= \begin{cases} \frac{1}{2\sqrt{A^2 - x_1^2}} & |x_1| < A \\ 0 & \text{otherwise} \end{cases} \cdot \left(\frac{1}{2} \delta(x_2 - \hat{x}_{2,1}(x_1)) + \frac{1}{2} \delta(x_2 - \hat{x}_{2,2}(x_1)) \right)$$

Partial description of Random Phase Cosine:

$x(t) = A \cos(\omega t + \theta)$ → (1) $M_x(t)$: Mean function

↳ $\theta \sim \text{Unif}[0, 2\pi)$

$$M_x(t_1) = E_{\theta} \{ x(t_1) \} = \frac{1}{2\pi} \int_0^{2\pi} A \cos(\omega t_1 + \theta) d\theta = 0$$

(2) Auto-correlation func:

$$R_x(t_1, t_2) = E_{\theta} \{ x(t_1) x(t_2) \}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} A^2 \cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta) d\theta$$

↓

$$\begin{aligned}
&= \frac{A^2}{2\pi} \int_0^{2\pi} \frac{\cos(\omega(t_1+t_2)+2\theta) + \cos(\omega(t_1-t_2))}{2} d\theta \quad (19) \\
&= \frac{A^2}{2\pi} \left[\frac{\cos(\omega(t_1-t_2))}{2} \cdot 2\pi + \int_0^{2\pi} \frac{\cos(\omega(t_1+t_2)+2\theta)}{2} d\theta \right] \\
&= \frac{A^2}{2} \cos(\omega(t_1-t_2))
\end{aligned}$$

Covariance function: $C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$.

Note that ① partial description is very easy to determine, since it involves expectation calculations.

② For the random phase cosine $\mu_x(t)$ is constant (independent of time) and $R_x(t_1, t_2)$ is a function of a time difference between independent variables t_1 and t_2 .

That is $R_x(t_1, t_2) = \text{func}(t_1 - t_2)$. Hence, say if we know $R_x(5, 0)$ that gives us info for $R(6, 1) = R(7, 2) = R(5, 0)$

or more generally $R_x(t_1 + \ell, t_2 + \ell) = R_x(t_1, t_2) \quad \forall \ell$ in this example. (This observation will be important in stationarity discussions.)

Discrete Random Processes:

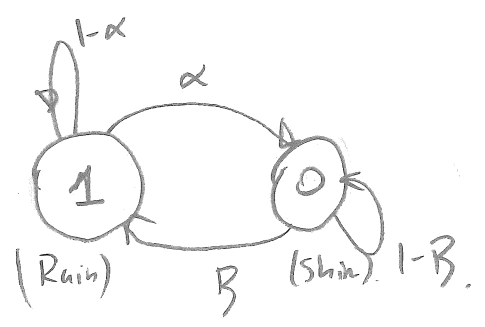
The previous descriptions of r.p.'s with continuous variables applies with no change to the discrete setting. We may consider that a discrete r.p. is formed by sampling of the continuous r.p. that is $\tilde{x}[n] = \tilde{x}(nT)$. The joint p.d.f and moment descriptions of $\tilde{x}[n]$ can be given as before. The only change is that process variable "n" is integer valued.

Some random processes are inherently discrete such as Markov chains. We can use our earlier knowledge of Markov chains for their joint p.d.f and moment descriptions.

Ex: (Rain or Shine Markov Chain)

$$x[n] = \begin{cases} 1 & \text{w.p. } P_0 \\ 0 & \text{w.p. } \overline{P_0} \end{cases}$$

\downarrow
 $1 - P_0$



$$P = \begin{bmatrix} 1-\alpha & \alpha \\ B & 1-B \end{bmatrix}$$

a) 1st order density at $n=n_0$

$$\begin{bmatrix} f_{X[n_0]}(x_{n_0}=1) & f_{X[n_0]}(x_{n_0}=0) \end{bmatrix} = [P_0 \quad \overline{P_0}] \cdot P^{n_0}$$

b) Joint density of $x[n_1]$ and $x[n_2]$.

Remember $f_{X(n_1), X(n_2)}(x_1, x_2) = \underbrace{f_{X(n_1)}(x_1)}_{\text{calculated before}} \cdot \underbrace{f_{X(n_2)|X(n_1)}(x_2|x_1)}_{\text{Conditional density}}$

Conditional density calculation:

①: $f_{X(n_2)|X(n_1)=1}(x_2|x_1=1) = \left[\begin{matrix} f_{X(n_2)}(x_2=1) & f_{X(n_2)}(x_2=0) \\ f_{X(n_2)|X(n_1)=1} & f_{X(n_2)|X(n_1)=1} \end{matrix} \right] = \left[\begin{matrix} \frac{f_{X(n_1)}(x_1=1)}{f_{X(n_1)}(x_1=1)} & \frac{f_{X(n_1)}(x_1=0)}{f_{X(n_1)}(x_1=1)} \end{matrix} \right] \cdot P^{n_2-n_1} = \left[\begin{matrix} 1 & 0 \end{matrix} \right]$

②: $f_{X(n_2)|X(n_1)=0}(x_2|x_1=0) = \left[\begin{matrix} f_{X(n_2)}(x_2=1) & f_{X(n_2)}(x_2=0) \\ f_{X(n_2)|X(n_1)=0} & f_{X(n_2)|X(n_1)=0} \end{matrix} \right] = \left[\begin{matrix} 0 & 1 \end{matrix} \right] \cdot P^{n_2-n_1}$

Example:

$$f_{X(5), X(8)}(x_5, x_8) = \left\{ \begin{matrix} \left[\begin{matrix} P_0 & \bar{P}_0 \end{matrix} \right]_1 \cdot P^5 \cdot \left[\begin{matrix} 1 & 0 \end{matrix} \right]_1 \cdot P^3, & x_5=1, x_8=1 \\ \left[\begin{matrix} P_0 & \bar{P}_0 \end{matrix} \right]_1 \cdot P^5 \cdot \left[\begin{matrix} 1 & 0 \end{matrix} \right]_2 \cdot P^3, & x_5=1, x_8=0 \\ \left[\begin{matrix} P_0 & \bar{P}_0 \end{matrix} \right]_2 \cdot P^5 \cdot \left[\begin{matrix} 0 & 1 \end{matrix} \right]_1 \cdot P^3, & x_5=0, x_8=1 \\ \left[\begin{matrix} P_0 & \bar{P}_0 \end{matrix} \right]_2 \cdot P^5 \cdot \left[\begin{matrix} 0 & 1 \end{matrix} \right]_2 \cdot P^3, & x_5=0, x_8=0 \end{matrix} \right.$$

$\left[\begin{matrix} x & y \end{matrix} \right]_1 \triangleq x$ (the 1st entry of vector)

$\left[\begin{matrix} x & y \end{matrix} \right]_2 \triangleq y$ (the 2nd entry of the vector)

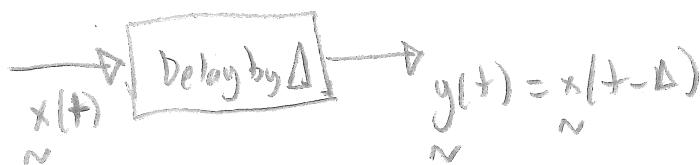
Stationary Random Processes:

The concept of steady-state in deterministic studies arise in linear system theory. In the study of LTI circuits, we know that there is DC and AC steady-state. For DC steady-state (^{50% of the} 1st order circuits), we say that it is sufficient to wait 5τ (5 time constants, $\tau = \{RC, \frac{L}{R}\}$) seconds, to reach steady-state. For AC steady-state, we assume that transients have died out and we have a response in the form $A\cos(\omega t + \phi)$.

The concept of AC-DC steady-state can be understood as the response of the circuit in a time-interval where the response does not "change" that is it ^{was} reached its "final form" and starts "repeating". (transients died)

We will extend this concept to random processes and define stationarity as follows

Stationarity:



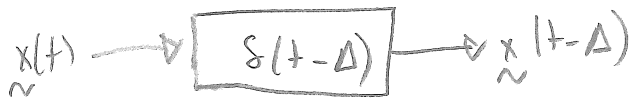
If the statistics of $\tilde{x}(t)$ and its delayed version $\tilde{y}(t) = \tilde{x}(t - \Delta)$ are the same for all Δ values, $\tilde{x}(t)$ is said to be stationary.

Note that, this is indeed a reasonable extension of the χ^2 family (23) steady-state concept to the r.p.'s.

Stationarity in joint density:

① 1st Order Stationarity:

For 1st order stationarity, we need to have



$$f_{\tilde{x}(t_1)}(x) \quad \text{and} \quad f_{\tilde{x}(t_2)}(x)$$

$t_2 = t_1 - \Delta$

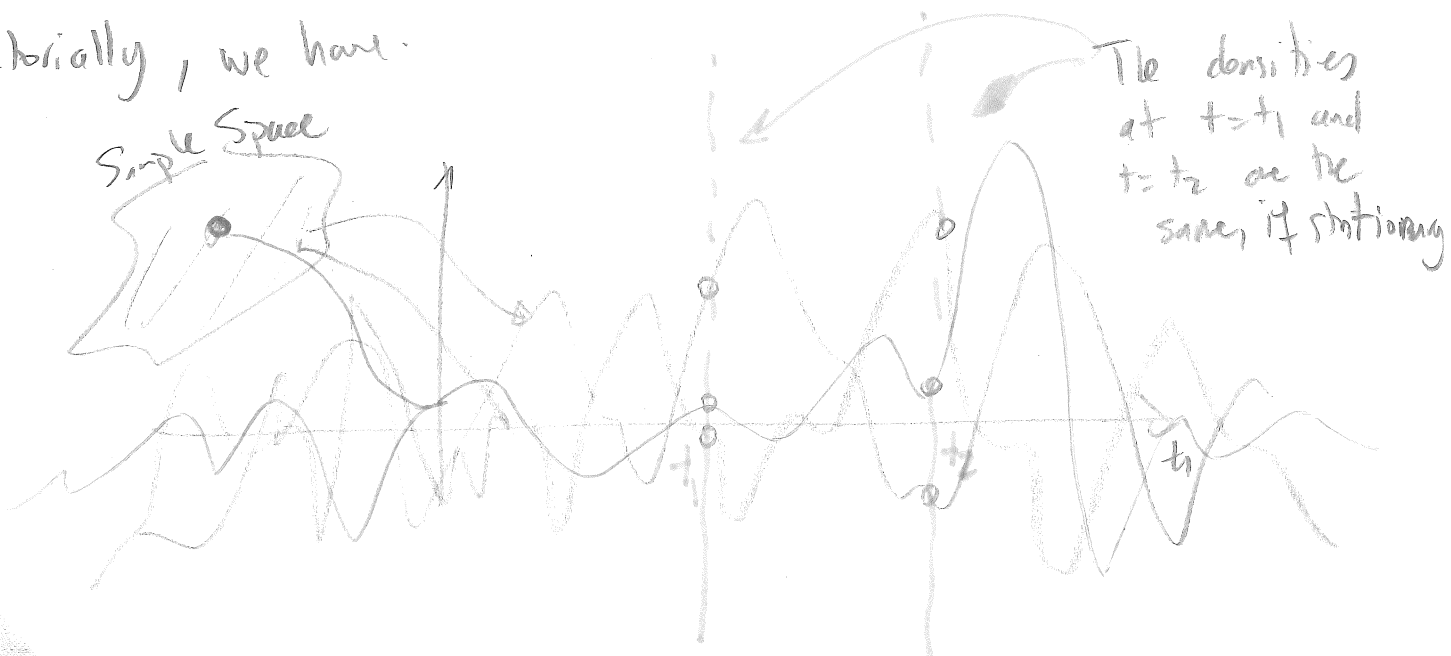
densities to be identical for all t_1 and t_2 values.

We can write this as

$$f_{\tilde{x}(t_1)}(x) = f_{\tilde{x}(t_2)}(x) \quad \forall (t_1, t_2)$$

(Note that we write the equality above we use the same process variable "x" on LHS and RHS of the equation.)

Pictorially, we have.



Ex (Random slope signal)

$$x(t) = \underset{\sim}{a} t \quad \swarrow \text{Unif } [0, 1]$$

$$\begin{aligned} x_1 = x(1) &\sim \text{Unif } [0, 1] \\ x_5 = x(5) &\sim \text{Unif } [0, 5] \\ x_{\frac{1}{2}} = x(\frac{1}{2}) &\sim \text{Unif } [0, \frac{1}{2}] \end{aligned}$$

densities at different sampling times are different

↓
Not 1st order stationary

Random phase cosine

$$x(t) = A \cos(\omega t + \underset{\sim}{\theta}) \quad \swarrow \text{Unif } [0, 2\pi]$$

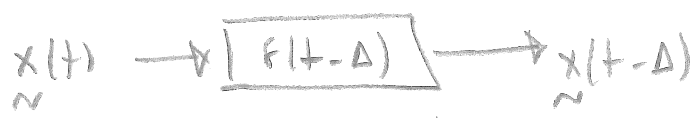
$$f_{x(t_1)}(x_1) = \begin{cases} \frac{1}{\pi \sqrt{A^2 - x_1^2}} & |x_1| < A \\ 0 & \text{otherwise} \end{cases}$$

density expression at time t_1 does not have any t_1 .

It is clear that $f_{x(t_1)}(x) = f_{x(t_2)}(x)$

↓
1st order stationary ✓

② 2nd Order Stationarity:



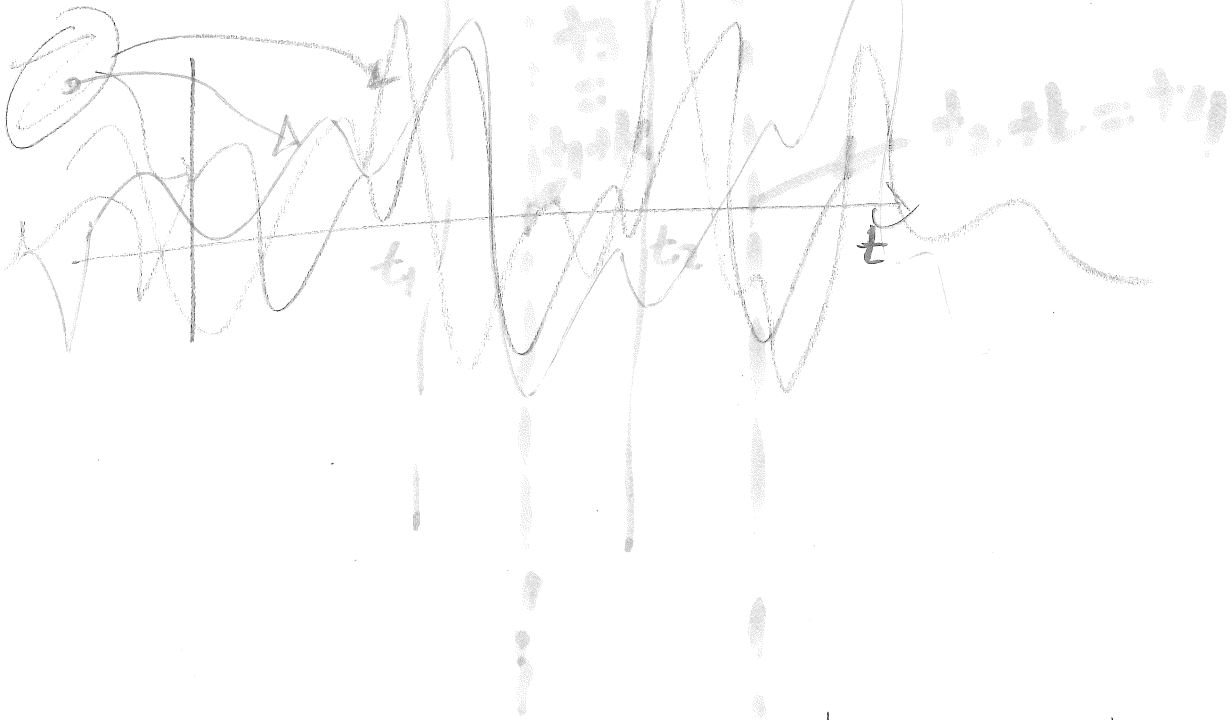
The 2nd order stationarity requires

$$f_{\mathcal{I}(t_1), \mathcal{I}(t_2)}(x_1, x_2) = f_{\mathcal{I}(t_1-\Delta), \mathcal{I}(t_2-\Delta)}(x_1, x_2)$$

for all (t_1, t_2) and Δ

Note that this requires the same joint density when both sampling instants t_1 and t_2 are shifted by the same amount Δ .

Pictorially,



$$\left. \begin{matrix} x_1 = x(t_1) \\ x_2 = x(t_2) \end{matrix} \right\} \rightarrow \left\{ \begin{matrix} x_1 = x(t_1 + k) \\ x_2 = x(t_2 + k) \end{matrix} \right.$$

(t_3 and t_4 are offset from t_1 and t_2 by the same delay value!!)

If joint densities are identical for all "k" time offset

2nd Order stationary.

Hence, for 2nd order stationarity, the joint density of $\underset{\sim}{x}(t_1)$ and $\underset{\sim}{x}(t_2)$ should not depend on t_1 and t_2 parameters, but their difference $z = t_1 - t_2$.

time separation between observations

(Since 2nd order stationarity requires valid for all Δ)

$$f_{\underset{\sim}{X}(t_1), \underset{\sim}{X}(t_2)}(x_1, x_2) = f_{\underset{\sim}{X}(t_1 - \Delta), \underset{\sim}{X}(t_2 - \Delta)}(x_1, x_2) \Big|_{\Delta = t_2} = f_{\underset{\sim}{X}(t_1 - t_2), \underset{\sim}{X}(0)}(x_1, x_0)$$

Random phase cosine is an example for 2nd order stationary r.p.s. (Check the joint density expression p. 18).

③ Nth Order Stationarity:

If $f(x_1, x_2, \dots, x_N)$
 $\mathbb{I}(t_1), \mathbb{I}(t_2), \mathbb{I}(t_3), \dots, \mathbb{I}(t_N) = f(x_1, x_2, \dots, x_N) \quad (*)$
 $\mathbb{I}(t_1 - \Delta), \mathbb{I}(t_2 - \Delta), \dots, \mathbb{I}(t_N - \Delta)$

for all (t_1, t_2, \dots, t_N) and $\Delta \rightarrow x(t)$ is Nth order stationary.

Notes: ① By marginalization operation, we can see that

If $x(t)$ is Nth order stationary, then it is also stationary of order (N-1) and all smaller orders.

② Set $\Delta = t_1$ in (*) and observe that Nth order stationary processes depend not on (t_1, t_2, \dots, t_N) ; but on $(t_2 - t_1, t_3 - t_1, t_4 - t_1, \dots, t_N - t_1)$.

③ For Nth order stationary processes, we may assume that $t_1 = 0$, by considering that we shift the first sampling time

process by t_1 seconds to the left (delay of $\Delta = -t_1$ seconds) without any loss of generality.

Ex Random phase cosine

$$x(t) = A \cos(\omega t + \theta)$$

θ : Unif. $[0, 2\pi)$

$$x(t) \rightarrow \boxed{\delta(t-\Delta)} \rightarrow A \cos(\omega t - \underbrace{\omega\Delta + \theta}_{\phi}) = y(t)$$

Let's call $\phi = \theta - \omega\Delta$ \leftarrow constant \rightarrow the density of ϕ is also uniform in $[-\omega\Delta, -\omega\Delta + 2\pi)$;

but since $\cos(x)$ is periodic with 2π , the shift of uniform density to another interval does change the process definition

$$x(t) = A \cos(\omega t + \theta) = A \cos(\omega t + \phi)$$

\uparrow Unif. $[0, 2\pi)$ \uparrow Unif. $[-\omega\Delta, -\omega\Delta + 2\pi)$.

(Think about the realizations of the process.)

Hence, Random phase cosine is N^{th} order stationary for all N .

Strict Sense Stationary: If $x(t)$ is N^{th} order stationary for all N values $\rightarrow x(t)$ is said to be strict sense stationary (SSS).

Random phase cosine is an example.

Summary:

* The stationarity in joint densities require invariance of the joint densities to the data collection starting time, say t_1 .

* For joint density stationary processes, we may assume that the first data collection time $t_1 = 0$ without any loss of generality. This leads to

a) $f_{X(t_1)}(x_1) = \text{func}(x_1) \quad \forall t_1$ for 1st order stationary

b) $f_{X(t_1), X(t_2)}(x_1, x_2) = \text{func}(x_1, x_2, t_1 - t_2) \quad \forall t_1, t_2$ for 2nd order stationary

* In many practical cases, the "time origin" of the process is randomized, that is the "starting time" becomes an arbitrary time via a randomization process. The random phase cosine is an example. The starting phase value, i.e. phase of $A \cos(\omega t + \theta)$ at $t=0$, is randomized. This can be seen as equivalent to process starting time randomization. (Since

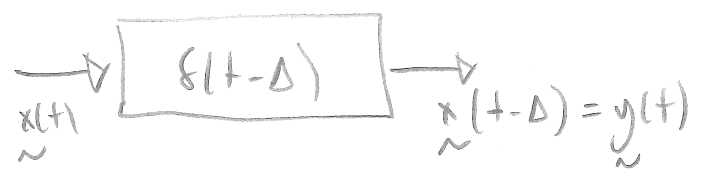
$A \cos(\omega t + \theta) = A \cos(\omega(t + \frac{\theta}{\omega}))$ period $\frac{2\pi}{\omega}$
 $\theta \in \mathcal{N} \cup \mathcal{U} \neq [0, T)$

Stationarity in moment/partial description:

Stationarity in joint pdf requires calculation of joint densities which can be practically difficult. An alternate form of stationarity can be given in terms of partial description namely moment descriptions:

① Stationarity in the mean function / Stationarity in 1st moment

$\tilde{x}(t) : r.p. \rightarrow M_x(t) = E\{x(t)\} : \text{mean function.}$



Stationarity requires the description of $\tilde{y}(t) = \tilde{x}(t - \Delta)$ to be invariant "Δ" (delay) parameter. From mean function characterization, this is equivalent to

$M_x(t) = M_y(t) = M_x(t - \Delta) \quad \forall \Delta$
mean func. at input mean func. at output

Hence, $M_x(t) = M_x(t - \Delta) = c \leftarrow \text{constant}$ for stationarity in the mean func.

(2) Stationarity in 2nd moment:

Remember that, we have

$R_x(t_1, t_2) = E\{x(t_1)x(t_2)\}$: auto-correlation func.

$C_x(t_1, t_2) = E\{[x(t_1) - M_x(t_1)][x(t_2) - M_x(t_2)]\}$: auto-covariance func.
 $= R_x(t_1, t_2) - M_x(t_1)M_x(t_2)$

Considering the stationarity requirement for 2nd order moments:

$R_x(t_1, t_2) = R_x(t_1, t_2) = R_x(t_1 - \Delta, t_2 - \Delta) \quad \forall \Delta$
 \uparrow
 $E\{x(t_1 - \Delta)x(t_2 - \Delta)\}$

The requirement of

$R_x(t_1, t_2) = R_x(t_1 - \Delta, t_2 - \Delta) \quad \forall \Delta$

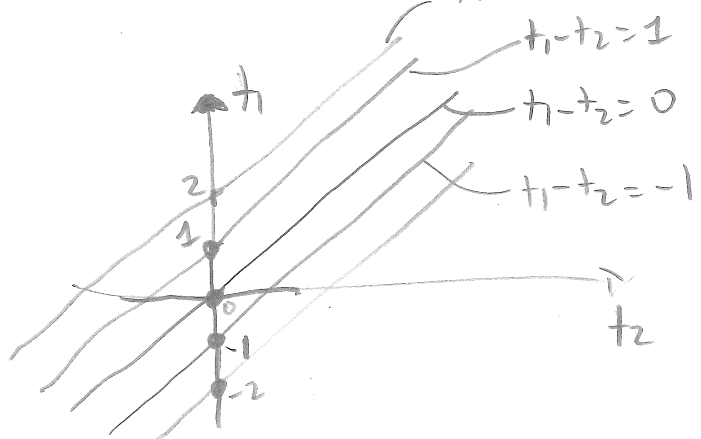
results in

$R_x(t_1, t_2) = R_x(t_1 - \Delta, t_2 - \Delta) \Big|_{\Delta = t_2} = R_x(t_1 - t_2, 0) = \text{func}(t_1 - t_2)$

that is

$R_x(t_1, t_2) = \text{func}(t_1 - t_2)$

Pictorially,



$R_x(t_1, t_2)$ func. does not vary on straight line shown!

Similarly, stationarity in covariance func. requires

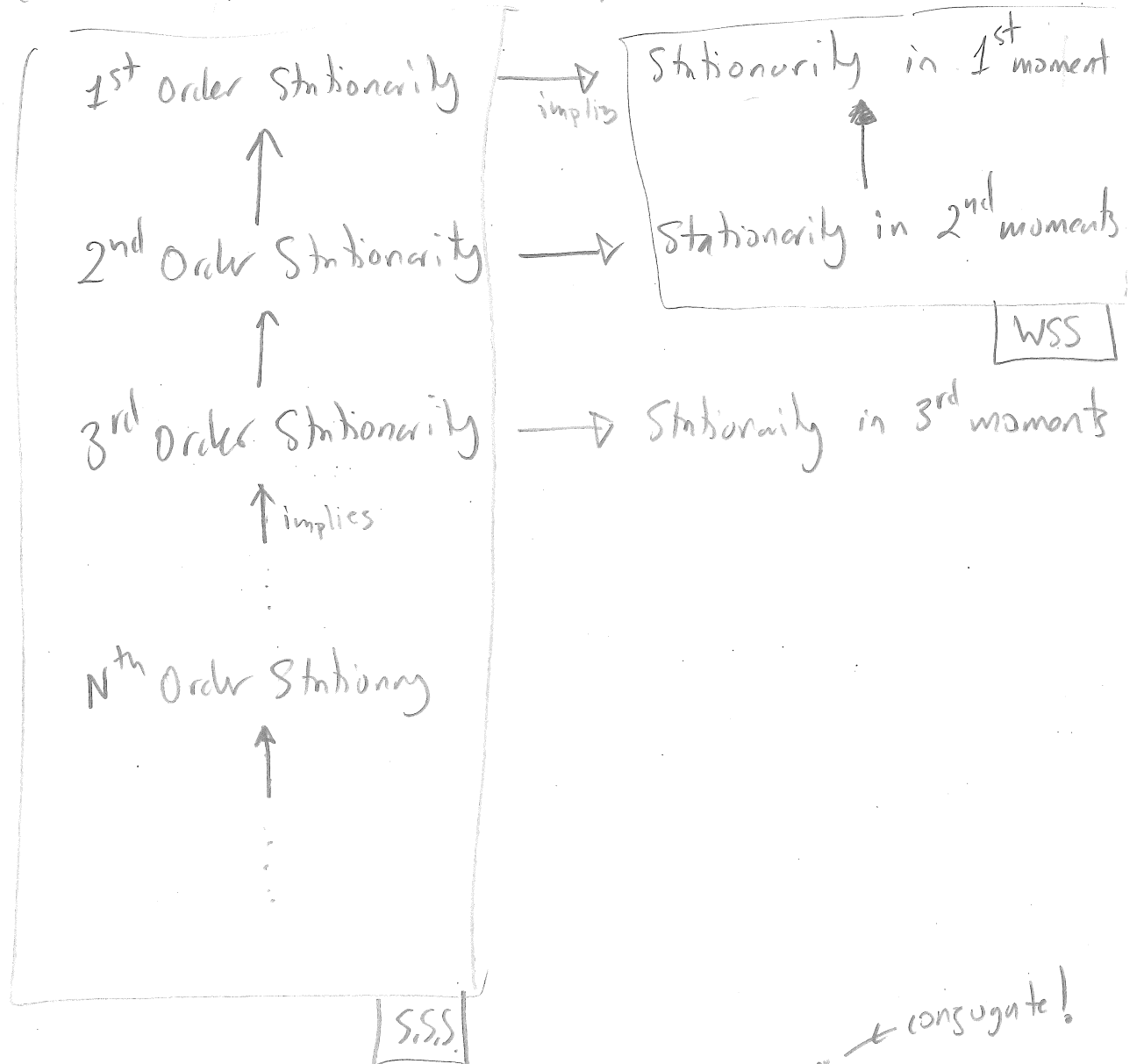
$$C_x(t_1, t_2) = \text{func}(t_1 - t_2).$$



Notes (1) Stationarity in higher order moments can also be defined; but not frequently used.

(2) Wide Sense Stationarity (WSS): A process is called wide sense stationary, if process is stationary in the mean and auto-correlation and covariance functions.

(3)



(4) If $x(t)$ is complex valued, $R_x(t_1, t_2) \triangleq E\{x(t_1)x^*(t_2)\}$ so that

$$R_x(t_0, t_0) = E\{|x(t)|^2\} \geq 0$$

← conjugate!

Ex: ① $x(t) = at$ (Random slope signal)
 $\sim \text{Unif. } [0,1]$

$$\mu_{x(t)} = E\{a\}t = \frac{t}{2} \neq c$$

Not stationary in mean func
 Not WSS.

② $x(t) = A \cos(\omega t + \theta)$ \rightarrow We know that $x(t)$ is SSS; hence, it should also be WSS. Let's verify.

$$\mu_{x(t)} = E\{A \cos(\omega t + \theta)\} = A E\{\cos(\omega t + \theta)\} = 0 \quad \checkmark$$

$$R_x(t_1, t_2) = E\{A \cos(\omega t_1 + \theta) A \cos(\omega t_2 + \theta)\} = \frac{A^2}{2} \cos(\omega(t_1 - t_2)) = \text{func}(t_1 - t_2)$$

(p.19) ↑ Dec. 30 8th min.

③ $x(t) = \sum_{k=0}^{\infty} a_k e^{j\omega_0 k t}$
 a_k : independent r.v.'s (complex valued)

$x(t)$ is a periodic waveform whose Fourier series coefficients are random.

i) $\mu_{x(t)} = E\left\{ \sum_{k=0}^{\infty} a_k e^{j\omega_0 k t} \right\} = \sum_{k=0}^{\infty} E\{a_k\} e^{j\omega_0 k t} = \frac{0}{M_a} \sum_{k=0}^{\infty} e^{j\omega_0 k t} = 0 \quad \checkmark$

ii) $R_x(t_1, t_2) = E\left\{ \left(\sum_{k_1=0}^{\infty} a_{k_1} e^{j\omega_0 k_1 t_1} \right) \left(\sum_{k_2=0}^{\infty} a_{k_2} e^{j\omega_0 k_2 t_2} \right)^* \right\}$

$$= \sum_{k_1} \sum_{k_2} E\{a_{k_1} a_{k_2}^*\} e^{j\omega_0(k_1 t_1 - k_2 t_2)}$$

$$= \sum_{k_1} E\{|a_{k_1}|^2\} \sum_{k_2} \delta[k_2 - k_1] e^{j\omega_0(k_1 t_1 - k_2 t_2)} = \sum_{k_1} E\{|a_{k_1}|^2\} e^{j\omega_0 k_1(t_1 - t_2)}$$

So, $x(t)$ is WSS

$$= e^{j\omega_0 k_1(t_1 - t_2)} \quad \checkmark$$

Gaussian Processes:

$\underline{x}(t)$ is called a Gaussian process if N^{th} order joint density of the process can be written in the form:

$$f(\underline{x}(t_1), \underline{x}(t_2), \dots, \underline{x}(t_N)) = \frac{1}{(\sqrt{2\pi})^N |\underline{C}_x|^{1/2}} \cdot e^{-\frac{1}{2}(\underline{x} - \underline{M}_x)^T \underline{C}_x^{-1} (\underline{x} - \underline{M}_x)}$$

Remember
 $\underline{x}_k = \underline{x}(t_k) \triangleq \underline{x}(t_k)$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$\underline{M}_x = E\{\underline{x}\}$
 \underline{M}_x
 mean vector

$$\begin{bmatrix} \underline{x}(t_1) \\ \underline{x}(t_2) \\ \vdots \\ \underline{x}(t_N) \end{bmatrix} = \begin{bmatrix} M_x(t_1) \\ M_x(t_2) \\ \vdots \\ M_x(t_N) \end{bmatrix}$$

$M_x(t)$: mean function

$\underline{C}_x = E\{\dots\}$
 \underline{C}_x
 covariance matrix

$$\begin{bmatrix} \underline{x}(t_1) - M_x(t_1) \\ \underline{x}(t_2) - M_x(t_2) \\ \vdots \\ \underline{x}(t_N) - M_x(t_N) \end{bmatrix} \quad [x(t_1) - M_x(t_1)] \dots [x(t_N) - M_x(t_N)]$$

Notes: ① $[M_x]_k = M_x(t_k)$ ← k^{th} element of mean vector
 ← mean function evaluated at $t=t_k$.

$$[C_x]_{k,e} = \text{Cov}(\underline{x}(t_k), \underline{x}(t_e)) = E\{(\underline{x}_k - M_x(t_k))(\underline{x}_e - M_x(t_e))\} = C_x(t_k, t_e)$$

← k^{th} row, e^{th} column of covariance matrix

← covariance evaluated at (t_k, t_e)

(2) Observe that knowing 1st and 2nd moments, i.e. mean and covariance function is sufficient to write the joint p.d.f description.

(3) Special case for N=1

$$f_{x(t_1)} = \frac{1}{\sqrt{2\pi} \sigma_{x_1}} \cdot e^{-\frac{1}{2} \frac{(x - \mu_1)^2}{\sigma_{x_1}^2}}$$

$\leftarrow E\{x(t_1)\} = \mu(t_1)$
 $\leftarrow \sigma_{x_1}^2 = \text{Var}\{x(t_1)\} = \text{Cov}(x(t_1), x(t_1))$

(4) If process is WSS and Gaussian, then

$\mu_x(t) = \text{constant} = c \leftarrow$ constant mean function

$C_x(t_k, t_e) = \text{func}(t_k - t_e)$

Covariance is ^{only} a func. of "lag" variable $z = t_k - t_e$.

\rightarrow the Gaussian process is also SSS.

(5) Gaussian processes are important since

i) Linear operations on Gaussian processes results in processes which are also Gaussian. (Easy to prove. from moment generating functions)

ii) By central limit theorem, Gaussian dist./process is "limit" process when many independent effects additively combined

Ex: $\tilde{x}(t)$: is a Gaussian process with zero mean and auto-correlation function, $R_x(t_k, t_e) = \frac{\sin(\pi(t_k - t_e))}{\pi(t_k - t_e)}$

$= \text{sinc}(z)$
 $z \triangleq t_k - t_e$
 $\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$

- i) Is $\tilde{x}(t)$ WSS?
- ii) Determine a sampling rate T s.t. $x[n] = x(nT)$ samples are independent. Write the joint density for $x[1], x[2], x[3]$ for such a sampling period.

Soln

i) $\mu_x(t) = 0$ and $R_x(t_k, t_e) = \text{sinc}\left(\frac{z}{T}\right) = \text{sinc}(z)$

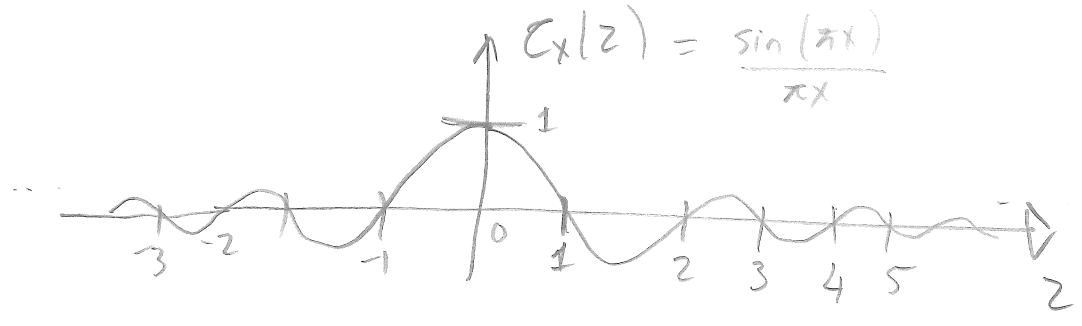
Hence, $\tilde{x}(t)$ is WSS.

ii) $x[n]$ and $x[m]$ are Gaussian distributed with

$x[n] \sim N(0, \sigma_{x_n}^2)$ and $x[m] \sim N(0, \sigma_{x_m}^2)$. (The case for $N=1$).
 $\sigma_{x_n}^2 = E\{(x(t_n) - 0)^2\} = \text{Cov}(x(t_n), x(t_n))$
 $= C_x(t_n, t_n)$
 $= R_x(t_n, t_n) + \cancel{\mu_x(t_n)^2}$
 $= \text{sinc}(0)$
 $= 1.$

Hence $x[n] \sim N(0, 1) \forall n$

$x[n], x[m]$ are jointly Gaussian dist. pair of r.v.'s,
 the r.v.'s have the covariance func.
 $C_x(t_k, t_e) = R_x(t_k, t_e) = \text{sinc}(t_k - t_e) = \text{sinc}(z)$



It is clear that $C_x(z) = 0$ $k \in \mathbb{I} \leftarrow$ integer.
 \downarrow
 $z = k$ ($k \neq 0$)

Hence, samples of the process $x(t)$ separated by 1 second (or its integer multiples) are uncorrelated, i.e. have zero covariance. Since samples are jointly Gaussian distributed, this is equivalent to samples being independent.

Let's select $T=1$, $x[n] = x[nT]$. Then,

$$\underline{x} = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \end{bmatrix} \quad \begin{matrix} \rightarrow \underline{\mu}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \rightarrow \underline{C}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$f_{\mathcal{X}}(x_1, x_2, x_3) = \frac{1}{(\sqrt{2\pi})^3} |\underline{C}_x|^{1/2} e^{-\frac{1}{2}(\underline{x} - \underline{\mu}_x)^T \underline{C}_x^{-1} (\underline{x} - \underline{\mu}_x)}$$

$$= \frac{1}{(\sqrt{2\pi})^3} \cdot 1 \cdot e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)}$$

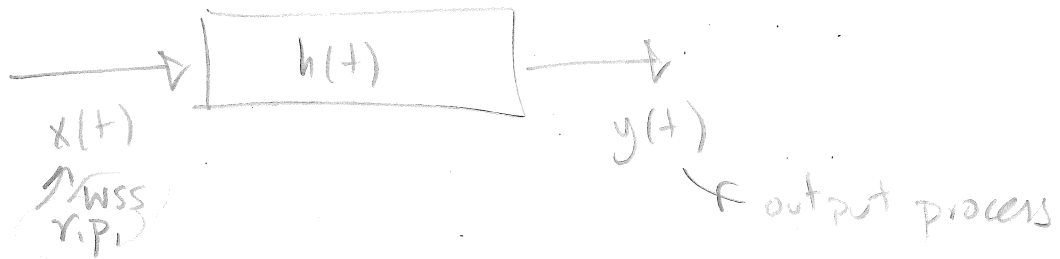
$$= \underbrace{f_{\mathcal{X}(1)}}_{N(x_1; 0, 1)} \cdot \underbrace{f_{\mathcal{X}(2)}}_{N(x_2; 0, 1)} \cdot \underbrace{f_{\mathcal{X}(3)}}_{N(x_3; 0, 1)}$$

definition of independence for r.v.s

joint density factorized into marginals

LTI Processing of WSS Processes:

We will show that LTI processing of an WSS process results in another WSS process at the output.



where

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(t') x(t-t') dt'$$

Before showing this important result, we define jointly WSS processes.

Jointly WSS processes:

$\tilde{x}(t)$ and $\tilde{y}(t)$ are jointly WSS if

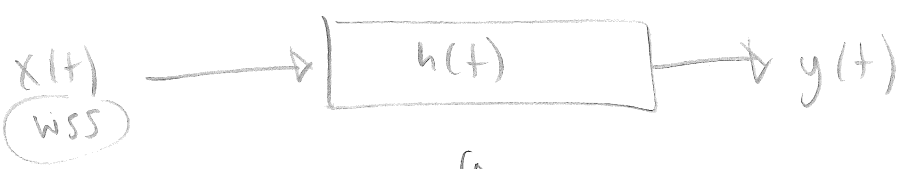
① $\tilde{x}(t)$ is WSS

② $\tilde{y}(t)$ is WSS

③ $R_{xy}(t_1, t_2) = E\{x(t_1)y(t_2)\} = \text{func}(z) \quad (z \triangleq t_1 - t_2)$.

(cross-correlation of $\tilde{x}(t)$ and $\tilde{y}(t)$)

Complex valued case: $R_{xy}(t_1, t_2) \triangleq E\{x(t_1)y^*(t_2)\}$



$y(t)$ is WSS iff
 (1) $M_y(t) = c$
 (2) $R_y(t_1, t_2) = \text{func}(t_1 - t_2)$

(1) $M_y(t) \stackrel{?}{=} \text{constant}$

$$y(t) = \int_{-\infty}^{\infty} h(t') x(t-t') dt'$$

$$E\{y(t)\} = \int_{-\infty}^{\infty} h(t') E\{x(t-t')\} dt' = M_x \int_{-\infty}^{\infty} h(t') dt'$$

$M_x = M_x \cdot H(0)$
 (constant since $x(t)$ is WSS)
 $H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$
 $H(0) = \int_{-\infty}^{\infty} h(t) dt$

Hence $M_y(t) = M_x \cdot \overbrace{H(0)}^{\text{D.C. gain of filter}} = \text{constant}$

(2) $R_y(t_1, t_2) \stackrel{?}{=} \text{func}(t_1 - t_2)$

Let's find $R_{xy}(t_1, t_2)$ first

$$\begin{aligned} R_{xy}(t_1, t_2) &= E\{x(t_1) y^*(t_2)\} \\ &= E\left\{x(t_1) \int_{-\infty}^{\infty} h(t') x^*(t_2 - t_1 - t') dt'\right\} \\ &= \int_{-\infty}^{\infty} h^*(t') E\{x(t_1) x^*(t_2 - t_1 - t')\} dt' \\ &= \int_{-\infty}^{\infty} h^*(t') r_x(t_2 - t_1 - t') dt' \end{aligned}$$

Not a func. of "t" only "z"

$$= \int_{-a}^a h^*(-t'') r_x(z-t'') dt'' \quad (t'' = -t') \quad (39)$$

$$= h^*(-z) * r_x(z)$$

hence,

$$R_{xy}(t, t-z) = h^*(-z) * r_x(z) = r_{xy}(z)$$

↑
auto-correlation
func. of $x(t)$

$$R_y(t_1, t_2) = ?$$

$$R_y(t_1, t_2) = E\{y(t_1) y^*(t_2)\}$$

$$= E\left\{ \left(\int_{-a}^a h(t') x(t-t') dt' \right) y^*(t_2) \right\}$$

$$= \int_{-a}^a h(t') E\{x(t-t') y^*(t_2)\} dt'$$

$$R_{xy}(t_1, t_2) = r_{xy}(z-t')$$

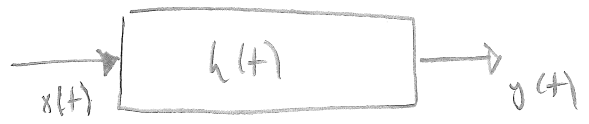
$$= \int_{-a}^a h(t') r_{xy}(z-t') dt'$$

Not a func. of t'

$$= h(z) * r_{xy}(z)$$

$$R_y(t_1, t_2) = r_y(\underbrace{t_1 - t_2}_z) = r_y(z) = h(z) * h^*(-z) * r_x(z)$$

Ex 1



$x(t)$: zero-mean WSS process with auto-correlation func. $r_x(z)$.

Find the moment characterization of the output process.

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{other} \end{cases}$$

Soln. $y(t)$ is a WSS since $x(t)$ is WSS and processing is LTI.

i) $M_y(t) = H(0) \cdot M_x = 0$

ii) $R_y(t_1, t_2) = r_y(z) = r_x(z) * h(z) * h^*(-z)$

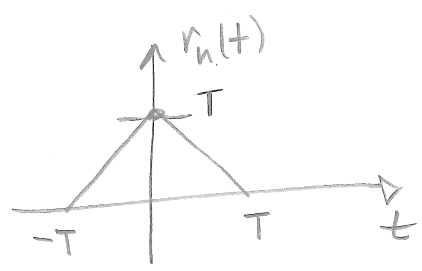
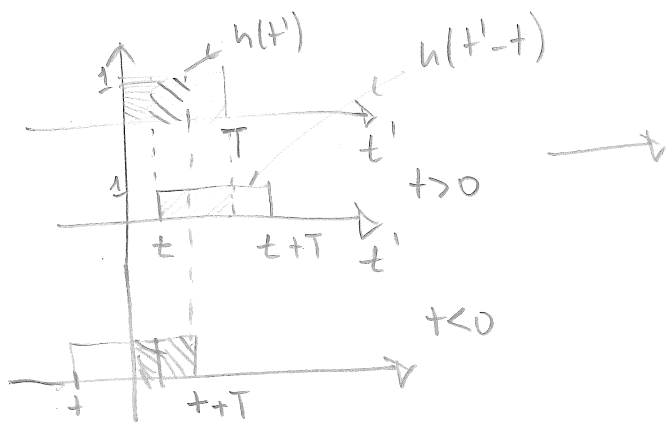
Notes:

① $h(t) * h^*(-t) \triangleq r_h(t)$ ← deterministic auto-correlation func.

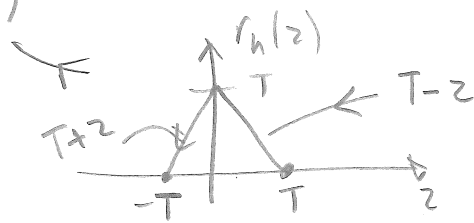
$$r_h(t) = \int_{-\infty}^{\infty} h(t') h^*(-(t-t')) dt'$$

$$r_h(t) \triangleq \int_{-\infty}^{\infty} h(t') h^*(t'-t) dt'$$

Note: $r_h(t) = r_h^*(-t)$



$$r_y(z) = r_x(z) * r_h(z)$$



(41)

(2) If $\tilde{x}(t)$ in the example is given as Gaussian process, $y(t)$ is also Gaussian with mean function $\mu_y(t) = 0$ and covariance function $R_y(t_1, t_2) = r_y(t_1 - t_2)$.

Let's assume that $r_x(z) = \delta(z)$, then $r_y(z) = \overbrace{\delta(z)}^{r_x(z)} * r_h(z) = r_h(z)$.

Then the joint density of $\underbrace{y(t)}_{Y_A}, \underbrace{y(t+\frac{T}{2})}_{Y_B}, \underbrace{y(t+\frac{3T}{4})}_{Y_C}$ is

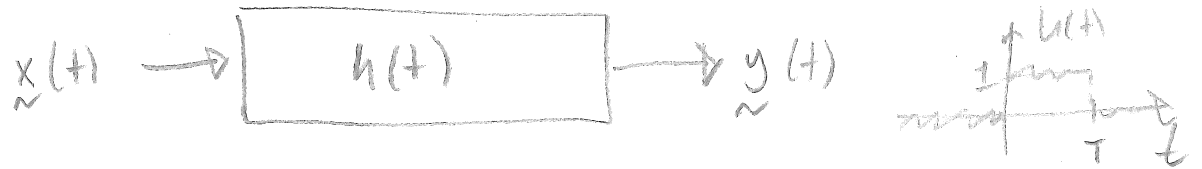
$$f_{Y_A, Y_B, Y_C}(y_a, y_b, y_c) = \frac{1}{(2\pi)^{3/2} |C_y|^{1/2}} e^{-\frac{1}{2} [y_a \ y_b \ y_c] C_y^{-1} \begin{bmatrix} y_a \\ y_b \\ y_c \end{bmatrix}}$$

where

$$C_y = \begin{bmatrix} E\{Y_A^2\} & E\{Y_A Y_B\} & E\{Y_A Y_C\} \\ E\{Y_B Y_A\} & E\{Y_B^2\} & E\{Y_B Y_C\} \\ E\{Y_C Y_A\} & E\{Y_C Y_B\} & E\{Y_C^2\} \end{bmatrix} = \begin{bmatrix} r_y(0) & r_y(-\frac{T}{2}) & r_y(-\frac{3T}{4}) \\ r_y(\frac{T}{2}) & r_y(0) & r_y(-\frac{T}{4}) \\ r_y(\frac{3T}{4}) & r_y(\frac{T}{4}) & r_y(0) \end{bmatrix}$$

$$= \begin{bmatrix} T & T/2 & T/4 \\ T/2 & T & 3T/4 \\ T/4 & 3T/4 & T \end{bmatrix}$$

We see from the last example that the processing below (42)



where $\tilde{x}(t)$ is a zero-mean ^{WSS} Gaussian process with auto-correlation $r_x(z) = \delta(z)$;

yields an output process $y(t)$ which is also WSS Gaussian process with zero mean; but auto-correlation $r_y(z)$.

Since $r_y(z)$ is ; all samples of $y(t)$ which

are separated by more than T seconds are uncorrelated; hence independent. But samples which lie in a T -second window are not independent.

Considering LTI filtering operation with $h(t)$, we note that $y(t_x)$ is nothing but sum/integral of all input in $[t_x - T, t_x]$; hence $y(t_x)$ and $y(t_y)$ do not process the same input sample $x(t)$ if $|t_y - t_x| > T$. This shows that $y(t_y)$ and $y(t_x)$ are independent from each other for this example.

The last example shows that by LTI filtering of WSS, we can "reshape" the input by changing its moment descriptions. We continue with the definition of white noise which is the simplest process that can be utilized for the synthesis of other WSS processes.

White-noise:

A process $x(t)$ is called white-noise, if

$$(1) \mu_x(t) = 0,$$

$$(2) R_x(t_1, t_2) = f(t_1) \delta(t_1 - t_2).$$

Note that (2) says that the samples of white-noise process is uncorrelated for any $t_1 \neq t_2$ pair. Note that $R_x(t_1, t_1) = C_x(t_1, t_1) = f(t_1) \leftarrow \text{VAR}\{x(t_1)\}$ is a function of t_1 . Hence, the process variance can change from t_1 to t_2 .

Special case, $x(t)$ is WSS white-noise

$$(1) \mu_x(t) = 0,$$

$$(2) R_x(t_1, t_2) = \sigma_x^2 \delta(t_1 - t_2).$$

← The process variance is constant!

Properties of Auto-correlation function of WSS processes: (44)

① Hermitian symmetry: $R_x(z) = R_x^*(-z)$

Since

$$R_x(z) = E\{x(t)x^*(t-z)\}$$

and

$$[R_x(-z)]^* = [E\{x(t)x^*(t+z)\}]^*$$

$$= E\{x(t+z)x^*(t)\} = R_x(z)$$

② Positive semi-definiteness:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y(t_1) E\{x(t_1)x^*(t_2)\} y^*(t_2) dt_1 dt_2 \geq 0 \quad \forall y(t)$$

Since

$$E\left\{ \left[\int x(t_1)y(t_1) dt_1 \right] \left[\int x^*(t_2)y^*(t_2) dt_2 \right]^* \right\}$$

← conjugate

$$= E\left\{ \left| \int x(t_1)y(t_1) dt_1 \right|^2 \right\} \geq 0 \quad \forall y(t)$$

Note: $x(t)$: WSS $\rightarrow \int \int y(t_1) R_x(t_1-t_2) y^*(t_2) dt_1 dt_2 \geq 0, \forall y(t)$

③ $R_x(0) \geq 0$ (Since $E\{x(t)x^*(t)\} \geq 0$)

④ $R_x(0) \geq |R_x(z)| \quad \forall z$

Let's prove this for the real valued processes for simplicity.

Proof: $y(t) = x(t) - x(t-\Delta) = x(t) * \underbrace{(\delta(t) - \delta(t-\Delta))}_{h(t)}$

$$r_y(z) = r_x(z) * h(z) * h^*(-z) \quad (\text{Why?})$$

$\begin{matrix} \uparrow^{(-1)} & \uparrow^{(2)} & \uparrow^{(-1)} \\ -\Delta & 0 & \Delta \end{matrix}$
 $\leftarrow r_h(z)$: deterministic auto-correlation

$$r_y(z) = r_x(z) * r_n(z) = 2r_x(z) - r_x(z+\Delta) - r_x(z-\Delta) \quad (45)$$

Since

$$r_y(0) \geq 0 \longrightarrow 2r_x(0) > r_x(-\Delta) + r_x(\Delta)$$

↓

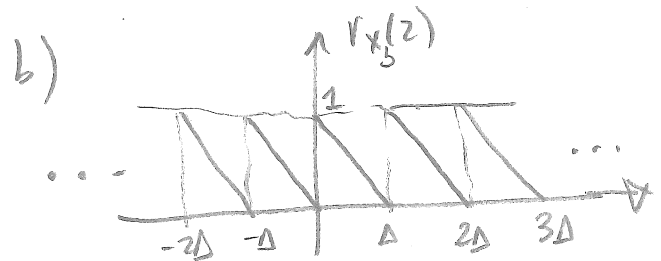
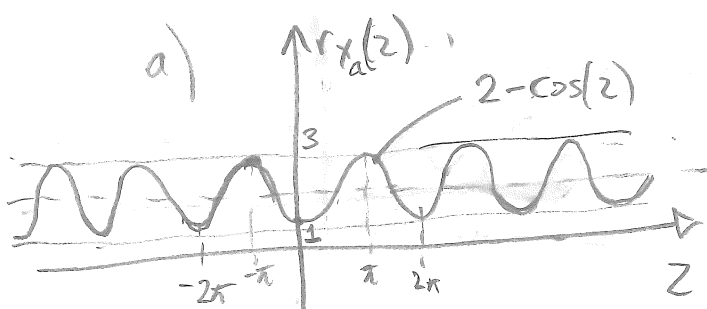
$$r_x(\Delta) = r_x(-\Delta) \quad (\text{even symmetry})$$

$$r_x(0) > r_x(\Delta), \quad \forall \Delta$$

Also need to show $r_x(0) > -r_x(\Delta)$ for the claim, try $x(t) = \delta(t) + \delta(t-\Delta)$!

Hence, from these properties, we can see some of the conditions such as symmetry, positive definiteness etc. that are automatically satisfied by valid auto-correlation functions. These properties enable us to eliminate some functions as potential auto-correlation functions.

Ex: $x(t)$ is a real valued ^{WSS} process. Determine whether each one of the following can be the auto-correlation function of this process



Ans: a) $r_{xa}(z)$ can not be an auto-cor. function, since all auto-cor. functions satisfy $r_x(0) \geq r_x(\Delta) \quad \forall \Delta$; i.e. the peak value is the value at the lag $z=0$. This required property is not satisfied by $r_x(a)$.

b) $r_x(b)$ is not a valid auto-correlation func.

Since $r_x(b) \neq r_x(-b)$, i.e. does not satisfy

the required symmetry property for valid auto-cor. functions.

Note: ① The properties given above are the necessary conditions for a valid auto-cor. func. That is (properties)

valid auto-corr function \rightarrow implies

- ① Symmetry
- ② Positive semi-definiteness
- ③ $R_x(0) \geq 0$
- ④ $R_x(0) \geq R_x(z) \forall z$.

If any one of the properties on the right side is not satisfied, say $R_x(0) = -5!$, then we are sure that this $r_x(z)$ is not a valid/true auto-correlation func.

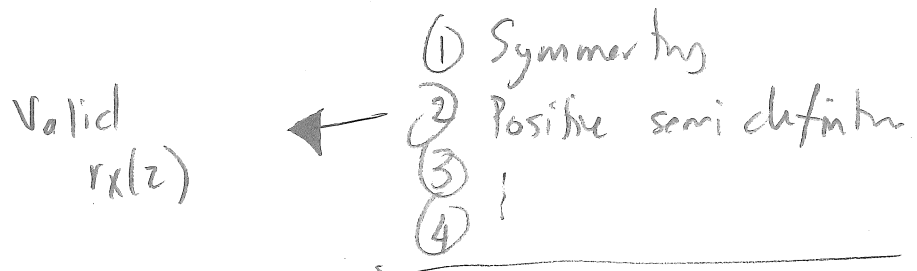
Hence, these properties are necessary conditions for a valid auto-cor. function.

② Let's assume that all necessary conditions for a valid auto-cor. func. is satisfied; can we say that $r_x(z)$ is indeed a valid auto-cor. function? The answer is No.

Right now, we only know

valid $r_x(z) \rightarrow$ ① Symmetry
② Positive semi, but not
③
④

but not, the reverse implication



Properties.

That is, we do not have any information that whether these properties are sufficient conditions for a valid auto-cor. seqn. (More on this after power spectral density) discussions.



$w[n]$: WSS discrete-time r.p.
 $h[n] = h_0 \delta[n] + h_1 \delta[n-1]$

Determine $R_x[n_1, n_2]$, $R_{wx}[n_1, n_2]$ and discuss WSS property of $x[n]$.

Soln: All previous results given for continuous valued r.p. and $h(t)$ function, is also valid in discrete time. The only change is that we do the convolutions in discrete-time!

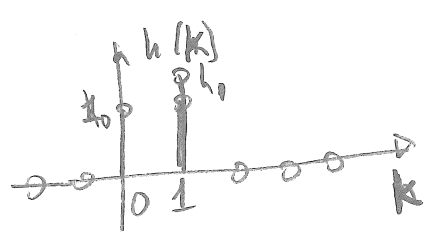
$k \triangleq n_1 - n_2$ ← lag variable

$$R_{wx}[n_1, n_2] = r_{wx}(k) = r_w[k] * h^*[-k]$$

$$R_x[n_1, n_2] = r_x(k) = r_w[k] * \underbrace{h[k] * h^*[-k]}_{r_h[k]}$$

and $x[n]$, $w[n]$ are jointly WSS.

In this example $h[n] = \delta[n] + \delta[n-1]$, that is



$$r_n[k] = h[k] * \underbrace{h^*[k]}_{h_2^*[k]} = \sum_{l=-\infty}^{\infty} h[l] h_2^*[k-l]$$

$$= \sum_{l=-\infty}^{\infty} h[l] h^*[l-k]$$

det. auto. cor.

$r_n[k]$ calculation

	$h[l] \Rightarrow$	$\begin{bmatrix} h_0 & h_1 & 0 & 0 & 0 \end{bmatrix}$	inner product $\Rightarrow r_n[0] =$	$h_0^2 + h_1^2$
	$h^*[l] \Rightarrow$	$\begin{bmatrix} h_0^* & h_1^* & 0 & 0 & 0 \dots \end{bmatrix}$		
	$h^*[l-1] \Rightarrow$	$\begin{bmatrix} 0 & h_0^* & h_1^* & 0 & \dots \end{bmatrix}$	inner product $\Rightarrow r_n[1] =$	$h_1 h_0^*$
	$h^*[l-2] \Rightarrow$	$\begin{bmatrix} 0 & 0 & h_0^* & h_1^* & \dots \end{bmatrix}$	inner product $\Rightarrow r_n[2] =$	0

Then, since $r_x[k] = r_x^*[k]$

$$r_n[k] = h_1^* h_0 \delta[k+1] + (h_0^2 + h_1^2) \delta[k] + h_1 h_0^* \delta[k-1]$$

The answer is then

$$r_{wx}[k] = r_w[k] * \underbrace{h^*[k]}_{(h_0^* \delta[k] + h_1^* \delta[k+1])} = h_0^* r_w[k] + h_1^* r_w[k+1]$$

$$r_x[k] = r_w[k] * \underbrace{h[k] * h^*[k]}_{r_n[k] \text{ (given above)}}$$

$$= h_1^* h_0 r_w[k+1] + (h_0^2 + h_1^2) r_w[k] + h_1 h_0^* r_w[k-1]$$

Revisiting Poisson Process:

Let $N(t)$ denote the number of arrivals in $(0, T]$.

$N(t) \sim \text{Poisson}(\lambda t)$

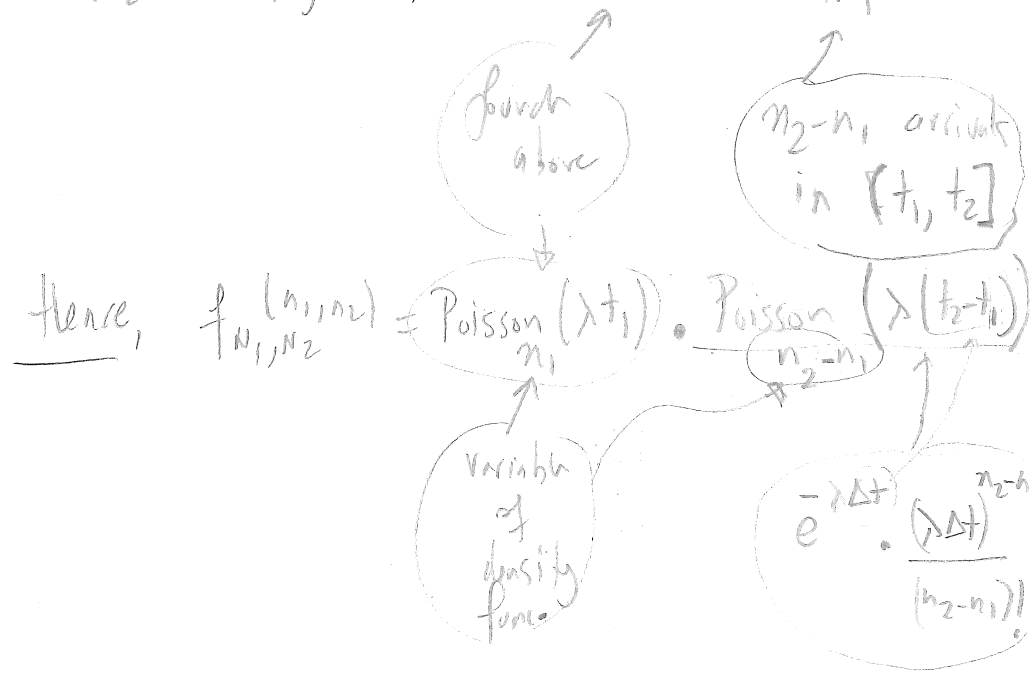
$Z \sim \text{Poisson}(\beta) \rightarrow P\{Z=n\} = e^{-\beta} \frac{\beta^n}{n!}$
 pmf.
 (1) $E\{Z\} = \beta$
 (2) $\text{Var}\{Z\} = \beta$
 variable of density func.

and arrivals in disjoint intervals are independent and the number arrivals in interval $(t_1, t_2]$ is also $N(t_2) - N(t_1) \sim \text{Poisson}(\lambda \Delta t)$ ($N(t)$ is Poisson and independent and stationary increments). $t_2 > t_1$

* Let's find joint density description of $N(t)$:

1st Order Density: $N(t_1) \sim \text{Poisson}(\lambda t_1)$ ✓

2nd Order Density: $\left. \begin{matrix} N_1 = N(t_1) \\ N_2 = N(t_2) \end{matrix} \right\} f_{N_1, N_2}(n_1, n_2) = f_{N_1}(n_1) \cdot f_{N_2|N_1}(n_2|n_1)$



Higher order densities can be found similarly.

Note: Not even 1st Order Stationary!

* Moment Description:

$N_1 \sim \text{Poisson}(\lambda t_1)$

50

1st
look

Mean function: $N_1 = N(t_1): M(t_1) = E\{N_1\} = \lambda t_1$

2nd
look

Variance function: $\text{Var}(N(t_1)) = \sigma_{N(t_1)}^2 = E\{\lambda t_1\}$

Auto-correlation function: $R(t_1, t_2) = E\{N(t_1)N(t_2)\}$

$= E\{N(t_1)[N(t_2) - N(t_1) + N(t_1)]\}$

increment from t_1 to t_2

$= E\{N(t_1)[\Delta_N^{t_1 \rightarrow t_2}] + N(t_1)\}$

$\text{Var}\{N_1\} + (E\{N_1\})^2$

$= E\{N(t_1)\} E\{\Delta_N^{t_1 \rightarrow t_2}\} + \lambda t_1 + (\lambda t_1)^2$

$= \lambda t_1 \cdot \lambda(t_2 - t_1) + \lambda t_1 + (\lambda t_1)^2$

$= \lambda t_1 + \lambda^2 t_1 t_2$

Hence

$$R(t_1, t_2) = \begin{cases} \lambda t_1 + \lambda^2 t_1 t_2 & , t_2 > t_1 \\ \lambda t_2 + \lambda^2 t_1 t_2 & , t_2 < t_1 \end{cases}$$

$= \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$

Note: For $t_2 < t_1$ analysis, you can interchange t_1 and t_2 variables in the $R(t_1, t_2)$ analysis for $t_2 > t_1$

Brief Information on Fourier Transform:

①

In this course, we will use the following definition of Fourier Transform.

$$x(t) \leftrightarrow X(f) \quad \leftarrow f \text{ (in Hertz)}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \quad \leftarrow \text{Forward F.T.}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df \quad \leftarrow \text{Inv. F.T.}$$

The main advantage of this definition is the absence of $\frac{1}{2\pi}$ factor in the inv. F.T. expression. This leads to several simplifications in transform pairs and eases the application of duality.

$$\boxed{x(t) \leftrightarrow X(f)} \\ \text{F.T.}$$

① $x(t-t_0) \leftrightarrow X(f) e^{-j2\pi ft_0}$ (Time shift)

② $x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$ (Scaling)

③ $x(t) e^{j2\pi f_0 t} \leftrightarrow X(f-f_0)$ (Freq. shift or modulation)

④ $X(f) \leftrightarrow x(-f)$ (Duality)

⑤ $x_1(t) * x_2(t) \leftrightarrow X_1(f) X_2(f)$ (Convolution)

⑥ $x_1(t) \cdot x_2(t) \leftrightarrow X_1(f) * X_2(f)$ (Time-multiplication)

Some Fourier Transform Pairs:

$$x(t) \longleftrightarrow X(f)$$

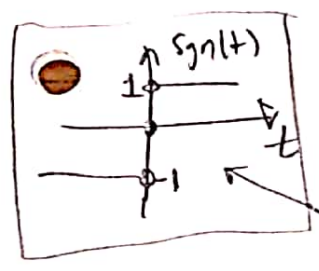
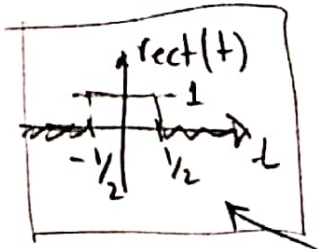
$$1 \longleftrightarrow \delta(f)$$

$$e^{j2\pi f_0 t} \longleftrightarrow \delta(f - f_0)$$

$$\text{rect}(t) \longleftrightarrow \text{sinc}(f)$$

$$\text{sinc}(t) \longleftrightarrow \text{rect}(f) \quad (\text{By duality})$$

$$\text{sinc}(f) \triangleq \frac{\sin(\pi f)}{\pi f}$$



$$u(t) \longleftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\text{sgn}(t) \longleftrightarrow \frac{1}{j\pi f}$$

$$\frac{1}{\pi t} \longleftrightarrow -j \text{sgn}(f)$$

← (Hilbert Transform related)

Power Spectral Density:

①

We have observed that LTI processing of stochastic WSS processes yields an output process with the auto-correlation function of

$$r_y(z) = r_x(z) * h(z) * h^*(z^*)$$

Considering the relation of convolution operation and Fourier transforms, the idea of utilizing Fourier transforms for output auto-correlation calculation immediately pops up. This is indeed correct, but there is more into this topic than the computational considerations.

Given a WSS process with auto-correlation function $r_x(z)$, the Fourier transform of $r_x(z)$, i.e.

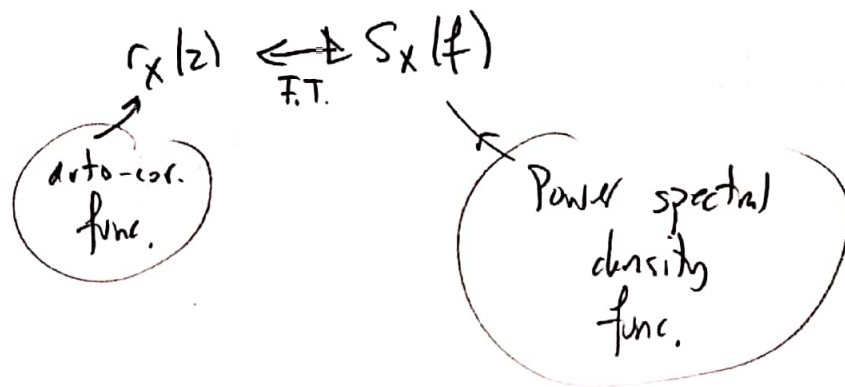
$$S_x(f) = \int_{-\infty}^{\infty} r_x(z) e^{-j2\pi fz} dz$$

is called the power spectral density of $x(t)$.

Similarly for WSS processes with power spectral density $S_x(f)$; it is possible to get $r_x(z) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi fz} df$.

Inverse Fourier Transform

Hence, we define the following as a Fourier Transform pair: ②



A careful eye must have noticed the word "density" in the "power spectral density function" title given to the Fourier transform of $r_x(z)$. The first few properties will reveal, why this function is called a density function.

Properties of Power Spectral density:

① Area under psd function:

$$r_x(0) = \int_{-\infty}^{\infty} S_x(f) df$$

This equation immediately follows from $r_x(z) \xleftrightarrow{\text{F.T.}} S_x(f)$ relation. That's, the area of the function in one of the domains is identical to the value of the function in the other domain at the origin.

Since, $r_x(0) = E\{ |x(t)|^2 \}$ is the avg. power of the process, we can say that the area under $S_x(f)$ is the ^{ensemble} average power of the process.

(2) $S_x(f)$ is real valued.

(3)

Proof We will use Hermitian-symmetry of $r_x(z)$ for the proof.

$$S_x(f) = \int_{-N}^N r_x(z) e^{-j2\pi f z} dz$$

Taking conjugate of both parts,

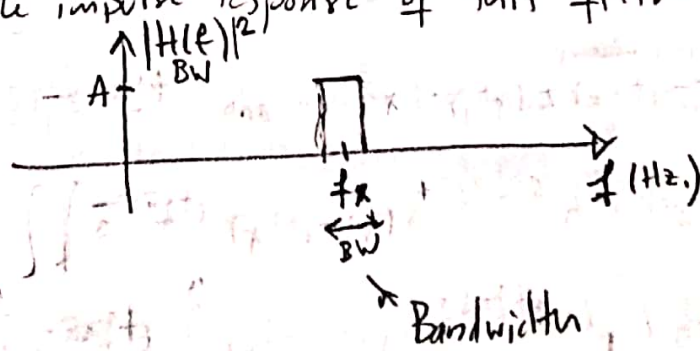
$$\begin{aligned} S_x^*(f) &= \int_{-N}^N r_x^*(z) e^{j2\pi f z} dz \\ &= \int_{-N}^N r_x^*(-z') e^{-j2\pi f z'} dz' \quad \leftarrow z' = -z \\ &= \int_{-N}^N r_x(z') e^{-j2\pi f z'} dz' \quad \leftarrow r_x(z) = r_x^*(-z) \\ &= S_x(f). \end{aligned}$$

(3) $S_x(f) \geq 0$.

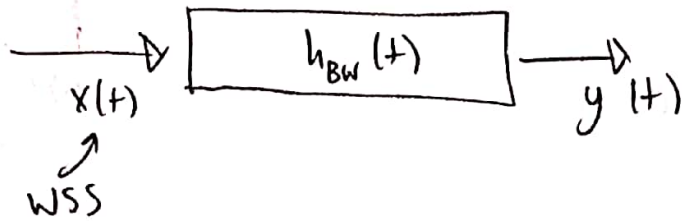
(This property is closely related positive semi definiteness of $r_x(z)$.)

Proof: Let's assume that $S_x(f_x) < 0$ for some f_x .

Then, we construct a bandpass filter with center freq. f_x and denote the impulse response of this filter as $h_{BW}(t)$.



Now,



$$\begin{aligned} x(t) &\leftrightarrow X(f) \quad (4) \\ x^*(-t) &\leftrightarrow X^*(f) \end{aligned}$$

We know that

$$\begin{aligned} r_y(z) &= r_x(z) * h_{BW}(z) * h_{BW}^*(-z) \\ \downarrow \\ S_y(f) &= S_x(f) \cdot |H_{BW}(f)|^2 \\ &= S_x(f) \cdot |H_{BW}(f)|^2 \end{aligned}$$

We know that $r_y(0) = \int_{-\infty}^{\infty} S_y(f) df \geq 0$

total avg. power of output process

$$= \int_{f_x - \frac{BW}{2}}^{f_x + \frac{BW}{2}} S_x(f) \cdot A \geq 0$$

$$\approx S_x(f_x) \cdot A \cdot BW$$

Note that the approximation to the integral becomes more and more accurate as $BW \rightarrow 0$ and $S_x(f_x)$ is continuous at f_x .

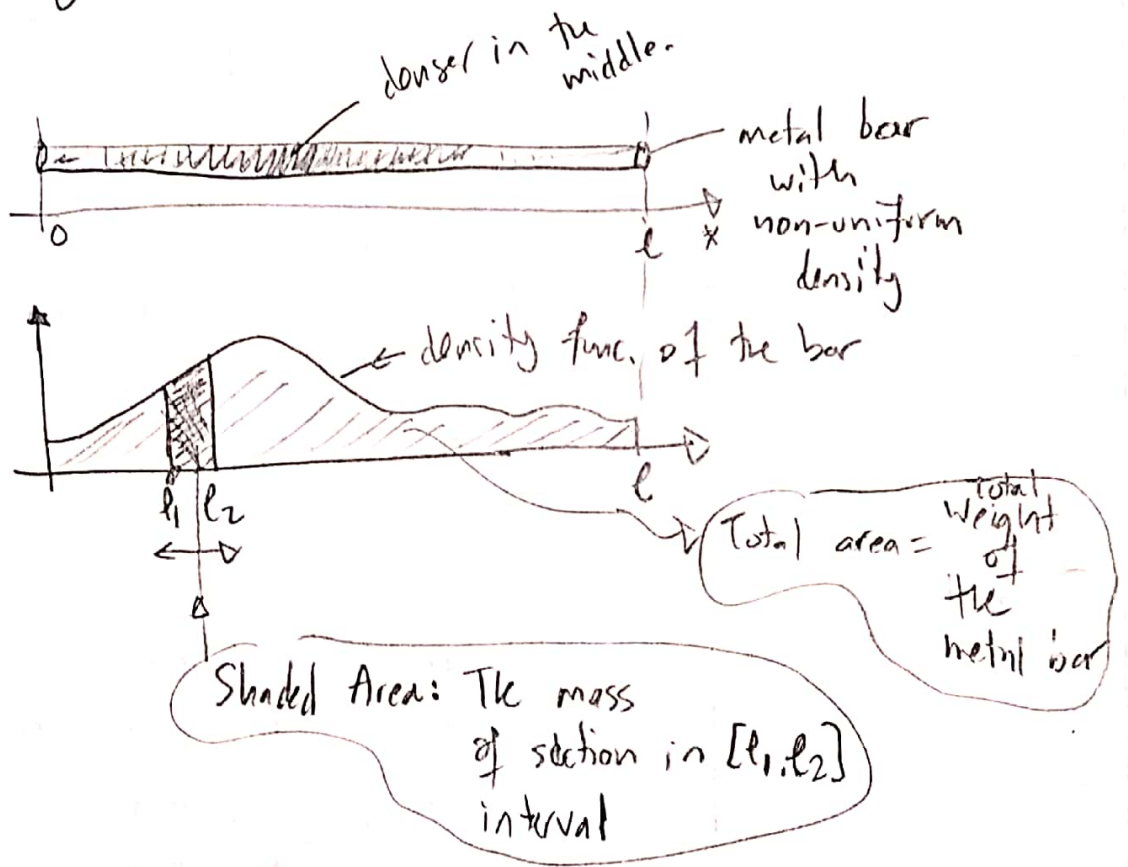
At the beginning, we have claimed that $S_x(f_x) < 0$, but this contradicts with $r_y(0) \geq 0$.

Hence, there is no frequency value f_x for which $S_x(f_x) < 0$.

Summarizing

$S_x(f)$ is a real, non-negative valued function, whose area under it is equal to the average power of the WSS process $x(t)$.

Hence, $S_x(f)$ shows the distribution of power across the spectrum. It is indeed a density function, i.e. a function showing the distribution of "mass" along an axis.



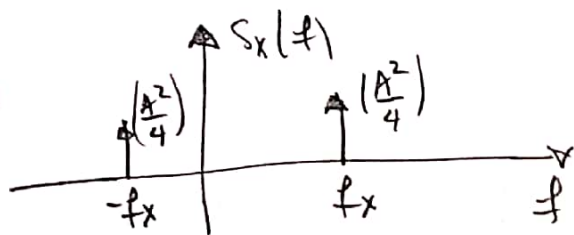
Ex: (1) $x(t) = A \cos(2\pi f_x t + \theta)$ (Random phase cosine) (6)
 $\theta \sim \text{Unif. } [0, 2\pi)$

We know that $x(t)$ is WSS with $r_x(z) = \frac{A^2}{2} \cos(\omega_x z)$.

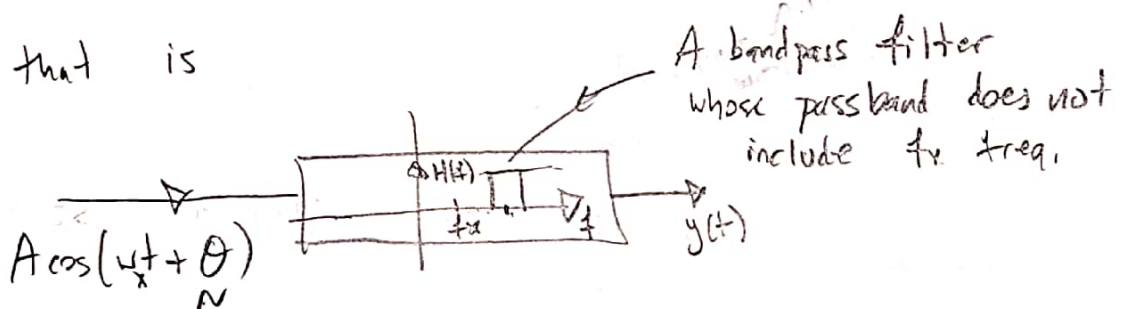
$\omega_x = 2\pi f_x$

$$S_x(f) = \mathcal{F} \left\{ \frac{A^2}{2} \cos(2\pi f_x z) \right\}$$

$$= \frac{A^2}{4} [\delta(f - f_x) + \delta(f + f_x)]$$



Hence, the power of this process is concentrated to $\pm f_x$ frequencies, that is



Random phase cosine

The output p.s.d. $S_y(f) = S_x(f) |H(f)|^2 = 0$. Hence, there is no output power if filter passband does not include frequency f_x . This should make sense considering the realization of the random phase cosine process.

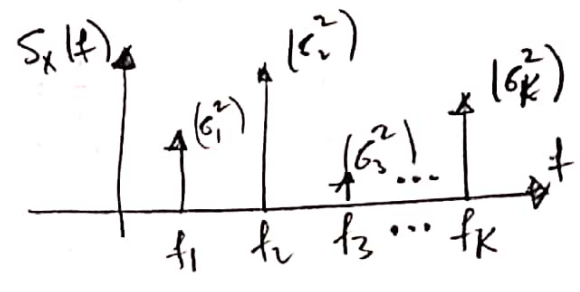
Ex: $x(t) = \sum_{k=1}^K a_k e^{j2\pi f_k t}$

a_k : zero-mean and uncorrelated r.v.'s with variance σ_k^2

Then $R_x(z) = \sum_{k=1}^N \sigma_k^2 e^{j2\pi f_k z}$ \longleftrightarrow $S_x(f) = \sum_{k=1}^N \sigma_k^2 \delta(f - f_k)$
F.T.

Note that $R_x(0) = \sum_{k=1}^K \sigma_k^2$

total power



distribution of total power in the spectrum.

Notes: (i) LTI filtering of white-noise (WSS) with a filter having magnitude response $|H(f)|$ results in an output process with power spectral density shaped in $|H(f)|^2$.

Hence, by designing filters, we can synthesize random processes with a desired $S_x(f)$ or $R_x(z)$.

In many practical modeling problems, $\hat{r}_x(z)$ or $\hat{S}_x(f)$ is estimated from collected data and an LTI filter with white-noise input is designed s.t. the filter output process has approximately the observed $\hat{r}_x(z)$ or $\hat{S}_x(f)$.

② Previously, we have studied some necessary cond. ⑧
 for auto-correlation func. Here, we have found another
 necessary condition

$$r_x(z) \text{ a valid auto-cor. func.} \longrightarrow \underbrace{S_x(f) \geq 0 \quad \forall f}_{\text{new necessary cond.}}$$

Different from other necessary conditions, the non-negativity
 of $S_x(f)$ is also sufficient for the validity
 of auto-cor. func. That is,

$$r_x(z) \text{ a valid auto-cor. func.} \longleftarrow \underbrace{S_x(f) = F\{r_x(z)\} \geq 0 \quad \forall f.}_{\text{Sufficient cond.}}$$

hence, we have

$$r_x(z) \text{ a valid auto-cor. func.} \iff \underbrace{S_x(f) = F\{r_x(z)\} \geq 0 \quad \forall f}_{\text{necessary and sufficient cond. for a valid } r_x(z).}$$

The sufficiency cond. is proved by constructing a WSS process for an arbitrary $g(f) \geq 0$ function whose power spectral density is $g(f)$, see Papoulis (10.24 p.322). Sec. 10.24 p.322. $\forall f$

③ Power spectral density of a r.p. allows us to classify the processes as low pass, High pass, Bandpass processes similar to the classification of filters. (9)

Different from filters, if a significant amount of process power resides, say, in low freq. band (a freq. band containing zero) that process is a low-pass process.

For filters, the impulse response of the filter has finite energy and if depending on the distribution of energy across the spectrum, filters are classified. (This is called energy spectrum density (ESD).)

Bandpass Processes, Bandpass Process Representation:

Before we discuss the bandpass signal representation, let's focus on another tool that will be needed along this road, the Hilbert transform.

$$\begin{array}{ccc} \xrightarrow{x(t)} & \boxed{h(t) = \frac{1}{\pi t}} & \xrightarrow{\hat{x}(t) = \text{H.T.}\{x(t)\}} \end{array}$$

Hence,

$$\hat{x}(t) = \int_{-\infty}^{\infty} x(z) \frac{1}{\pi(t-z)} dz.$$

(Hilbert Transform of $x(t)$)

If $x(t) \xleftrightarrow{\text{F.T.}} X(f)$ and $\frac{1}{\pi t} \xleftrightarrow{\text{F.T.}} -j \operatorname{sgn}(f)$

a known F.T. pair

$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$

Then, $\hat{X}(f) = X(f) \cdot -j \operatorname{sgn}(f)$ by convolution theorem of F.T.

↑

F.T. of $\{x(t)\}$

Now, it is easy to see that,

Observe $(-j \operatorname{sgn}(f))^2 = -1$
 $(f \neq 0)$



Hence,

$$\hat{X}(t) = \int_{-\infty}^{\infty} x(z) \cdot \frac{1}{\pi(t-z)} dz$$

$$x(t) = \int_{-\infty}^{\infty} \hat{X}(z) \cdot \frac{1}{\pi(z-t)} dz$$

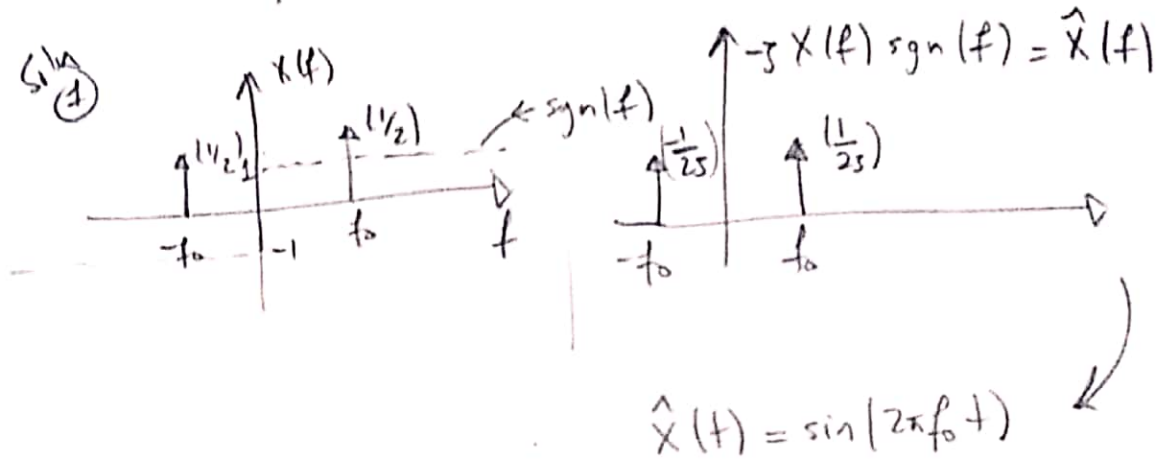
} Hilbert Transform Forward and Inverse relation expression.

Note that $h(t) = \frac{1}{\pi t}$ is not absolutely summable impulse response, hence, the H.T. as implemented by filtering results in a filtering operation with an unstable filter!

The H.T. forward and inverse expression are valid in the distribution sense which is not very important for our purposes; Since we will use the H.T. as a tool to facilitate other calculations. (11)

Ex: Let $x(t) = \cos(2\pi f_0 t)$. Find $\hat{x}(t)$.

Solu. (1)



Solu (2) $X(f) = \frac{1}{2} (\delta(f-f_0) + \delta(f+f_0))$

$\hat{X}(f) = \frac{1}{2} (-j \text{sgn}(f)) [\delta(f-f_0) + \delta(f+f_0)]$ (Assum. $f_0 > 0$)

$= \frac{1}{2j} \underbrace{\text{sgn}(f_0)}_{\uparrow 1} \delta(f-f_0) + \frac{1}{2j} \underbrace{\text{sgn}(-f_0)}_{\uparrow -1} \delta(f+f_0)$

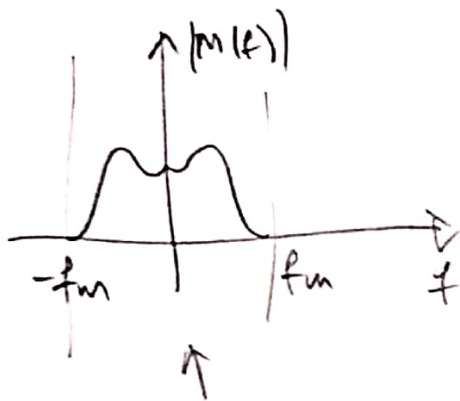
$= \frac{1}{2j} [\delta(f-f_0) - \delta(f+f_0)]$

$\left. \begin{matrix} \downarrow \\ \uparrow \end{matrix} \right\} F^{-1}\{ \}$

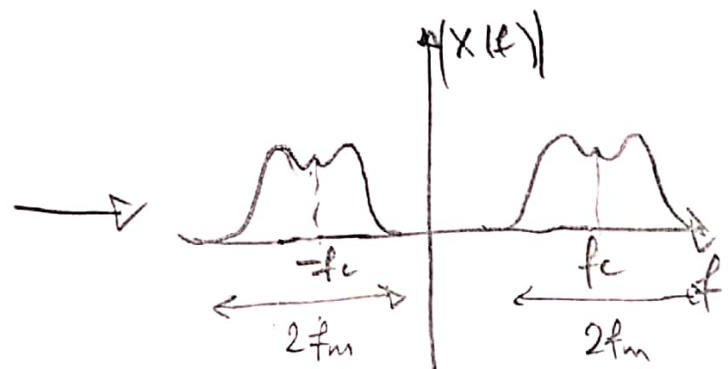
$\hat{X}(t) = \sin(2\pi f_0 t)$

Single Side Band Modulation

In communications applications, the message signal (12)
is modulated to a carrier frequency suitable for
the propagation of the message. The message can be
a song and this message can be modulated to 6 MHz
to reach far away listeners ^(of Voice of Turkey radio) in China.



Spectrum of
message
"song"



modulated message

$$x(t) = m(t) \cos(2\pi f_c t)$$

Note that, since (1) $m(t)$ is real valued $\rightarrow |M(f)|$ symmetric around zero

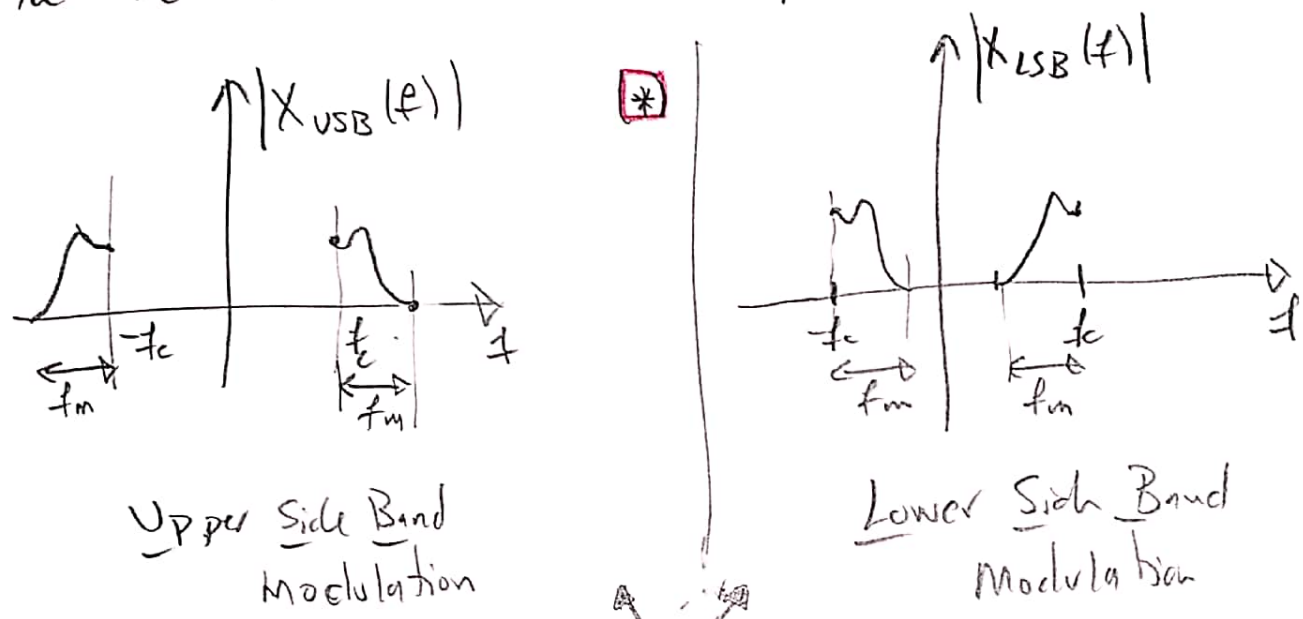
(2) $|X(f)|$ is then symmetric around f_c .

Hence, there is redundancy in the band $[f_c - f_m, f_c + f_m]$,

that is we know that $X(f_c + u) = X^*(f_c - u)$ (why)

$$|u| < f_m.$$

We can improve the spectral efficiency, if we can only send one of the side-bands, i.e. the one below or above f_c .



Single Side Band Modulation Schemes

Claim: $X_{USB}(t) = m(t) \cos(2\pi f_c t) - \hat{m}(t) \sin(2\pi f_c t)$

$X_{LSB}(t) = m(t) \cos(2\pi f_c t) + \hat{m}(t) \sin(2\pi f_c t)$

Proof:

$$X_{USB}(f) = \frac{M(f)}{2} * [\delta(f-f_c) + \delta(f+f_c)]$$

$$\frac{M(f) * (-j \operatorname{sgn}(f))}{2j} * [\delta(f-f_c) - \delta(f+f_c)]$$

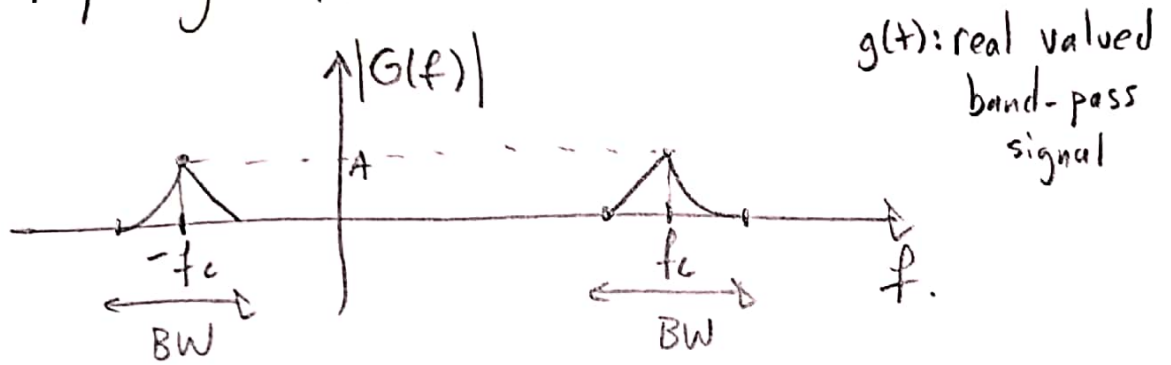
$$= \frac{M(f-f_c)}{2} \underbrace{(1 + \operatorname{sgn}(f-f_c))}_{= \begin{cases} 2, & f > f_c \\ 0, & f < f_c \end{cases}} + \frac{M(f+f_c)}{2} \underbrace{(1 - \operatorname{sgn}(f+f_c))}_{= \begin{cases} 2, & f < -f_c \\ 0, & f > -f_c \end{cases}}$$

(compare $X_{USB}(f)$ expression with the sketch in \boxtimes for $|X_{USB}(f)|$).

Representation of Band Pass Signals:

(14)

A passband signal, typically a message signal modulated to a high frequency, occupies a bandwidth around a chosen frequency " f_c ".



The information is clearly contained in the frequency band/interval shown by the BW (Bandwidth) above.

Our first goal is to represent this ^{real-valued} bandpass signal as an equivalent low pass (but complex valued) signal.

① Bandpass Signal \rightarrow low Pass Equivalent:

Step ①: Form the analytic/pre-envelope signal

$$g_+ \triangleq g(t) + j\hat{g}(t)$$

analytic signal (pre-envelope) \leftarrow $H.T. \{ \hat{g}(t) \}$

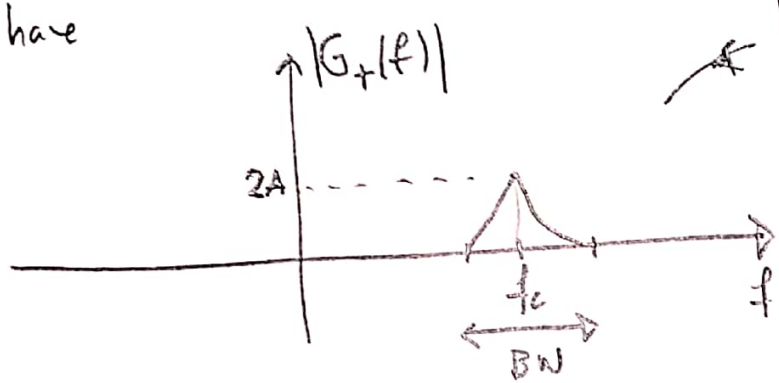
To understand $g_+(t)$, let's evaluate its F.T.

$$G_+(f) = G(f) + j\hat{G}(f) \qquad \hat{G}(f) = -j \operatorname{sgn}(f) G(f)$$

$$= (1 + \operatorname{sgn}(f)) G(f)$$

$$= \begin{cases} 2G(f), & f > 0 \\ 0, & f < 0 \end{cases}$$

Hence, we have



Negative freq. components are eliminated!

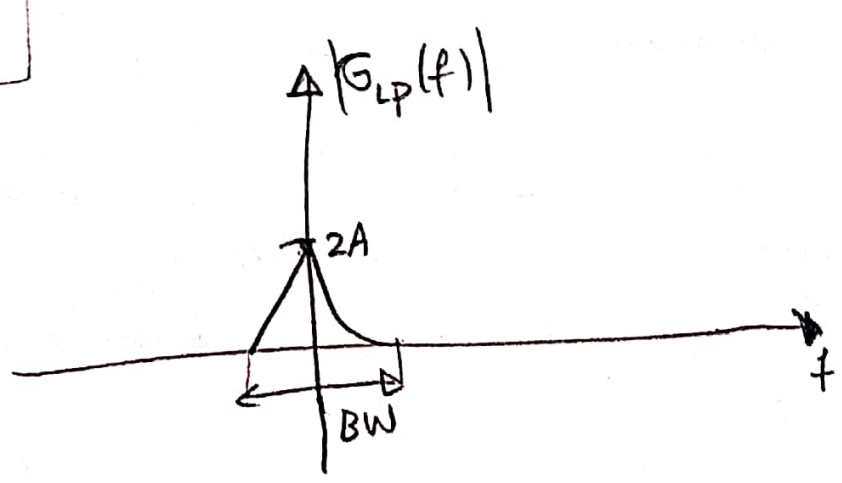
Step 2: Shift $G_+(f)$ to DC frequency to get the low pass equivalent signal:

$$g_{LP}(t) \triangleq g_+(t) e^{-j2\pi f_c t}$$

Freq. Shift towards the negative frequencies ($f_c > 0$)

Low pass equivalent signal

$$G_{LP}(f) = G_+(f + f_c)$$



Note: ① $g_{LP}(t)$ is a complex valued signal in general, since $|G_{LP}(f)|$ does not have even symmetry in spectrum. (This should also explain why we focus also on negative frequencies or complex valued signals in signal processing. The answer is such signals are related to the low pass equivalents of band-pass signals.)

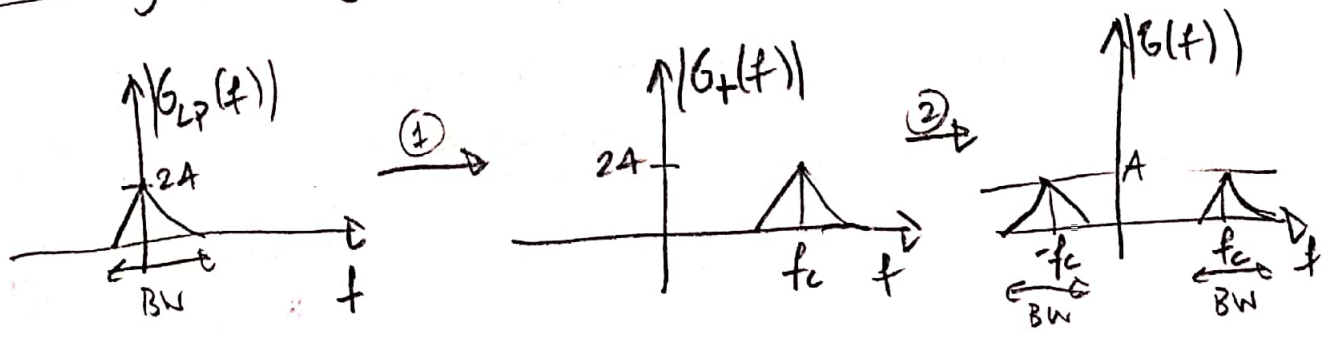
② The "center" frequency does not have to be in the middle of the bandwidth of the band-pass signal. The center frequency is arbitrary in this process. It can even be outside of the bandwidth!

Low-Pass Equivalent \rightarrow Bandpass Signal:

Band-pass signal

Given $g_{LP}(t)$ and f_c , how can we get $g(t)$?

Answer: By following ^{the same} steps in reverse!



Let's do this algebraically:

$$g_+(t) = g_{LP}(t) e^{j2\pi f_c t}$$

$$g_{LP}(t) = g_I(t) + j g_Q(t)$$

Real part of $g_{LP}(t)$ (17)

Complex value

imag. part.

analytic signal retrieved (1)

$$g(t) = \text{Re}\{g_+(t)\}$$

Remember that (2)

$$g_+(t) = g(t) + j\hat{g}(t)$$

Hence,

$$g(t) = \text{Re}\{g_{LP}(t) e^{j2\pi f_c t}\}$$

$g_I(t) + j g_Q(t)$

$$g(t) = g_I(t) \cos(2\pi f_c t) - g_Q(t) \sin(2\pi f_c t)$$

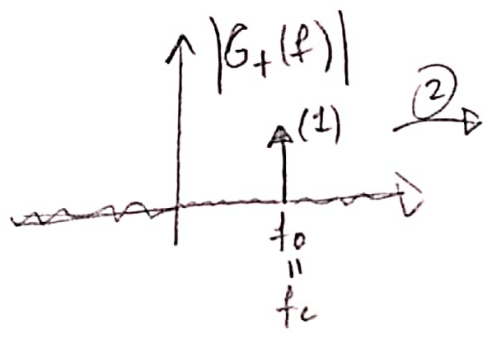
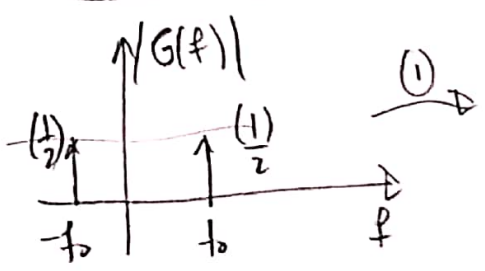
where $g_I(t)$, $g_Q(t)$ are inphase, quadrature phase components of low pass equivalent signal $g_{LP}(t)$, respectively.

The expression $g(t) = g_I(t) \cos(2\pi f_c t) - g_Q(t) \sin(2\pi f_c t)$ is called the canonical representation of band-pass signals and it is very important in both theory and practice.

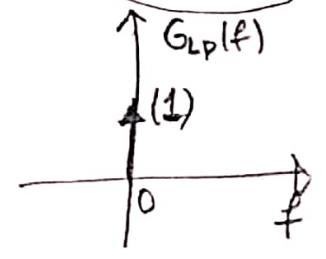
Ex: $g(t) = \cos(2\pi f_0 t)$

i) Find the low pass equivalent of $g(t)$ for $f_c = f_0$.

Graphically:



↑
representation
freq. for
band-pass signal



$G_{LP}(t) = 1$

From canonical representation:

$g(t) = g_I(t) \cos(2\pi f_c t) - g_Q(t) \sin(2\pi f_c t)$

↑
any band-pass
signal

In this case, $g(t) = \cos(2\pi f_0 t)$ and $f_c = f_0 \rightarrow$

$$\left. \begin{aligned} g_I(t) &= 1 \\ g_Q(t) &= 0 \end{aligned} \right\}$$

$$G_{LP}(t) = g_I(t) + jg_Q(t) = 1$$

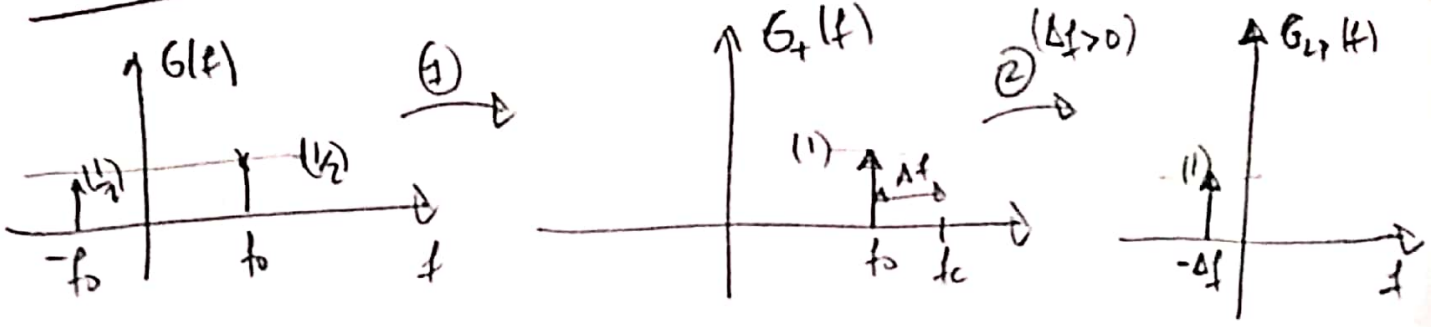
ii) Repeat the same for $f_c = f_0 + \Delta f$

$\Delta f > 0$

(Typically Δf is very small in comparison to f_0)

(It may correspond to a mismatch in frequency to f_0)

Graphically:



$$g_{LP}(t) = e^{-j2\pi\Delta f t}$$

$$= \underbrace{\cos(2\pi\Delta f t)}_{g_I(t)} - j \underbrace{\sin(2\pi\Delta f t)}_{-g_Q(t)}$$

i.e. $g_I(t) = \cos(2\pi\Delta f t)$
 $g_Q(t) = -\sin(2\pi\Delta f t)$

From canonical rep.

low band pass signal \rightarrow $g(t) = g_I(t) \cos(2\pi f_c t) - g_Q(t) \sin(2\pi f_c t)$ (canonical rep.)

Let, $g(t) = \cos(2\pi f_0 t) = \cos(2\pi \overbrace{f_c - \Delta f}^{f_0} t)$

$$= \underbrace{\cos(2\pi\Delta f t)}_{g_I(t)} \cos(2\pi f_c t) + \underbrace{\sin(2\pi\Delta f t)}_{-g_Q(t)} \sin(2\pi f_c t)$$

By comparing with the canonical rep., we get the same result. (Remember $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$).

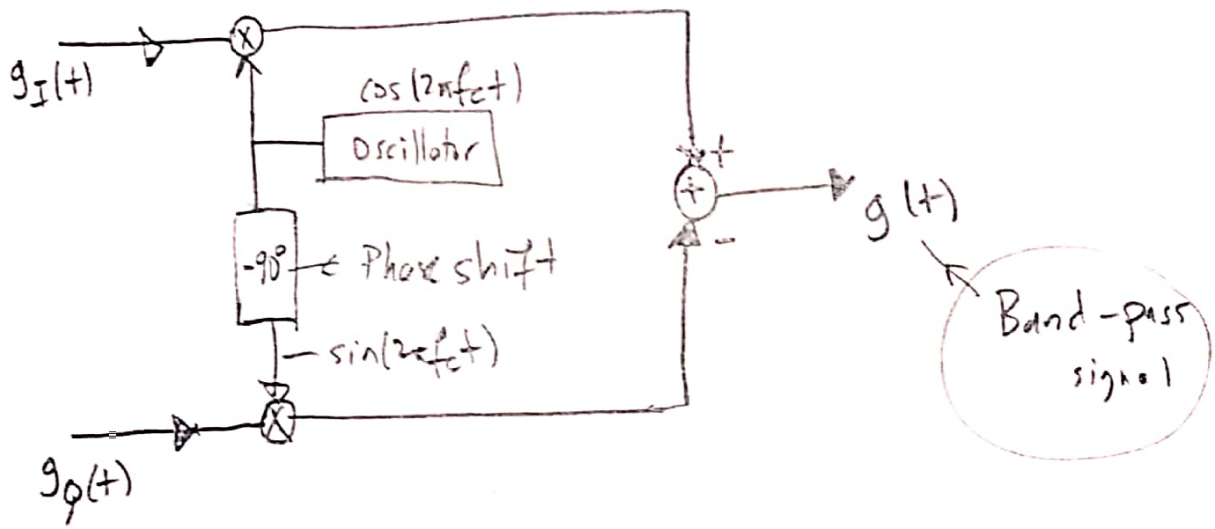
From practical view point:

Generation of Bandpass Signals
(ie modulation)

Given $g_I(t)$: inphase signal

$g_Q(t)$: quadrature phase signal

Q. How to generate bandpass signal?

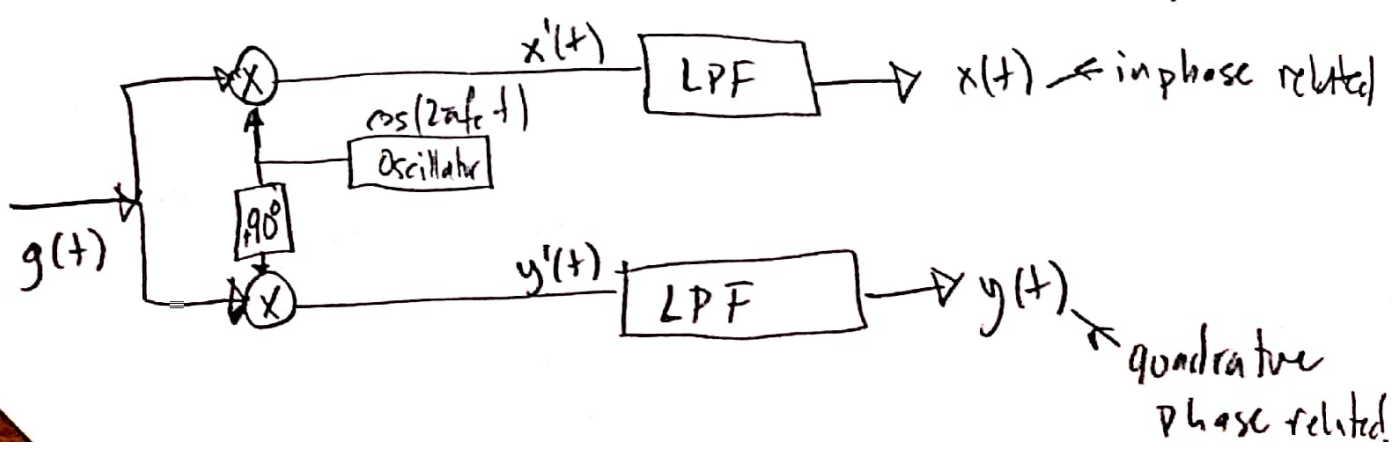


The above is a direct implementation of canonical representation.

Generation of I/Q signals
(i.e. demodulation)

Given $g(t)$ a band-pass signal,

Q. How to generate I/Q signals?



$$x'(t) = g_I(t) \cos(2\pi f_c t)$$

$$= g_I(t) \frac{\cos^2(2\pi f_c t)}{\frac{1 + \cos(4\pi f_c t)}{2}} - g_Q(t) \frac{\sin(2\pi f_c t) \cos(2\pi f_c t)}{\frac{\sin(4\pi f_c t)}{2}}$$

$$= \frac{g_I(t)}{2} + (\text{High freq. components})$$

low pass filtering $x'(t)$ gives $\frac{g_I(t)}{2} = x(t)$

Similarly,

$$y'(t) = g(t) [-\sin(2\pi f_c t)]$$

$$= g_I(t) \cos(2\pi f_c t) \sin(2\pi f_c t) + g_Q(t) \frac{\sin^2(2\pi f_c t)}{\frac{1 - \cos(4\pi f_c t)}{2}}$$

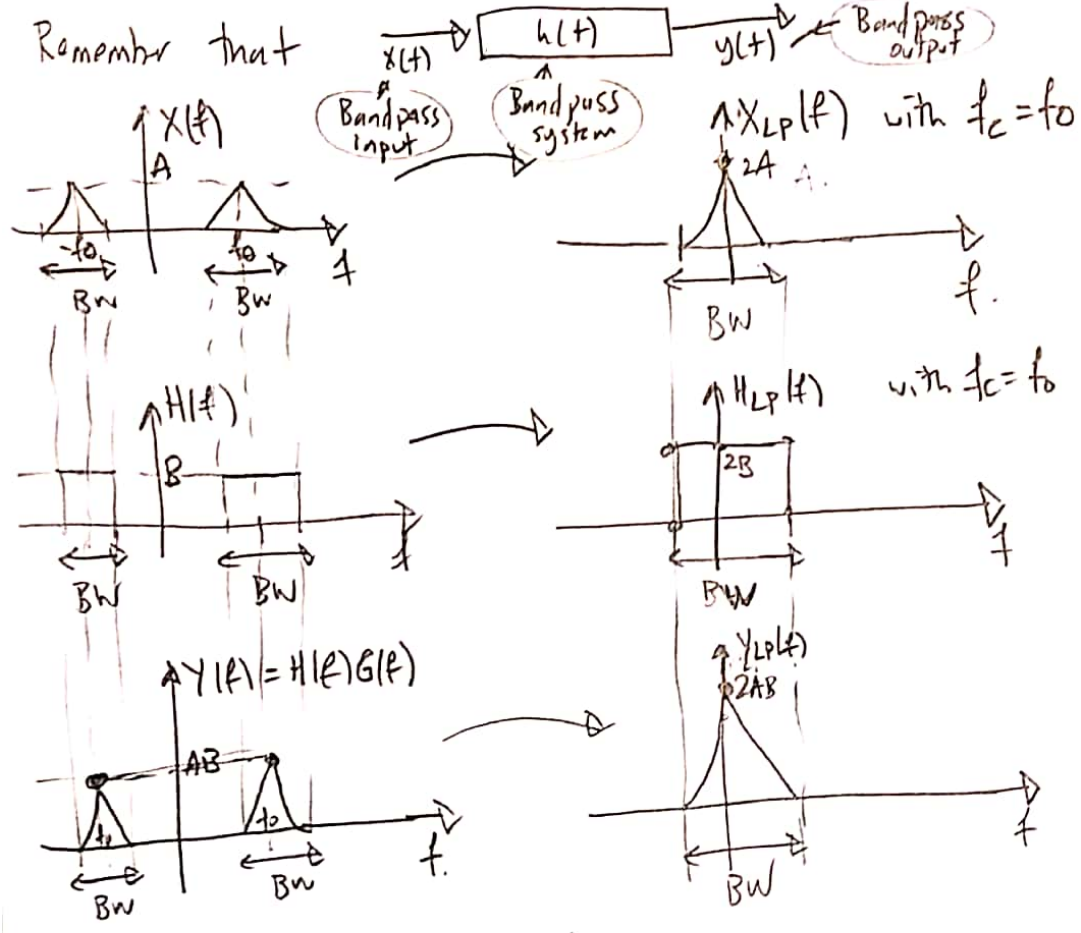
$$= \frac{g_Q(t)}{2} + (\text{High freq. terms})$$

Representation of Bandpass Systems

A band-pass system is an LTI system whose frequency response is centered around a non-zero carrier frequency.

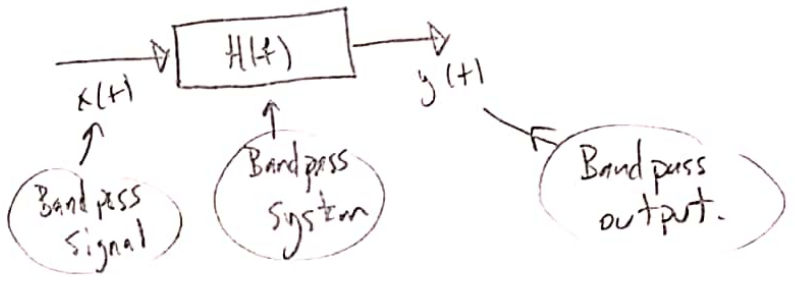
We extend our earlier results for bandpass signal representation to band pass systems by considering the impulse response of a band-pass system and its low pass equivalent.

Remember that



From above graphs, we can observe that

The relation $Y(f) = H(f)X(f)$, that is



results in the following for the low pass equivalent:

$$Y_{LP}(f) = \frac{1}{2} H_{LP}(f) \cdot X_{LP}(f)$$

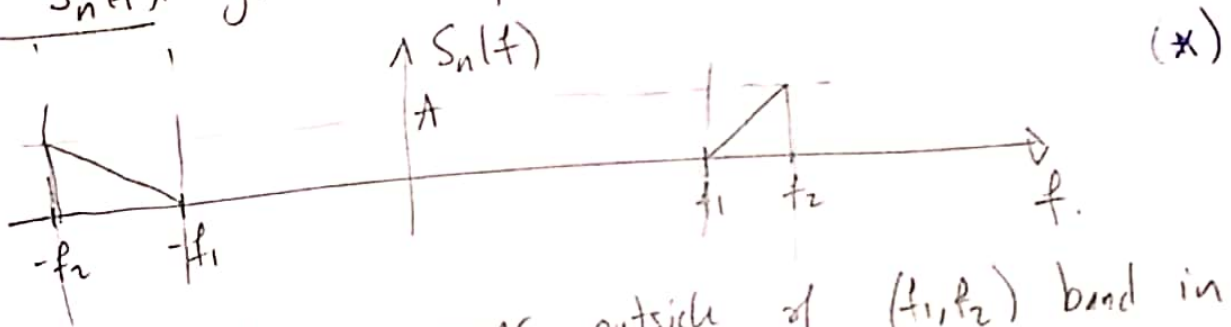
The factor of $\frac{1}{2}$ is required, since the spectrum of low-pass equivalent is "doubled" (multiplied by two) when compared with the corresponding spectrum of the band-pass signal. $H_{LP}(f)X_{LP}(f)$ correctly gets the "shape" of

low pass equivalent for $Y(f) = H(f)X(f)$; but scaling by 2

is done twice when $H_{LP}(f) X_{LP}(f)$ is evaluated; hence, a factor of $1/2$ is needed. to get $Y_{LP}(f)$ from $H_{LP}(f) X_{LP}(f)$ product.

Representation of Bandpass WSS Processes:

Let $n(t)$ be a zero mean, band-pass WSS process with power spectral density $S_n(f)$. given as follows.



Note that $n(t)$ has no power outside of (f_1, f_2) band in the spectrum. Our goal is to represent this r.p. with an equivalent low-pass r.p.

- Step (1): Generate analytic process, $n_+(t) = n(t) + j\hat{n}(t)$
- Step (2): Generate low-pass equivalent process, $n_{LP}(t) = n_+(t) e^{-j2f_c t}$ (with a choice of f_c)

Note that these steps are the same as generating a low-pass equivalent of deterministic signals. Since, a realization of a band-pass process is a deterministic signal, we may think that we are applying the low-pass equivalent generation process

to the realizations of a band-pass process and generating a new process called the low-pass equivalent process.

We would like to answer questions like whether $\tilde{n}_{LP}(t)$ is WSS, the auto-correlation/power spectral density of $\tilde{n}_{LP}(t)$, the auto-cor. of $\tilde{n}_I(t) = \text{Re}\{\tilde{n}_{LP}(t)\}$; the cross-correlation of $\tilde{n}_I(t)$ and $\tilde{n}_Q(t)$ etc.

Before this discussion, let's remember some basic facts:

① $y(t)$ is also WSS

② $x(t), y(t)$ are jointly WSS

$r_y(z) = r_x(z) * h(z) * h^*(1-z)$

③ $r_{xy}(z) = r_x(z) * h^*(1-z)$

$r_{yx}(z) = r_{xy}^*(1-z) = h(z) * r_x(z)$

F.T.

④ $S_y(f) = S_x(f) |H(f)|^2$

$S_{xy}(f) = S_x(f) H^*(f)$

$S_{yx}(f) = H(f) \cdot S_x(f)$

Basic Facts:

Now, let's start with step-1

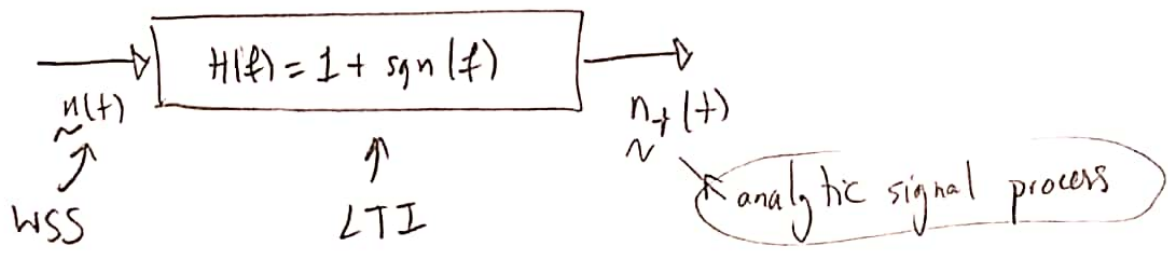
Step ①: Generate the analytic process

$$\tilde{n}_+(t) = \tilde{n}(t) + j\hat{\tilde{n}}(t)$$

↑
F.T.

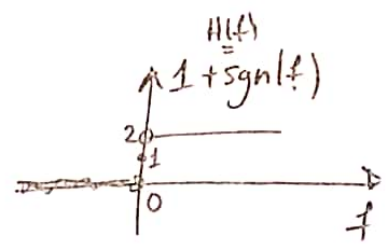
$$N_+(f) = N(f) + j(\text{sgn}(f))N(f)$$

$$N_+(f) = (1 + \text{sgn}(f))N(f)$$

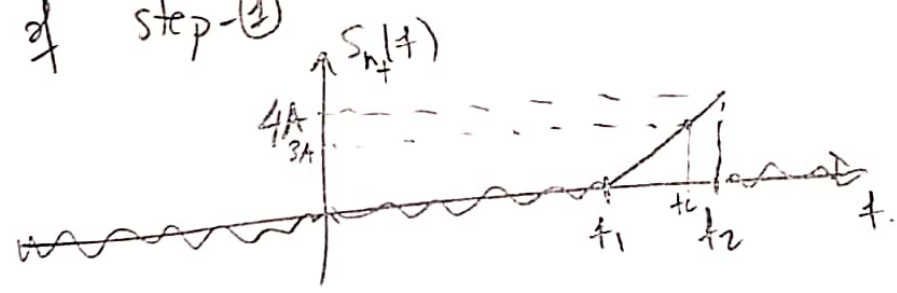


Since $\tilde{n}_+(t)$ is LTI filtering of $\tilde{n}(t)$; it is WSS and has the power spectrum density of

$$S_{n_+}(f) = |H(f)|^2 S_n(f) = \begin{cases} 4S_n(f), & f > 0 \\ 0, & f < 0 \end{cases}$$



Then, sketch (**) in page 23 has the following p.s.d. sketch at the end of step-(1)



(**)

Step-(2) : Generation of low-pass equivalent process

$$\tilde{n}_{LP}(t) = \tilde{n}_+(t) e^{-j2\pi f_c t}$$

(f_c : a chosen representation freq. for low-pass equivalent rep.)

Let's calculate the auto-correlation of $\tilde{n}_{LP}(t)$:

$$\begin{aligned} r_{n_{LP}}(\tau) &= E\{\tilde{n}_{LP}(t) \tilde{n}_{LP}^*(t-\tau)\} \\ &= E\{\tilde{n}_+(t) \tilde{n}_+^*(t-\tau) e^{-j2\pi f_c \tau}\} \\ &= r_{n_+}(\tau) e^{-j2\pi f_c \tau} \end{aligned}$$

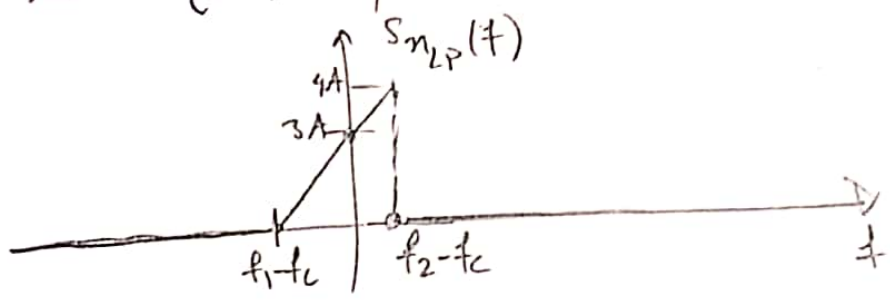
Not a func. of " t ", so $\tilde{n}_{LP}(t)$ is stat. in auto-correlation.

Then its p.s.d is

$$S_{n_{LP}}(f) = F \left\{ r_{n_+}(z) e^{-j2\pi f_c z} \right\}$$

$$= S_{n_+}(f + f_c)$$

Then, sketch (***) in p. 25 after step (2) becomes



So, we have seen that $n_{LP}(t)$ is a WSS process with zero mean (it is easy to check this given the information that the band-pass r.p. is also zero-mean) WSS process with p.s.d as given/can be found, as shown above.

Next, we would like to find the auto-corr/power spectral density of I/Q components of low-pass equivalent signal.

Remember that, $\tilde{n}_{LP}(t) = \tilde{n}_I(t) + j\tilde{n}_Q(t)$. We, now,

know that $\tilde{n}_{LP}(t)$ is WSS. How about $\tilde{n}_I(t), \tilde{n}_Q(t)$ processes.

This is our next question.

$$\tilde{n}_I(t) = \text{Re} \left\{ \tilde{n}_{2P}(t) \right\} = \text{Re} \left\{ n(t) e^{-j 2\pi f_c t} \right\}$$

$n(t) + j \hat{n}(t)$

$$\tilde{n}_I(t) = \tilde{n}(t) \cos(2\pi f_c t) + \hat{\tilde{n}}(t) \sin(2\pi f_c t)$$

$$\tilde{n}_Q(t) = \hat{\tilde{n}}(t) \cos(2\pi f_c t) - \tilde{n}(t) \sin(2\pi f_c t)$$

Similarly, \rightarrow

Note that:

$$\begin{bmatrix} \tilde{n}_I(t) \\ \tilde{n}_Q(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(2\pi f_c t) & \sin(2\pi f_c t) \\ -\sin(2\pi f_c t) & \cos(2\pi f_c t) \end{bmatrix}}_{R(t)} \begin{bmatrix} \tilde{n}(t) \\ \hat{\tilde{n}}(t) \end{bmatrix}$$

$$= R(t) \cdot \begin{bmatrix} \tilde{n}(t) \\ \hat{\tilde{n}}(t) \end{bmatrix}$$

Now,

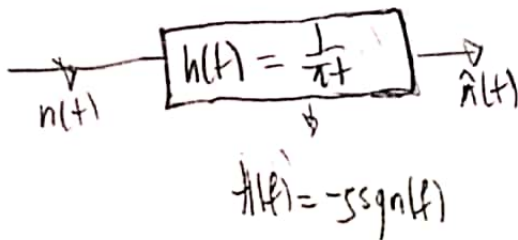
$$\mathbb{E} \left\{ \begin{bmatrix} \tilde{n}_I(t) \\ \tilde{n}_Q(t) \end{bmatrix} \begin{bmatrix} \tilde{n}_I(t-z) & \tilde{n}_Q(t-z) \end{bmatrix} \right\} = \begin{bmatrix} R_{\tilde{n}_I \tilde{n}_I}(z) & R_{\tilde{n}_I \tilde{n}_Q}(z) \\ R_{\tilde{n}_Q \tilde{n}_I}(z) & R_{\tilde{n}_Q \tilde{n}_Q}(z) \end{bmatrix}$$

$$= R(t) \mathbb{E} \left\{ \begin{bmatrix} \tilde{n}(t) \\ \hat{\tilde{n}}(t) \end{bmatrix} \begin{bmatrix} \tilde{n}(t-z) & \hat{\tilde{n}}(t-z) \end{bmatrix}^T \right\} R^T(t-z)$$

$$\begin{bmatrix} R_{\tilde{n}_I \tilde{n}_I}(z) & R_{\tilde{n}_I \tilde{n}_Q}(z) \\ R_{\tilde{n}_Q \tilde{n}_I}(z) & R_{\tilde{n}_Q \tilde{n}_Q}(z) \end{bmatrix} = R(t) \begin{bmatrix} R_n(z) & R_{\hat{n}}(z) \\ R_{\hat{n}n}(z) & R_{\hat{n}}(z) \end{bmatrix} R^T(t-z)$$

From Basic Facts (p.24)

Since



$$\begin{aligned} S_{\hat{n}}(f) &= S_n(f) | -j \text{sgn}(f) | = S_n(f) \\ S_{\hat{n}n}(f) &= -j \text{sgn}(f) S_n(f) \\ S_{n\hat{n}}(f) &= S_n(f) [-j \text{sgn}(f)]^* \end{aligned}$$

$$\begin{aligned} \rightarrow R_n^{\wedge}(z) &= F^{-1}\{S_n(f)\} = R_n(z) \\ \rightarrow R_{nn}^{\wedge}(z) &= F^{-1}\{S_{nn}^{\wedge}(f)\} = H_oT\{R_n(z)\} = \hat{R}_n(z) \\ \rightarrow R_{n\hat{n}}^{\wedge}(z) &= F^{-1}\{S_{n\hat{n}}^{\wedge}(f)\} = -H_iT_i\{R_n(z)\} = -\hat{R}_n(z) \end{aligned}$$

Hence,

$$\left[\begin{array}{c|c} R_{nI}(z) & R_{N_I N_p}(z) \\ \hline R_{N_p N_I}(z) & R_{N_p}(z) \end{array} \right] = R(t) = \left[\begin{array}{c|c} R_n(z) & -\hat{R}_n(z) \\ \hline \hat{R}_n(z) & R_n(z) \end{array} \right] = R^T(t-z)$$

$$= \begin{bmatrix} \cos(2\pi f_c t) & \sin(2\pi f_c t) \\ -\sin(2\pi f_c t) & \cos(2\pi f_c t) \end{bmatrix} \left[\begin{array}{c|c} R_n(z) & -\hat{R}_n(z) \\ \hline \hat{R}_n(z) & R_n(z) \end{array} \right] \left[\begin{array}{c|c} (c) & -s \\ \hline (s) & (c) \end{array} \right]$$

$$= \left[\begin{array}{c|c} R_n(z) \cos(2\pi f_c z) + \hat{R}_n(z) \sin(2\pi f_c z) & R_n(z) \sin(2\pi f_c z) - \hat{R}_n(z) \cos(2\pi f_c z) \\ \hline -R_n(z) \sin(2\pi f_c z) + \hat{R}_n(z) \cos(2\pi f_c z) & R_n(z) \cos(2\pi f_c z) + \hat{R}_n(z) \sin(2\pi f_c z) \end{array} \right]$$

This concludes the derivation of

$$(*) \begin{cases} (1) R_{n_I}(z) = R_{n_Q}(z) = R_N(z) \cos(2\pi f_c z) + \hat{R}_N(z) \sin(2\pi f_c z) \\ (2) R_{N_I N_Q}(z) = -R_{N_Q N_I}(z) = R_N(z) \sin(2\pi f_c z) - \hat{R}_N(z) \cos(2\pi f_c z) \end{cases}$$

where $R_N(z)$ is the auto-cor. of band-pass r.p.

$$\hat{R}_N(z) = H.T. \{ R_N(z) \}$$

and

f_c : representation freq. for low pass equivalent.

We see that $R_{n_I}(z)$, $R_{n_Q}(z)$ and $R_{n_I n_Q}(z)$ expressions have no dependency on "t", hence $n_I(t)$ and $n_Q(t)$ are jointly

WSS.

The expressions (1) and (2) in (*) above are easier to keep in mind in terms of its Fourier transform. let's calculate the

P.S.D of $r_{n_I}(z)$ (or $r_{n_Q}(z)$).

$$\begin{aligned} S_{n_I}(f) &= F\{r_{n_I}(z)\} = \left(S_N(f) \left(\frac{\delta(f-f_c) + \delta(f+f_c)}{2} \right) + (-j \operatorname{sgn}(f) S_N(f)) \frac{\delta(f-f_c) - \delta(f+f_c)}{j2} \right) \\ &= \frac{1}{2} \left[\underbrace{(1 - \operatorname{sgn}(f-f_c))}_{= \begin{cases} 0, & f > f_c \\ 2, & f < f_c \end{cases}} S_N(f-f_c) + \underbrace{(1 + \operatorname{sgn}(f+f_c))}_{= \begin{cases} 2, & f > -f_c \\ 0, & f < -f_c \end{cases}} S_N(f+f_c) \right] \end{aligned}$$

We assume

$f_c \gg BW$
 \wedge (BW of band pass process)

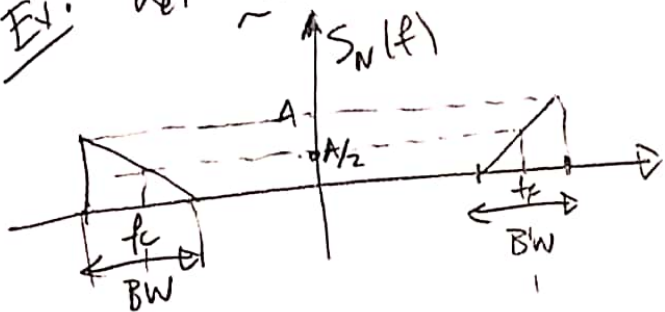
$$= \begin{cases} S_N(f-f_c) + S_N(f+f_c), & -f_c < f < f_c \\ 0, & \text{Other,} \end{cases}$$

Similarly,

$$\begin{aligned}
 S_{N_I N_Q}(f) &= \mathcal{F}\{R_{N_I N_Q}(z)\} \\
 &= S_N(f) * \frac{(\delta(f-f_c) - \delta(f+f_c))}{j2} - (j \operatorname{sgn}(f) S_N(f)) * \left(\frac{\delta(f-f_c) + \delta(f+f_c)}{2} \right) \\
 &= \frac{1}{j2} \left[\underbrace{[1 - \operatorname{sgn}(f-f_c)]}_{= \begin{cases} 0, & f > f_c \\ 2, & f < f_c \end{cases}} S_N(f-f_c) - \underbrace{[1 + \operatorname{sgn}(f+f_c)]}_{= \begin{cases} 2, & f > -f_c \\ 0, & f < -f_c \end{cases}} S_N(f+f_c) \right] \\
 &= \frac{1}{j} \cdot \begin{cases} S_N(f-f_c) - S_N(f+f_c), & -f_c < f < f_c \\ 0, & \text{Other.} \end{cases}
 \end{aligned}$$

Also, $S_Q(f) = S_I(f)$.
 $S_{N_Q N_I}(f) = -S_{N_I N_Q}(f)$.

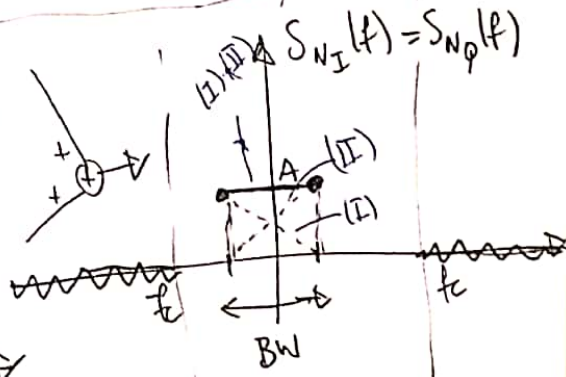
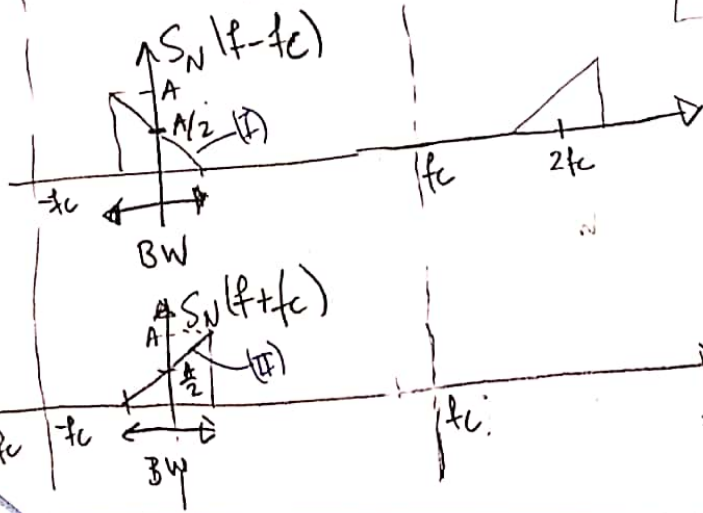
Ex: Let $x(t)$ be zero-mean WSS band-pass process with $S_N(f)$



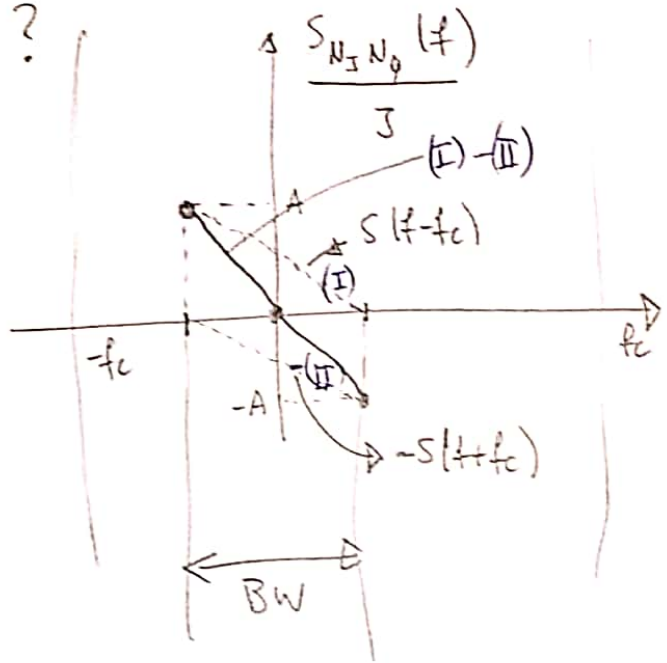
Find $S_{N_I}(f)$, $S_{N_Q}(f)$, $S_{N_I N_Q}(f)$.

$$S_{N_I}(f) = \begin{cases} S_N(f-f_c) \sqrt{(f_c-f)} & f > f_c \\ S_N(f+f_c) \sqrt{(f+f_c)} & f < -f_c \\ 0 & \text{elsewhere} \end{cases}$$

Soln.



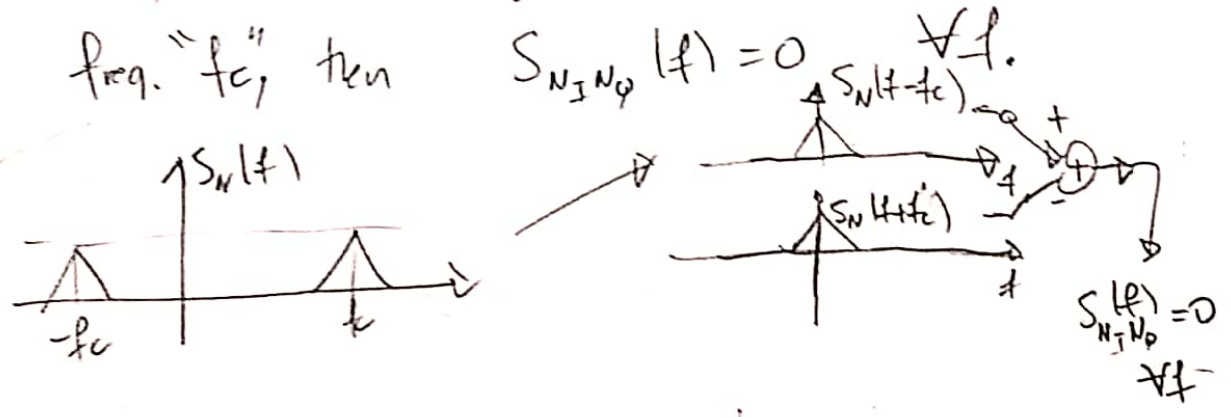
$S_{N_I N_Q}(f) = ?$



S_{01} , $S_{N_I}(f) = S_{N_Q}(f)$ is formed by sum of $+/-$ freq. components of $S_N(f)$ after modulating to low pass region $(-f_c < f < f_c)$.

$S_{N_I N_Q}(f)$ is formed by dif. of $+/-$ freq. components of $S_N(f)$ after modulating to the low-pass region.

Notes: (1) If $S_N(f)$ is symmetric around representation freq. " f_c ", then $S_{N_I N_Q}(f) = 0 \forall f$.



(2) If $\tilde{n}(t)$ is a band-pass ^{zero-mean} Gaussian process

→ $n_I(t)$ and $n_Q(t)$ are jointly WSS Gaussian processes,

(Since linear operations on Gaussian processes preserve Gaussianity).

(3) If $\tilde{n}(t)$ is a band-pass ^{zero-mean} Gaussian process and $S_n(f)$ is symmetric around $f=f_0$ and $f_c=f_0$, i.e. representation freq. for low pass equivalent is the symmetry center (as in Note ④) then $\tilde{n}_I(t)$ and $\tilde{n}_Q(t)$ are Gaussian and independent and identically distributed (iid).

Since, $S_{N_I N_Q}(f) = 0$ ← (from Note ④) → $r_{N_I N_Q}(z) = 0$

↓
 $\tilde{n}_I(t)$ and $\tilde{n}_Q(t-z)$
are uncorrelated

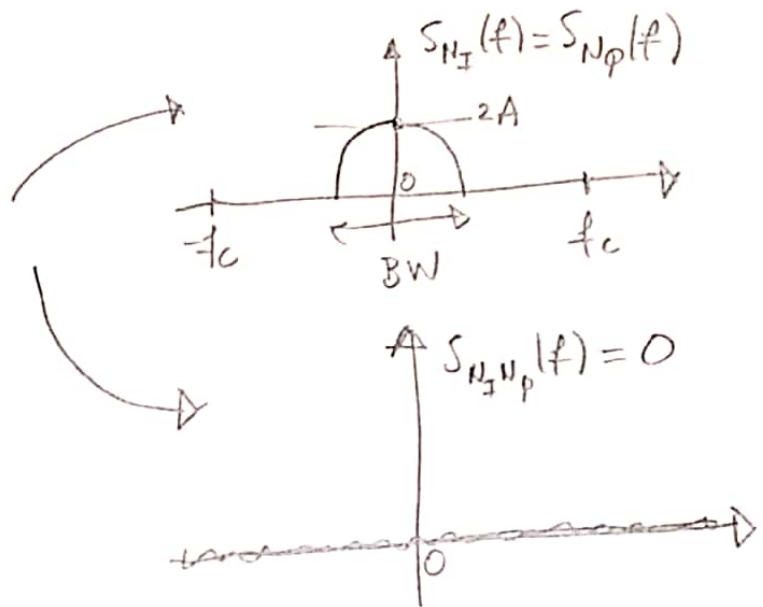
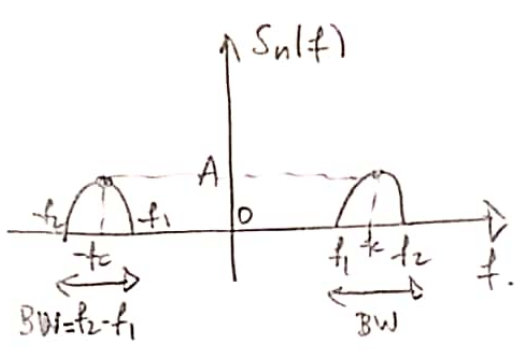
↓
 $n_I(t_1), n_Q(t_2)$ independent
(since Gaussian)
∀ t_1, t_2

Also, $S_{N_I}(f) = S_{N_Q}(f) \rightarrow r_{N_I}(z) = r_{N_Q}(z) \rightarrow n_I(t)$ and $n_Q(t)$ have the same auto-cor. (and zero-mean)

↓
 $n_I(t), n_Q(t)$ are Gaussian dist. with identical autocor. func. ← $n_I(t)$ and $n_Q(t)$ are identically dist.

In many practical applications, the low pass equivalent of noise process is assumed to be Gaussian with iid $n_I(t_1)$ and $n_Q(t_2)$ parts for all t_1 and t_2 .

In pictures, we have

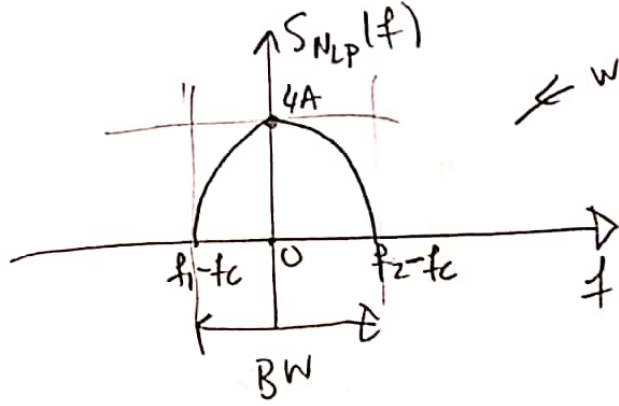


$$\tilde{n}_{LP}(t) = \tilde{n}_I(t) + j\tilde{n}_Q(t) \rightarrow r_{\tilde{n}_{LP}}(z) = E\{\tilde{n}_{LP}(t)\tilde{n}_{LP}^*(t-z)\}$$

$$= r_{n_I}(z) + r_{n_Q}(z) + j(r_{n_Q n_I}(z) - r_{n_I n_Q}(z))$$

(Since $S_{N_I N_Q}(f) = 0$)

$$S_{\tilde{n}_{LP}}(f) = 2S_{N_I}(f) = 2S_{N_Q}(f)$$



which is the result we expect from p. 25-26.