Phys 430 - 1^{st} HOMEWORK

1. Consider special dices that have 12 faces rather than 6. Suppose you throw N such dice. Let d_i be the value of the i^{th} dice, and let $d = \sum_{i=1}^{N} d_i$. Let $\rho_N(x)$ be the probability that $d = x$ for a system of N dice.

   (a) Show that

   $$\rho_N(x) = \sum_{i=1}^{12} \rho_1(i)\rho_{N-1}(x-i) \tag{1}$$

   (b) Calculate and sketch $\rho_3(x)$

   (c) Using a computer, draw the graph of $\frac{\rho_N(x)}{\rho_N^{max}}$ versus $\frac{x}{N}$, where ρ_N^{max} is the maximum value of $\rho_N(x)$, for $N = 3, 4, 20$ and see that the probability distribution gets narrower as N increases.

   (d) Estimate the relative fluctuation of d, i.e. $\frac{\Delta f}{\langle f \rangle}$ for $N = 3, 4,$ and 20

2. Consider two different quantum systems: system A and system B. Both of the systems can be only in two different states. If the wave function of the whole system is given by

   $$\Psi = a_1\Psi_A^1\Psi_B^1 + a_2\Psi_A^1\Psi_B^2 + a_3\Psi_A^2\Psi_B^2 + a_4\Psi_A^2\Psi_B^1 \tag{2}$$

   where $|a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2 = 1$ and Ψ_X^i ($X = A,\ B,\ i = 1,\ 2$) is the wavefunction of the i^{th} state of the X system. What is the quantum statistical matrix of system A? What is the quantum statistical matrix of system B?

3. Consider a mixture consisting of two different kinds of molecules, molecule A and molecule B. These molecules can undergo the reaction $nA \leftrightarrow mB + energy$ where the energy released is ϵ and n, m are positive integers. Let n_A be the number of A molecules per unit volume and n_B be the number of B molecules per unit volume. Assume that the molecules are distributed uniformly. The A molecules can be treated to form a subsystem of the whole system, similarly for B molecules. In this problem we will neglect the internal structure of the molecules, and hence treat them as point like particles. This will limit the approach to temperatures $T << \epsilon$

(a) The total number of molecules and the energy available for thermalization is no longer conserved, since the reaction changes the number of molecules and is exothermic. What are the new conserved quantities? (Hint: if the reaction would have been $2O_3 \leftrightarrow 3O_2$, it would not be the total number of molecules but the total number of atoms that is conserved)

(b) Show that in equilibrium, the temperature of both of the subsystems (that formed by the A molecules, and that formed by the B molecules) are equal.

(c) Write down the total entropy of the system.

(d) By equating the derivative of the total entropy with respect to the changing number of particles to zero, derive a relation for the particle densities of the A and B type molecules, in equilibrium (Let the equilibrium temperature be denoted by T_{eq}, and sketch the general behaviour of the relation that you have found.)

(e) Show that if the equilibrium temperature is raised by some external influence, the density of B molecules in equilibrium gets reduced.

4. Consider the four stages of the Carnot cycle. For each of the stages, calculate the work done, the heat exchange and the change in the energy of the Carnot engine.