1. In the class, while we were discussing the continuous symmetries, we said that grouping the states into irreducible representation, the matrix representation of the angular momentum operator becomes block diagonal, each block corresponding to a different l. Find the block corresponding to $l = 2$. For this aim, choose your basis states as:$\{|m = +2\rangle,\ |m = +1\rangle,\ |m = 0\rangle,\ |m = -1\rangle,\ |m = -2\rangle,\ \}$ and follow these steps:

   (a) In this basis, L_z is diagonal, write down its matrix.

   (b) Define the raising and lowering operators $L_\pm = \frac{1}{\sqrt{2}}(L_x \mp iL_y)$. The the mn component of $L_\pm$ is given by

   $$(L_\pm)_{mn} = \langle m|L_\pm|n\rangle = \hbar\sqrt{(l \mp n)(l \pm n + 1)}\delta_{m,n\pm1} \tag{1}$$

   What are the 5×5 matrices corresponding to $L_\pm$?

   (c) Express L_x and L_y in terms of $L_\pm$, and find the explicit expressions for the 5×5 matrices corresponding to L_x and L_y

2. The wave function for the lowest lying baryons can be written as the product of four pieces: space, spin, flavor and color as

   $$\Psi = \psi_{space}\psi_{spin}\psi_{flavor}\psi_{color} \tag{2}$$

   The lowest energy states are expected to have $l = 0$ (any rotation would in general cost additional energy), and hence ψ_{space} part is a symmetric function. The color piece ψ_{color} is completely anti-symmetric under the exchange of any two color indices (for reasons which will be discussed lated in the lecture). Hence, under the exchange of identical fermions, the product of the spin and the flavor piece has to be symmetric. Since we have three quarks in the baryons, we have to find irreducible representations for the product of three fundamental representations. For this aim follow the following strategy:

   (a) First, combine two fundamental representations and reduce the product to irreducible representation by symmetrizing and anti-symmetrizing:

   $$q_i q_j = T_{\{ij\}} + T_{[ij]} \tag{3}$$

where the curly/square brackets show symmetric/antisymmetric combination. What are the explicit expressions for $T_{\{ij\}}$ and $T_{[ij]}$? How many independent values does each have for $SU(2)$ (i.e. $i,\ j = 1,\ 2$)? For $SU(3)$ (i.e $i,\ j = 1,\ 2,\ 3$) (Note the the sum has to be N^2 for $SU(N)$)

(b) Now, add the third fundamental representation to each one of the T's separately and separate the product into their irreducible representations:

$$\begin{aligned} T_{\{ij\}}q_k &= T_{\{ijk\}} + T_{\{ij\}k} \\ T_{[ij]}q_k &= T_{[ijk]} + T_{[ij]k} \end{aligned} \tag{4}$$

where $T_{\{ijk\}}$ and $T_{[ijk]}$ are respectively the totally antisymmetric and symmetric combinations given as:

$$\begin{aligned} T_{\{ijk\}} &= \frac{1}{3}\left(T_{\{ij\}}q_k - T_{\{ik\}}q_j - T_{\{kj\}}q_i\right) \\ T_{[ijk]} &= \frac{1}{3}\left(T_{[ij]}q_k + T_{[ik]}q_j + T_{[kj]}q_i\right) \end{aligned} \tag{5}$$

and the mixed symmetry representations are the rest of the corresponding products. What are the dimensions of each of these representations for $SU(2)$ and $SU(3)$? Write down the explicit expression in terms of q_i of all these tensors (both those that have a definite symmetry, and those that have a mixed symmetry)

(c) The Δ resonance that we have discussed in the lecture is known to have isospin 3/2, we know this because in the nucleon-pion scattering the $I = 3/2$ amplitude dominates at the Δ resonance energy. Considering Δ^+ which has the same quark content as the proton, what is its flavor wave function? (it belongs to the representation that has dimension 4 since the Δ can be in 4 different isospin states corresponding to 4 possible values of the third component of isospin)

(d) What is the spin of Δ baryons? Note that the Δ baryon can be considered as the bound state of three identical fermions with various isospin states. Hence, the product of flavor and spin wavefunctions have to be completely symmetric. In the previous section, you have already shown that the flavor part is symmetric.

Thus the spin wavefunctions also has to be completely symmetric. What is the total spin of this state?

(e) The proton is known to have spin-1/2. What possible spin wavefunctions can the proton have? (there are two spin-1/2 representation, write down their explicit expressions)

(f) Can the proton have isospin-3/2?

(g) There are two possible (iso)spin-1/2 wavefunctions. The flavor-spin piece of the proton wavefunction has to be completely symmetric. Considering the first two quarks only, there are two possible flavor-spin products that are symmetric. None of the two have a definite symmetry property under the exchanges involving the third quark. Write down the linear combination of these two wavefunctions such that the linear combination is totally symmetric.

Questions From the Book

3. Show that the reaction $\pi^- + d \to n + n + \pi^0$ cannot occur for pions at rest

4. Show that, for pions with zero relative orbital angular momentum, the combination $\pi^+\pi^-$ is an eigenstate of CP with eigenvalue $+1$, and $\pi^+\pi^-\pi^0$ is an eigenstate of CP with eigenvalue -1

5. Show that a scalar meson cannot decay to three pseudoscalar mesons in a parity conserving process

6. Find a relation between the total cross-sections (at a given energy) for the reactions

$$\begin{aligned} \pi^- p &\to K^0\Sigma^0 \\ \pi^- p &\to K^+\Sigma^- \\ \pi^+ p &\to K^+\Sigma^+ \end{aligned} \tag{6}$$

(K has total isospin $I = 1/2$ and Σ baryons have isospin $I = 1$)

7. In which isospin states can (*a*) $\pi^+\pi^-\pi^0$, (*b*) $\pi^0\pi^0\pi^0$ exist? (*Hint:* First write the isospin function for a pair of pions, and then combine each with a third pion.)

8. State which of the following decays of the ρ meson ($J^P = 1^-$, $I = 1$) are allowed by the strong or electromagnetic interactions:

$$\begin{aligned} \rho^0 &\rightarrow \pi^+\pi^- \\ &\rightarrow \pi^0\pi^0 \\ &\rightarrow \eta^0\pi^0 \\ &\rightarrow \pi^0\gamma \end{aligned}$$

where the η is a pseudoscalar, isosinglet meson.