PHYS 430 - Second Midterm
29 April 2006

Name and Surname:
Student ID:
Department:
Signature:

INSTRUCTIONS

Write all your steps with explanations of why you do those steps. The questions might contain extra information or too few information. If the question does not contain sufficient information, make necessary assumptions, stating why those assumptions are necessary.
YOU HAVE 4 HOURS WITH A POSSIBILITY OF EXTENSION

1. Comment on the following concepts (Just writing equations will not gain you any points): (3 points each, 18 points total)

   Canonical Distribution

   Grand canonical Distribution

   Chemical Potential

   Boltzmann Distribution

   Fermi-Dirac and Bose-Einstein Distributions

   Equipartition Theorem

2. Consider the classical ideal gas which is made up of point like particles with no internal structures and whose energy momentum relation is given by $\epsilon = \alpha p^n$ where α is some constant and n is some integer. Assume that the system is confined to a volume V, is at a temperature T and has a chemical potential μ. Using the Grand Canonical Distribution (you will not get any points if you use canonical distribution, even if all the rest of your steps and your results are true.) (30 points)

   (a) Calculate the partition function Z and obtain Ω(15 points)

   (b) Calculate F, S, E and W. (5 points)

   (c) Calculate the equation of state of the system. (5 points)

   (d) Calculate C_V and show that for $n = 2$, the results that you have obtained reduces to the result of the classical equipartition theorem results. (5 points)

	ϵ_1	ϵ_2	ϵ_3	$\sum_i \epsilon_i$	E	E'
1						
2						
3						
4						
5						
6						
7						
8						

Table 1:

E	Degeneracy

Table 2:

3. Consider 3 dipoles that can have the energies $+\epsilon$ or $-\epsilon$ only. Let ϵ_i denote the energy of the i^{th} dipole. Assume that the energy of the system can be written in the form

$$E = -m^2 \left(\sum_i \epsilon_i \right)^2 + \lambda^2 \left(\sum_i \epsilon_i \right)^4 \tag{1}$$

where m and λ are some positive constants. (42 points)

(a) Fill Table 1 with the possible states of the system. In total, the system can be in 8 different states. (for the time being leave the last column empty) (5 points)

(b) In Table 2, write the possible values of energy that the system can have and also their degeneracy. (2 points)

(c) If the system has temperature T, calculate its partition function (3 points), its energy E (2 points) and $\langle \sum_i \epsilon_i \rangle$ (5 points)

(d) What is

$$\lim_{T\to 0}\langle \sum_i \epsilon_i \rangle \tag{2}$$

(5 points)

(e) Now assume that you add the term $\alpha \sum_i \epsilon_i$ to the energy of the system for some positive constant α. Fill the last column of Table 1 with the new energy values of the system. (5 points)

(f) For the new system, calculate $\langle \sum_i \epsilon_i \rangle$ as a function of α (10 points)

(g) In your new expression of $\langle \sum_i \epsilon_i \rangle$ for the modified system, first take the limit $T \to 0$ and then the limit $B \to 0$ and show that the value that you obtain is different from the value you have obtained in part (d). If you take the limits in the other order, you would have obtained the result of part (d). Hence the limits do not commute. (10 points)

(h) Note that in the original system, without the additional term, the energy does not change if you replace every ϵ_i with $-\epsilon_i$. It is said that the system has a symmetry. Under this symmetry $\sum_i \epsilon_i$ changes sign. $\langle \sum_i \epsilon_i \rangle$ is invariant under this symmetry only if $\langle \sum_i \epsilon_i \rangle = 0$ as you should have found in part (d). If you add the additional term, this symmetry is no longer respected, and hence $\langle \sum_i \epsilon_i \rangle$ acquires a non-zero value. And this is reflected in the value of $\langle \sum_i \epsilon_i \rangle$ in the limit $T \to 0$ at some non-zero value of α. If you take the limit $\alpha \to 0$ after the limit $T \to 0$, although the energy acquires the symmetry, $\langle \sum_i \epsilon_i \rangle$ still remains non-zero violating the symmetry. It is said that the symmetry of the system is broken.

4. In the class, we had shown that the free energy F of a classical ideal gas made up of particles with internal degrees of freedom can be written as:

$$F = -NT \ln \left[\frac{eV}{N} \left(\frac{mT}{2\pi\hbar^2} \right)^{\frac{3}{2}} \sum_k e^{-\beta\epsilon_k} \right] \tag{3}$$

where k specifies the internal state of the system and ϵ_k is the internal energy. Without knowing the structure of the molecule, it is not possible to calculate the sum. But note that the sum depends only on the temperature T of the system. Hence the free energy of the system can be written as

$$F = -NT \ln \frac{eV}{N} + Nf(T) \tag{4}$$

for some unknown function $f(T)$. Let the specific heat of the system be written as $C_V = Nc_V$ where c_V is the specific heat per particle. Assume that c_V is constant. (20 points)

(a) Show that

$$f(T) = -c_V T \ln T - \chi T + \epsilon_0 \tag{5}$$

where χ and ϵ_0 are some constants. (10 points)

(b) Calculate S, E, Ω, μ for this. (10 points)

You can use the following formulas/definitions without deriving them:

$$\begin{aligned}
dE &= TdS - PdV + \mu dN \\
dF &= -SdT - PdV + \mu dN \\
dW &= TdS + VdP + \mu dN \\
d\Phi &= -SdT + VdP + \mu dN \\
d\Omega &= -SdT - PdV - Nd\mu \\
F &= E - ST \ ; \quad W = E + PV \ ; \quad \Phi = E - ST + PV \ ; \quad \Omega = F - \mu N \\
S &= \ln \Delta\Gamma(E) \ ; \quad \Delta\Gamma(E) = \Delta E \frac{\partial}{\partial E} \Gamma(E) \\
\ln N! &\simeq N \ln N - N \\
\int_0^\infty x^n e^{-x} &= n! \\
\beta &= \frac{1}{T}, \qquad k = 1
\end{aligned}$$

For anything else, you need to derive it.