PHYS 430 - SECOND MIDTERM

Name and Surname:
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Department:
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1. For bosonic and fermionic gases, we have obtained the first correction to the equation of state, and we have shown that

$$PV = NT\left(1 \pm A\frac{V}{N}\right)$$

where A is some positive constant and $+$ sign corresponds to fermions and the $-$ sign corresponds to bosons. That is, the pressure of a fermion gas is higher than the pressure of the classical gas which is higher than the pressure of a boson gas. Explain this result in terms of exchange effects of identical particles. (10 points)

2. When we were discussing the fermionic and bosonic systems, the occupation number of a state which has energy ϵ approached the classical maxwell distribution when $e^{\beta(\epsilon-\mu)} >> 1$ for both the fermions and the bosons. What does this limit correspond to physically for bosons and for fermions? How should you adjust the external parameter in order to obtain this limit? (For example, for a real gas, we know that it approaches the ideal case when we rarefy the gas.) (For a given system, since you don't have control over which level the particles should go, saying that energy should be large is not something that we have control of.) (20 points)

3. Consider a degenerate gas (i.e. at $T = 0$) of fermions. Suppose that the relation between the momentum and the energy of the fermions is given by $\epsilon = cp$ where p is the magnitude of their momenta. (This energy relation corresponds to ultra relativistic particles for which the rest mass energy can be neglected.) Calculate the pressure of this gas and express you result in terms of V, N, $\hbar$ and c. (20 points)

4. Consider a classical gas which has the energy momentum relation $\epsilon = cp$ where p is the magnitude of the momentum. Calculate the free energy F, the entropy S, the energy E and the specific heat at constant pressure c_P for this gas. (20 points)

5. Let us consider a hypothetical system defined as follows: Consider the corners of a square, and denote them A, B, C, and D. A is the top

left corner, B is the top right corner, C is the bottom right corner and D is the bottom left corner. Suppose that each corner can be assigned a number $n_X = \pm 1$ ($X = A,\ B,\ C$, or D) randomly. Each different assignment to the numbers n_X defines a state of the system. We will define the energy of the system as

$$E = -\epsilon \sum_{<XX'>} n_X n_{X'}$$

where $< XX' >$ means that X and X' are corners that lie at the end of the same side. In other words, consider each one of four sides of the square. If, the corners connected by this side have the same number, subtract ϵ from the energy, if they have different numbers, then add ϵ to the energy. (Physically, you can think of 4 electrons sitting at the corners of the square. Their spin can be pointing either up, corresponding to assigning $+1$ to that corner, or pointing down, corresponding to assigning -1 to that corner. And here, we are considering only nearest neighbor spin interactions). Suppose that the system is in contact with a heat reservoir at a temperature T. (50 points)

a) Write down all possible assignment of numbers to the corners, in Table 1 using the first four columns, i.e. write all possible states that the system can be in. You should have 16 different possible assignments. For each state, calculate the energy and the sum of the numbers assigned at each corner and write them in the fifth and sixth columns of the same table. (5 points)

b) Fill Table 2. The first column is the possible energy values that the system can have, i.e. distinct values that you have on the fifth column of Table 1. The second column, degeneracy, is the different number of ways that the system can have that energy, i.e. the number of times that energy appears on the fifth column of Table 1. (5 points)

c) Calculate the partition function $Z = \sum_{states} e^{-\beta E_s}$ and obtain the free energy $F = -T \ln Z$ for the system. (5 points)

d) Calculate the average of the sum of the numbers, i.e. $\langle \sum n_X \rangle$ that is of the sixth column of Table 1. (Note that, different states having the same energy might have different $\sum n_X$.) (5 points)

e) Now, let's add another term to the energy so that the total energy now reads:

$$E' = -\epsilon \sum_{<XX'>} n_X n_{X'} - \alpha \sum_X n_X \tag{1}$$

i.e. subtract α from the energy for each corner which is assigned $+1$ and add α to the energy for each corner assigned -1. In the seventh column in Table 1, write the new energies for each of the states. (In the spin analogy to this system, this additional term corresponds to an external magnetic field.) And fill Table 3, with the distinct energy levels and their corresponding degeneracies (similar to Table 2). (5 points)

f) Show that $< \sum n_X >= -T\frac{\partial}{\partial\alpha}\ln Z$ (5 points)

g) Calculate the partition function and obtain $< \sum n_X >$ as a function of ϵ, α and T. First take the limit $T \to 0$ and then take the limit $\alpha \to 0$ (10 points)

h) When you take the limit $\alpha \to 0$, your modified system returns to the original system (if you consider the α term as a perturbation, you are turning of the perturbation, or, in the spin analogy, you are turning off the external magnetic field). The results that you obtain in part (d) using the original system directly, and the result that you obtain in part (g) are different. Discuss/Comment. Note also that, in the original energy expression, if you replace all n_X by $-n_X$, your energy does not change, that is to say, the system is invariant under this transformation. How does $< \sum n_X >$ behave under this symmetry? What can you say about the symmetry when you add the additional term? when you calculate $< \sum n_X >$ with the additional term?(10 points)

$(0 < \alpha << \epsilon)$

You can use the following formulas/definitions without deriving them:

$$\begin{aligned}
dE &= TdS - PdV + \mu dN \\
dF &= -SdT - PdV + \mu dN \\
dW &= TdS + VdP + \mu dN \\
d\Phi &= -SdT + VdP + \mu dN \\
F &= E - ST \ ; \quad W = E + PV \ ; \quad \Phi = E - ST + PV \\
S &= \ln \Delta\Gamma(E) \ ; \quad \Delta\Gamma(E) = \Delta E \frac{\partial}{\partial E}\Gamma(E) \\
\ln N! &\simeq N \ln N - N \\
\beta &= \frac{1}{T}, \qquad k = 1
\end{aligned}$$

For anything else, you need to derive it.

n_A	n_B	n_C	n_D	E	$\sum n_X$	E'

Table 1:

E	Degeneracy

Table 2:

E'	Degeneracy

Table 3: