

8th Homework
Due: 13 December 2007

1. Solve the following differential equations, given the initial conditions.

- (a) $y''(x) - 2y'(x) + y(x) = 0$ with $y(0) = 0, y'(0) = 1$.
- (b) $y''(x) + 2y'(x) + y(x) = 0$ with $y(0) = 0, y'(0) = 1$
- (c) $y''(x) + y(x) = 0$ with $y(0) = 0, y'(0) = 1$.
- (d) $y''(x) - y(x) = 0$ with $y(0) = 0, y'(0) = 1$

where $y'(x) = \frac{dy}{dx}$ and $y''(x) = \frac{d^2y}{dx^2}$ (*Hint:* Assume a solution of the form $y = e^{\alpha x}$ and solve for alpha)

2. The damped harmonic oscillator satisfies the differential equation

$$y''(t) + \gamma y'(t) + ky(t) = 0$$

where the ' denotes differentiation with respect to t . Without explicitly solving the differential equation, show that

$$\frac{d}{dt}\tilde{E} = 0$$

where

$$\tilde{E} = e^{\frac{\gamma}{m}t} \left[\frac{1}{2}my'(t)^2 + \frac{1}{2}ky(t)^2 + \frac{\gamma}{2}y(t)y'(t) \right]$$

3. A complex number is a number that can be written in the form $z = a + ib$ where a and b are some real numbers and $i^2 = -1$. a is called the real part of z , i.e. $a = \text{Re}(z)$, and b is called the imaginary part of z , i.e. $b = \text{Im}(z)$. If a complex number is given by $z = a + ib$ for some real numbers a and b , then the complex conjugate of z is defined as $z^* = a - ib$. The absolute value of z is defined as $|z| = \sqrt{z^*z}$. The exponent of z is given by

$$e^z = e^{a+ib} = e^a (\cos(b) + i \sin(b)) \tag{1}$$

- (a) Calculate the absolute values of $z = e^{ib}$, $z = 4 + i3$
- (b) Show that if $z = 1/(a + ib)$, then $|z| = \sqrt{a^2 + b^2}$, $\text{Re}(z) = \frac{a}{a^2 + b^2}$, $\text{Im}(z) = -\frac{b}{a^2 + b^2}$ (*Hint:* Multiply and divide z by z^*)

(c) For the given complex numbers z , find r and θ such that $z = re^{i\theta}$

$$z = 3 + i4$$

$$z = 3e^{i2} + e^{i\pi}$$

$$z = e^{2+i} + 2e^{1-i}$$

4. Consider two gravitating identical bodies of masses M each, separated by a distance R . Consider a smaller mass $m \ll M$ placed at the equilibrium point between the two larger masses. Suppose the smaller mass is displaced slightly by an amount x from the equilibrium position in the direction (i) perpendicular or (ii) parallel to the line joining the two larger masses. Find the equations governing the subsequent motion of the particle. Along which direction is the position of the particle a stable point for small disturbances? (*Hint:* For $x \ll 1$, $(1+x)^n \simeq 1 + nx + \frac{1}{2}n(n-1)x^2$ for all values of n)
5. Consider a platform of mass M at rest on a frictionless surface. On top of this platform, assume that there is a spring with spring constant k , placed horizontally. At one end of the spring, there is attached a body of mass m , and the other end is fixed on the platform. At $t = 0$, the mass is displaced horizontally from the equilibrium position by a distance A and then released with zero initial velocity. Write an expression for the subsequent motion of the platform. (*Hint:* In order to write an expression, you first have to choose a reference frame and place a coordinate grid on that frame. In your answer, you have to specify which coordinates and which reference frame you use. Ignore the friction forces between the platform and the body.)
6. Consider the wavefunction $y = (3.0 \text{ m}) \cos(5.0\tilde{x} - 8.0\tilde{t}) + (4.0 \text{ m}) \sin(5.0\tilde{x} - 8.0\tilde{t})$ where $\tilde{x} = x/(1 \text{ m})$ and $\tilde{t} = t/(1 \text{ s})$. This wave is a superposition of two wavefunctions. Show that this wave function can be written in the form $y = A \cos(5.0\tilde{x} - 8.0\tilde{t} + \delta)$. What are the values of A and δ ?
7. Suppose you take a loop of rope and make it rotate about its center at speed v . The centrifugal tendency of the segments of rope will then stretch it out along a circle of some radius R . What is the tension in the rope under these conditions? Show that the speed of transverse waves on the rope (relative to the rope) coincides with the speed of rotation v .