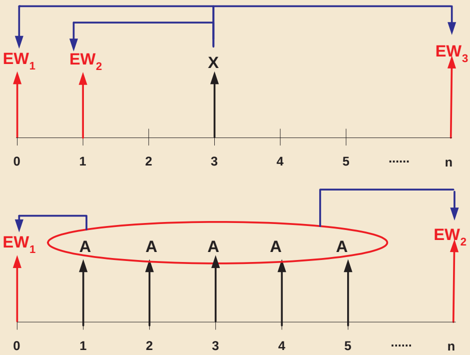
We learned:

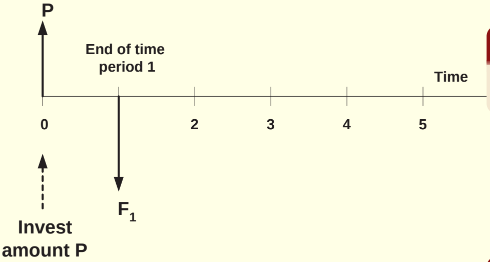
- \$100 cash today is preferred over \$100 a year from now
  - there is time value of money
  - in the form of willingness of banks, businesses, and people to pay interest for its use
- So this rises the question of how to compare various cash flows dispersed over the time horizon?



We learned:

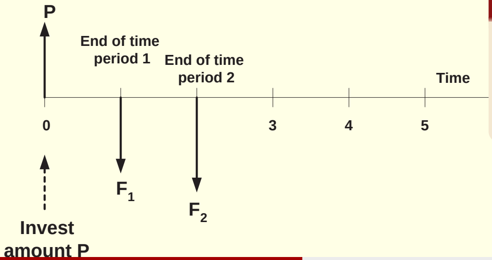
- So this rises the question of how to compare various cash flows dispersed over the time horizon?
    - Economic equivalence exists between cash flows that have the same economic effect and therefore be traded for one another
    - Economic Equivalence refers to the fact that: any cash flow - whether a single payment or a series of payments - can be converted to an equivalent cash flow at any point in time.
    - And that equivalence depends on the interest rate
- To convert the cash flows - whether a single payment or a series of payments - to an equivalent cash flow at any point in time, we need engineering economy factors
- This week we will focus on the derivations of the most commonly used engineering economy factors that take time value of money into account

The most fundamental factor in engineering economy is the one that determines the amount of money F accumulated after n periods from a single present worth P, with interest compounded one time per period.



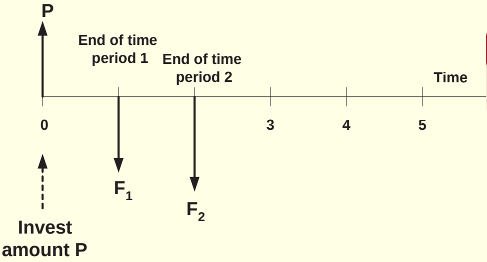
Note That

$$F_1 = P + Pi = P(1+i)$$



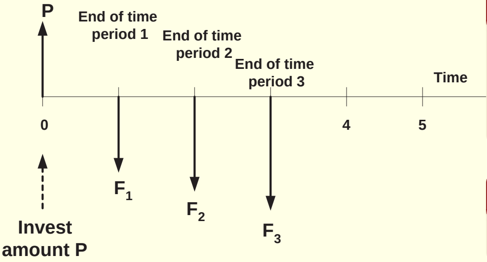
Note That

$$\begin{aligned} F_2 &= F_1 + F_1 i = F_1(1+i) \\ &= P(1+i)(1+i) \\ &= P(1+i)^2 \end{aligned}$$



Note That

$$F_2 = P(1+i)^2$$



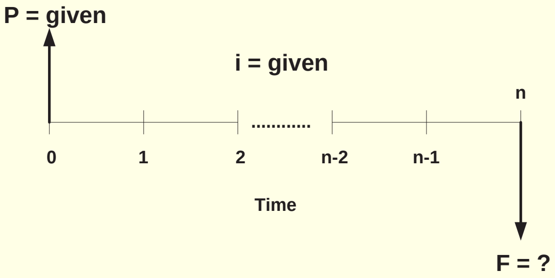
Note That

$$\begin{aligned} F_3 &= F_2 + F_2 i = F_2(1+i) \\ &= P(1+i)^2(1+i) \\ &= P(1+i)^3 \end{aligned}$$

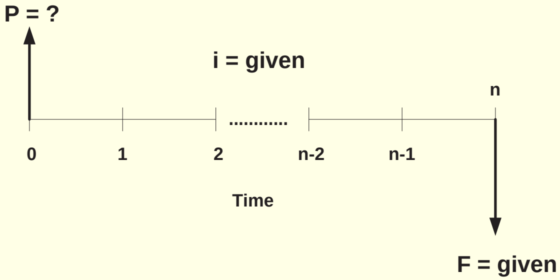
By Induction

$$F = P(1+i)^n$$

- The factor $(1+i)^n$ is called the single payment compound amount factor, or simply F/P factor.
- This conversion factor yields the future amount F of an initial amount P after n years at interest rate i, when it is multiplied by P.



- Reverse the situation to determine P value for stated amount F that occurs n periods in the future
- $\frac{1}{(1+i)^n}$ is the single payment present worth factor or the P/F
- This factor yields the present amount P of an given future amount F after n years at interest rate i, when it is multiplied by F.



- The two factors derived so far are for single payments.
- A standard notation is developed for all factors.
  - includes: two cash flow symbols, interest rate and number of periods
  - general form is $(X/Y, i, n)$
  - X is the cash flow that is sought
  - Y is the cash flow that is given
  - i is the interest rate in percent
  - n is the time periods involved
- $(F/P, 8\%, 20)$ represents the factor to calculate future value F accumulated in 20 periods at 8% interest rate for an initial 1 dollar invested at time zero.
- $P*(F/P, 8\%, 20)$ gives you the future F value.
- Tables of factors are available from 0.25 to 50% interest rates and time periods 1 to large n values.
- For a given interest rate, the factor value is found at the intersection of the factor name and n.
- $(P/F, 8\%, 10) = \frac{1}{(1+0.08)^{10}} = \frac{1}{1.08^{10}} = 0.4632$

Example

An industrial engineer receives a bonus of \$12,000, and decides to invest the whole amount to the bank for 20 years at 8% per each year. Find the amount amount of money he will get 20 years later.

By Formula

$F = ?$, $P = 12,000$, $i = 8$ percent and $n = 20$

$$\begin{aligned} F = P(1+i)^n &= 12,000(1+0.08)^{20} \\ &= 12,000(1.08)^{20} = 12,000(4.660957) = \$55,931.5 \end{aligned}$$

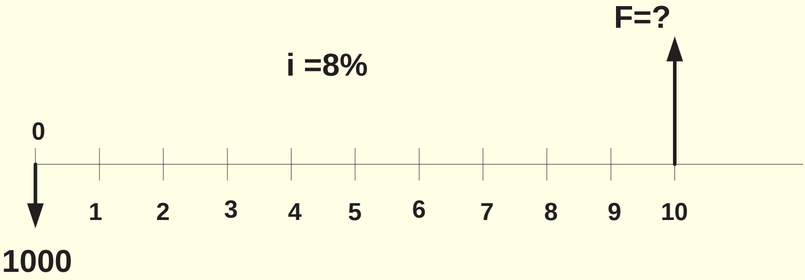
By Standard Factor

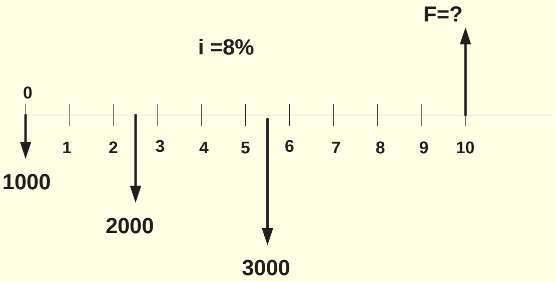
$F = ?$, $P = 12,000$, $i = 8\%$ per year and $n = 20$

$$\begin{aligned} F &= P(F/P, 8\%, 20) \\ &= 12,000(4.6610) = \$55,932 \end{aligned}$$

In Class Work 2

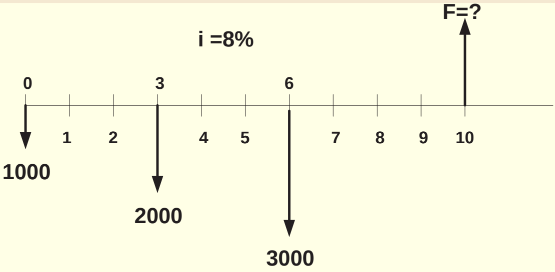
Foreman family decided to start investing to their son Eric's college fund. They started it by depositing \$1,000 on January 1, 2008 to the Washington Mutual Bank. They plan to deposit \$2,000 on June 31, 2010 and \$3,000 on June 31, 2013. If they manage to invest according to that plan, how much money will they have on on January 1, 2018 if the interest is taken as 8% per each year.





End-of-period convention

All cash flows occur at the end of the interest period.



By Formula:

$$
\begin{aligned}
F &= 3000(1.08)^4 + 2000(1.08)^7 + 1000(1.08)^{10} \\
&= \$9668.04
\end{aligned}
$$

OR

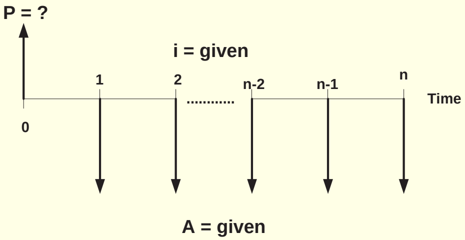
$$
\begin{aligned}
F &= \big[[2000 + 1000(1.08)^3](1.08)^3 + 3000\big](1.08)^4 \\
&= \$9668.04
\end{aligned}
$$

OR

By Standard Notation:

$$
\begin{aligned}
F &= 3000(F/P, 8\%, 4) + 2000(F/P, 8\%, 7) + 1000(F/P, 8\%, 10) \\
&= 3000(1.3605) + 2000(1.7138) + 1000(2.1589) \\
&= \$9668
\end{aligned}
$$

The equivalent present worth P of a uniform series A of end-of-period cash flows can be determined by considering each A value as a future worth F and calculating its present worth with the P/F factor.



$$
\begin{aligned}
P &= A\left[\frac{1}{(1+i)^1}\right] + A\left[\frac{1}{(1+i)^2}\right] + \cdots + A\left[\frac{1}{(1+i)^{n-1}}\right] + A\left[\frac{1}{(1+i)^n}\right] \\
&= A\left[\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right]
\end{aligned}
$$

$$\begin{aligned}
P &= A\left[\frac{1}{(1+i)^1}+\frac{1}{(1+i)^2}+\cdots+\frac{1}{(1+i)^{n-1}}+\frac{1}{(1+i)^n}\right]\\
\frac{P}{1+i} &= A\left[\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\cdots+\frac{1}{(1+i)^{n}}+\frac{1}{(1+i)^{n+1}}\right]\\
\frac{P}{1+i}-P &= A\left[\frac{1}{(1+i)^{n+1}}-\frac{1}{(1+i)^1}\right]\\
\frac{-iP}{1+i} &= A\left[\frac{1}{(1+i)^{n+1}}-\frac{1}{(1+i)^1}\right]\\
P &= \frac{A}{-i}\left[\frac{1}{(1+i)^n}-1\right]=A\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]
\end{aligned}$$

$\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]$ is the uniform series present worth factor, P/A factor used to calculate the equivalent present worth P in year 0 for a uniform end-of-period series of A values beginning at the end of period 1 and extending for n periods.

To reverse the situation, the present worth P is known and the equivalent uniform-series amount A is sought. The first A value occurs at the end of period 1. Then we have:

$$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right]$$

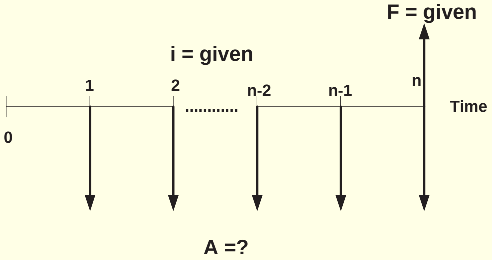
$\left[\frac{i(1+i)^n}{(1+i)^n-1}\right]$ is the capital recovery factor or A/P factor. It calculates the equivalent uniform annual worth A over n years starting from the end of year 1 for a given P in year 0.

CAUTION!

These formulas are derived with the present worth P and the first uniform annual amount A one period apart. Therefore, the present worth P must always be located one period prior to the first A.

Standard notations for these two factors are: $(P/A, i\%, n)$ and $(A/P, i\%, n)$.

This time, the cash flow diagram we are dealing with is:



We already know that:

$$
\begin{aligned}
A &= P\left[\frac{i(1+i)^n}{(1+i)^n-1}\right] \\
&= \frac{F}{(1+i)^n}\left[\frac{i(1+i)^n}{(1+i)^n-1}\right] \\
&= F\left[\frac{i}{(1+i)^n-1}\right]
\end{aligned}
$$

$\left[\frac{i}{(1+i)^n-1}\right]$ is the sinking fund factor or A/F factor, used to determine the annual series that is equivalent to a given future worth F.

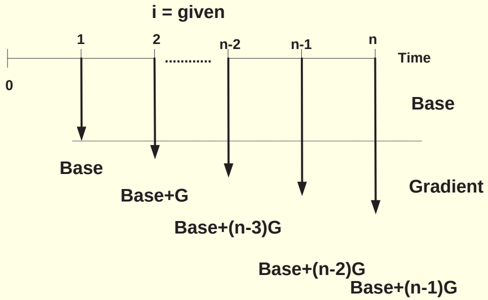
CAUTION!

The uniform series A begins at the end of period 1 and continues through the period of the given F. Therefore, the future worth F is at the same period as the last A.

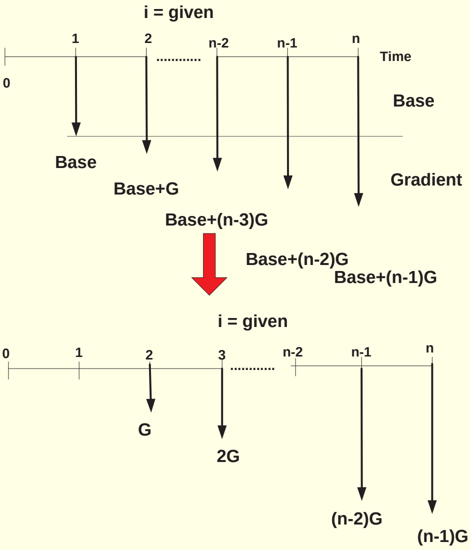
Obviously, $\left[\frac{(1+i)^n-1}{i}\right]$ is the F/A factor. When multiplied by the given uniform annual amount A, it yields the future worth of the uniform series.

The standard notations for these two factors are: $(F/A, i\%, n)$ and $(A/F, i\%, n)$.

- An arithmetic gradient is a cash flow that either increases or decreases by a constant amount.
- The cash flow, changes by the same amount each period.
- The amount of decrease or the increase is the gradient or G.
- Therefore, it is composed of base amount and gradient part.



- The cash flow in period n (CF_n) is given as: $CF_n =$ base amount $+(n-1)G$



CAUTION:Conventional Gradient

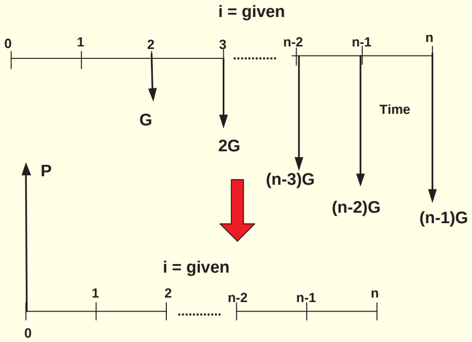
The gradient begins between years 1 and 2.

$$
\begin{aligned}
P &= G\left[\frac{1}{(1+i)^2}\right] + 2G\left[\frac{1}{(1+i)^3}\right] + \cdots + (n-2)G\left[\frac{1}{(1+i)^{n-1}}\right] \\
&+ (n-1)G\left[\frac{1}{(1+i)^n}\right] \\
&= G\left[\frac{1}{(1+i)^2} + \frac{2}{(1+i)^3} + \cdots + \frac{n-2}{(1+i)^{n-1}} + \frac{n-1}{(1+i)^n}\right] \\
P(1+i) &= G\left[\frac{1}{(1+i)} + \frac{2}{(1+i)^2} + \cdots + \frac{n-2}{(1+i)^{n-2}} + \frac{n-1}{(1+i)^{n-1}}\right] \\
Pi &= G\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}}\right] \\
&- G\frac{n-1}{(1+i)^n}
\end{aligned}
$$

$$
\begin{aligned}
Pi &= G\left[\frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-2}} + \frac{1}{(1+i)^{n-1}} + \frac{1}{(1+i)^n}\right] \\
&- G\frac{n}{(1+i)^n}
\end{aligned}
$$

$$P = \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$$

$\frac{1}{i}\left[\frac{(1+i)^n-1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]$ is the arithmetic-gradient present worth factor or P/G factor that converts the arithmetic gradient (without base amount) for n years to the present worth at year 0. The standard notation is $(P/G, i\%, n)$.



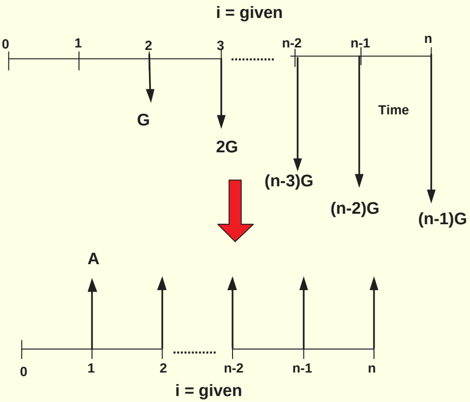
The equivalent uniform annual series A for an arithmetic gradient G is found by multiplying $(P/G, i\%, n)$ by (A/P,i%,n)

$$\begin{aligned} A &= G(P/G, i\%, n)(A/P, i\%, n) \\ &= G(A/G, i\%, n) \end{aligned}$$

In equation form:

$$\begin{aligned} A &= \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right]\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] \\ &= G\left[\frac{1}{i} - \frac{n}{(1+i)^n - 1}\right] \end{aligned}$$

$\left[\frac{1}{i} - \frac{n}{(1+i)^n-1}\right]$ is the arithmetic-gradient uniform-series factor and is identified by $(A/G, i\%, n)$. This factor converts:



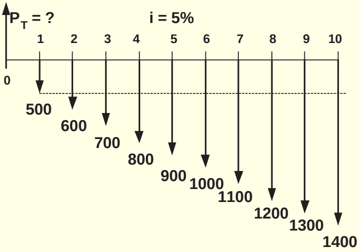
An F/G factor (arithmetic-gradient future worth factor)can be found by multiplying P/G and F/P factors. The resulting $(F/G, i\%, n)$ is:

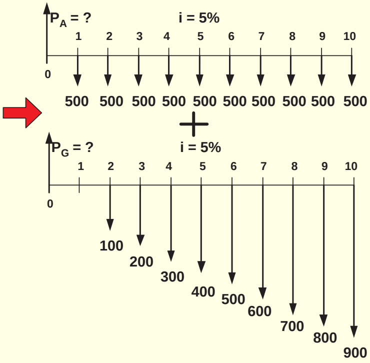
$$\begin{aligned} F &= G(P/G, i\%, n)(F/P, i\%, n) \\ &= G(F/G, i\%, n) \\ &= \frac{G}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right](1+i)^n \\ &= G\left[\frac{1}{i}\left(\frac{(1+i)^n - 1}{i} - n\right)\right] \end{aligned}$$

- The total present worth P_T of a gradient series must consider the base and the gradient separately.
- Let P_A be the present worth of the base amount; uniform series amount A starting from the end of period 1 extending through period n.
- Let P_G be the present worth of an increasing gradient and $-P_G$ be the present worth of an decreasing gradient.
- Therefore: $P_T = P_A + P_G$ and $P_T = P_A - P_G$ for increasing and decreasing gradient series, respectively.

Example

Three counties in Florida agreed to pool tax resources already designated for county-maintained bridge refurbishment. At a recent meeting, the county engineers estimated that a total of $500,000 will be deposited at the end of next year into an account for the repair of old bridges throughout the three-county area. Further, they estimate the deposits will increase by $100,000 per year for only 9 years thereafter, then cease. Determine the equivalent present worth if county funds earn interest at a rate of 5% per year.





$P_T = P_A + P_G$, we have
$G = 100$ and $A = 500$.

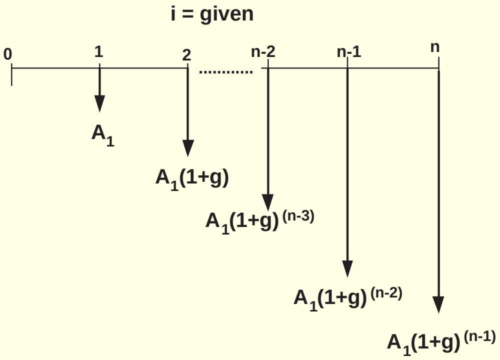
Solution by Standard Notation

$$\begin{aligned} P_T &= 500(P/A, 5\%, 10) + 100(P/G, 5\%, 10) \\ &= 500(7.7217) + 100(31.6520) = 7026,05 => \$7,026,050 \end{aligned}$$

Solution by Formulas

$$\begin{aligned} P_T &= 500\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] + 100\frac{1}{i}\left[\frac{(1+i)^n - 1}{i(1+i)^n} - \frac{n}{(1+i)^n}\right] \\ &= 500\left[\frac{(1.05)^{10} - 1}{0.05(1.05)^{10}}\right] + 100\frac{1}{0.05}\left[\frac{(1.05)^{10} - 1}{0.05(1.05)^{10}} - \frac{10}{(1.05)^{10}}\right] \\ &= 500(7.7217) + 100(31.6520) = 7026,05 => \$7,026,050 \end{aligned}$$

- Cash flow series that increase or decrease from period to period by constant percentage.
- g is the constant rate of change in decimal form, by which amounts change
- The series starts at the end of year 1, with amount A_1, which is not considered as a base amount



The total present worth P_g for the entire cash flow series can be derived as:

$$\begin{aligned} P_g &= A_1\left[\frac{1}{(1+i)}\right] + A_1\left[\frac{(1+g)}{(1+i)^2}\right] + \cdots + A_1\left[\frac{(1+g)^{n-2}}{(1+i)^{n-1}}\right] \\ &+ A_1\left[\frac{(1+g)^{n-1}}{(1+i)^n}\right] \\ &= A_1\left[\frac{1}{(1+i)} + \frac{(1+g)}{(1+i)^2} + \cdots + \frac{(1+g)^{n-2}}{(1+i)^{n-1}} + \frac{(1+g)^{n-1}}{(1+i)^n}\right] \end{aligned}$$

$$P_g\left[\frac{(1+g)}{(1+i)}\right] = A_1\left[\frac{1+g}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \cdots + \frac{(1+g)^{n-1}}{(1+i)^n} + \frac{(1+g)^n}{(1+i)^{n+1}}\right]$$

$$\begin{aligned} P_g\left[\frac{(1+g)}{(1+i)} - 1\right] &= A_1\left[\frac{(1+g)^n}{(1+i)^{n+1}} - \frac{1}{(1+i)}\right] \\ P_g(g-i) &= A_1\left[\frac{(1+g)^n}{(1+i)^n} - 1\right] \end{aligned}$$

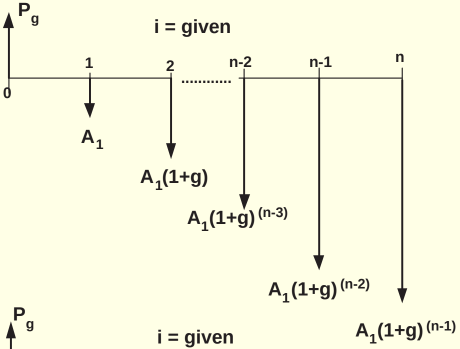
Therefore, when $i \neq g$;

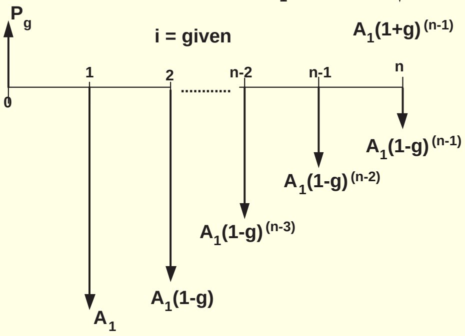
$$P_g = A_1 \left[\frac{\left[1 - \frac{(1+g)^n}{(1+i)^n}\right]}{(i-g)} \right]$$

when $i = g$;

$$\begin{aligned} P_g &= A_1 \left[\frac{1}{(1+i)} + \frac{1}{(1+i)} + \cdots + \frac{1}{(1+i)} + \frac{1}{(1+i)} \right] \\ &= A_1 \left[\frac{n}{(1+i)} \right] \end{aligned}$$

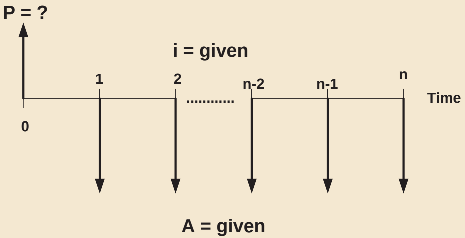
These two factors, depending on the case whether $g = i$ or not, will transform the geometric-gradient series starting at the end of period 1 with amount A_1 that is changing with constant rate g to the the present worth P_g at time 0. The standard notation for the factor is $(P/A, g, i, n)$.


Pg
i = given
0
1
2
....
n-2
n-1
n
A1
A1(1+g)
A1(1+g) (n-3)
A1(1+g) (n-2)
A1(1+g) (n-1)


Pg
i = given
0
1
2
....
n-2
n-1
n
A1(1-g) (n-1)
A1(1-g) (n-2)
A1(1-g) (n-3)
A1(1-g)
A1

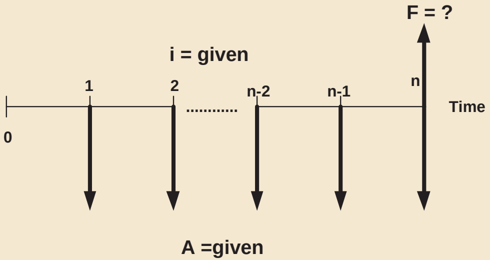
CAUTION on using P/A factor: $(P/A, i\%, n)$

The present worth P and the first uniform annual amount A are one period apart. Therefore, the present worth P must always be located one period prior to the first A.



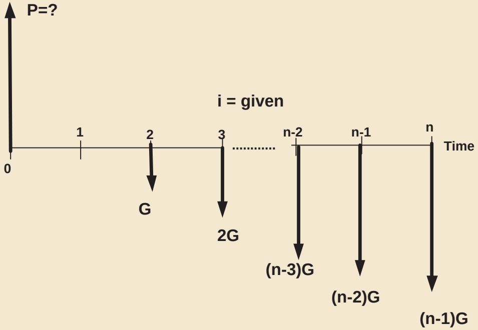
CAUTION on using F/A factor: $(F/A, i\%, n)$

The uniform series A begins at the end of period 1 and continues through the period n. Therefore, the future worth F is at the same period as the last A.



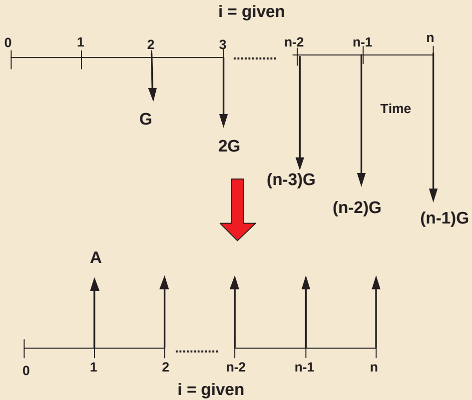
CAUTION on using P/G factor: : $(P/G, i\%, n)$

The gradient begins between years 1 and 2, continues through the period n. Therefore, the present worth of an arithmetic gradient will always be located two periods before the gradient starts.



CAUTION on using A/G factor: : $(A/G, i\%, n)$

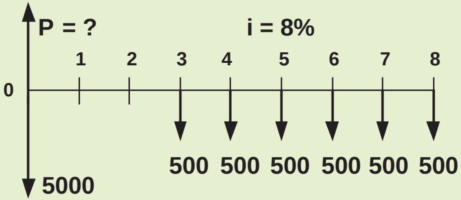
The gradient begins between years 1 and 2, continues through the period n. Therefore, the equivalent annual series of an arithmetic gradient will start from period 1 and continue through the period n.



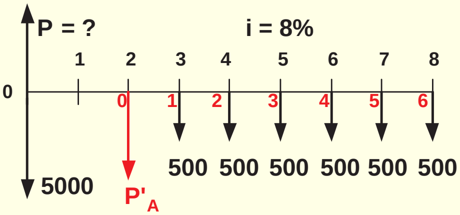
Most estimated cash flow series do not fit exactly the series for which the factors are derived so far. Therefore, we need to combine or shift the given series. Renumbering the cash flow diagram is a good practice.

Example

The engineering company just purchased new CAD software for \$5,000 now and annual payments of \$500 per year for 6 years, starting 3 years from now for annual upgrades. What is the present worth of the payments if the interest rate is 8% per year.



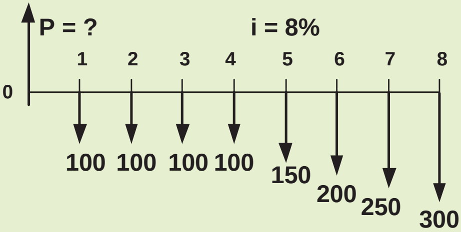
$P = 5000 + P_A$

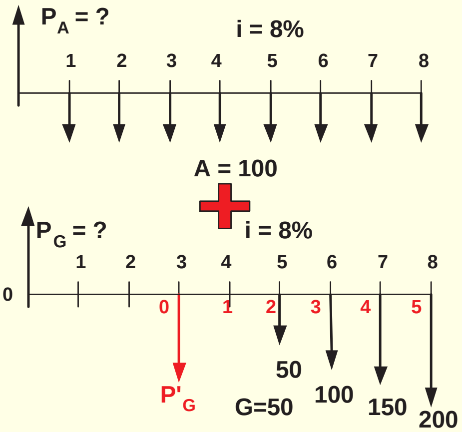


$$
\begin{aligned}
P &= 5000 + P'_A(P/F, 8\%, 2) \\
&= 5000 + 500(P/A, 8\%, 6)(P/F, 8\%, 2) \\
&= 5000 + 500(4.6229)(0.8573) \\
&= \$6981.60
\end{aligned}
$$

Example

The average inspection cost on robotics manufacturing line has been tracked for 8 years. Cost averages were steady at \$ 100 per completed unit for the first 4 years, but have increased consistently by \$50 per unit for each of the last 4 years. Find the present worth equivalent of the cost averages assuming that the applicable interest rate is 8% ?

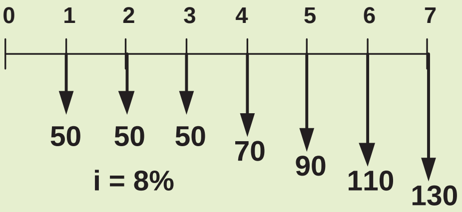




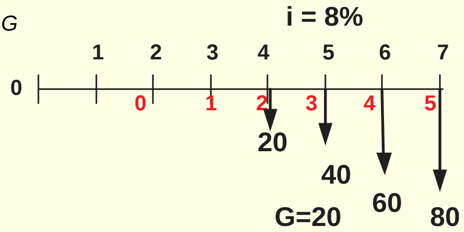
$$\begin{aligned} P &= P_A + P_G = P_A + P'_G(P/F, 8\%, 3) \\ &= 100(P/A, 8\%, 8) + 50(P/G, 8\%, 5)(P/F, 8\%, 3) \\ &= 100(5.7466) + 50(7.3724)(0.7938) = \$867.27 \end{aligned}$$

Example

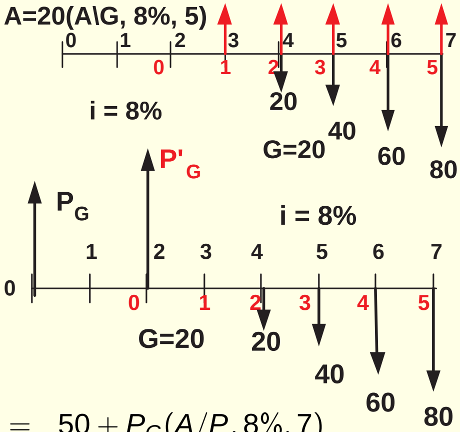
Compute the equivalent annual series in years 1 through 7 for the cash flow estimates given below when applicable interest rate is 8%.



$A = 50 + A_G$



Can we use the A/G factor?



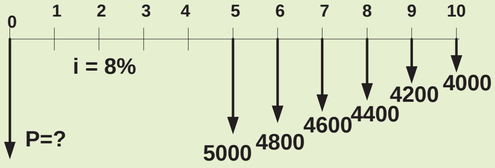
$$
\begin{aligned}
A &= 50 + P_G(A/P, 8\%, 7) \\
&= 50 + P'_G(P/F, 8\%, 2)(A/P, 8\%, 7) \\
&= 50 + 20(P/G, 8\%, 5)(P/F, 8\%, 2)(A/P, 8\%, 7) \\
&= 50 + 20(7.3724)(0.8573)(0.19207) = 74.2
\end{aligned}
$$

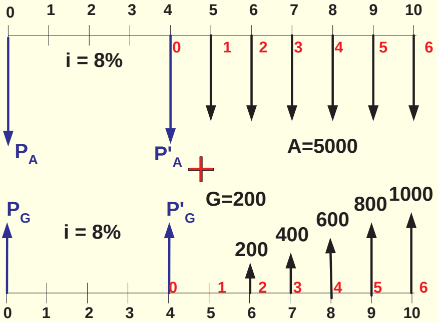
The use of arithmetic gradient factors is the same for increasing and decreasing gradients, except that in the case of decreasing gradients the following are true:

1. The base amount is equal to the largest amount, the amount at period 1.
2. The gradient amount is subtracted from the base amount.
3. $-G$ is used in the computations.

Example

Compute the present worth at year 0 at $i = 8\%$ per year for the cash flow shown below:





CAUTION on Signs of Cash Flows

So far we have all the cash flows on one side of the cash flow diagram. Therefore we do not consider the signs of the cash flows. Whenever we have cash flows on both sides of the cash flow diagram, it is good practice to take the ones on the upside as (+) and downside (-).

$$P = P_A - P_G \; (\textit{if you do not consider signs})$$

$$P = P_G - P_A \; (\textit{Considering upside } (+) \& \textit{downside } (-))$$

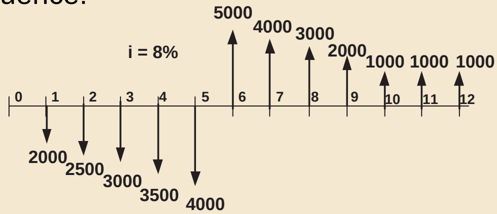
$$\begin{aligned} P_A &= P'_A(P/F, 8\%, 4) \\ &= 5000(P/A, 8\%, 6)(P/F, 8\%, 4) \\ &= 5000(4.6229)(0.7350) = \$16,989.1575 \end{aligned}$$

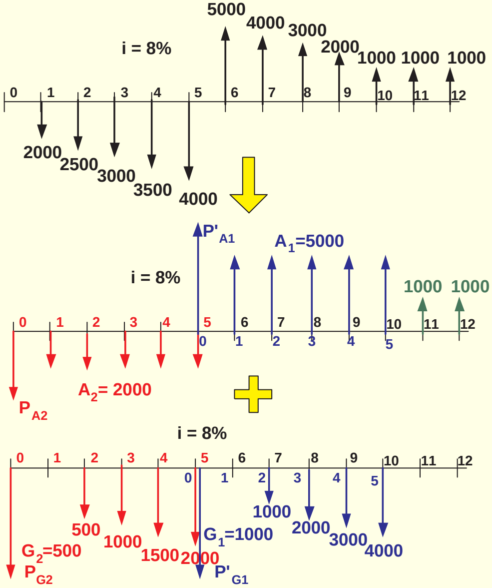
$$\begin{aligned} P_G &= P'_G(P/F, 8\%, 4) \\ &= 200(P/G, 8\%, 6)(P/F, 8\%, 4) \\ &= 200(10.5233)(0.7350) = \$1,546.7781 \end{aligned}$$

$$\begin{aligned} P &= P_A - P_G \\ &= 16,989.1575 - 1,546.7781 \\ &= \$15,442.3795 \text{ (}\textit{Must indicate it is pointing downward}\text{)} \\ P &= P_G - P_A \\ &= 1,546.7781 - 16,989.1575 \\ &= -\$15,442.3795 \text{ (}\textit{No need to indicate direction}\text{)} \end{aligned}$$

In Class Work 3

Find the present worth and equivalent annual series for the following cash flow sequence:





$$P = P_{GREEN} + P_{BLUE} - P_{RED}$$

$$\begin{aligned} P_{GREEN} &= 1000(P/F, 8\%, 11) + 1000(P/F, 8\%, 12) \\ &= 1000(0.4289) + 1000(0.3971) = 826 \end{aligned}$$

$$\begin{aligned} P_{BLUE} &= P_{A_1}(P/F, 8\%, 5) - P_{G_1}(P/F, 8\%, 5) \\ &= (P/F, 8\%, 5)[5000(P/A, 8\%, 5) - 1000(P/G, 8\%, 5)] \\ &= 5000(3.9927)(0.6806) - 1000(7.3724)(0.6806) = 8569.5 \end{aligned}$$

$$\begin{aligned} P_{RED} &= P_{A_2} + P_{G_2} = 2000(P/A, 8\%, 5) + 500(P/G, 8\%, 5) \\ &= 2000(3.9927) + 500(7.3724) = 11671.6 \end{aligned}$$

$P = 826 + 8569.5 - 11671.6 = -2276.1$
$A = -2276.1(A/P, 8\%, 12) = -2276.1(0.1327) = -302.04$