

- Along with the *PW* and *AW* criteria, the third primary measure of worth is **rate of return**
- Mostly preferred over *PW* and *AW* criteria. Ex: an alternative's worth can be reported as:
 - 15% rate of return on the investment (**EXACT!**)
 - will result in a net surplus of \$10,000 in terms of *PW* at 10% MARR (**return > 10%** is known only)
- ROR is also known as
 - Internal Rate of Return (IRR)
 - Return on Investment (ROI)

Rate of Return: Definition 1

is the rate (**-100% to ∞**) paid on the unpaid balance of borrowed money

The rate of return is calculated on the **unrecovered balance** not the initial amount of investment.

Example

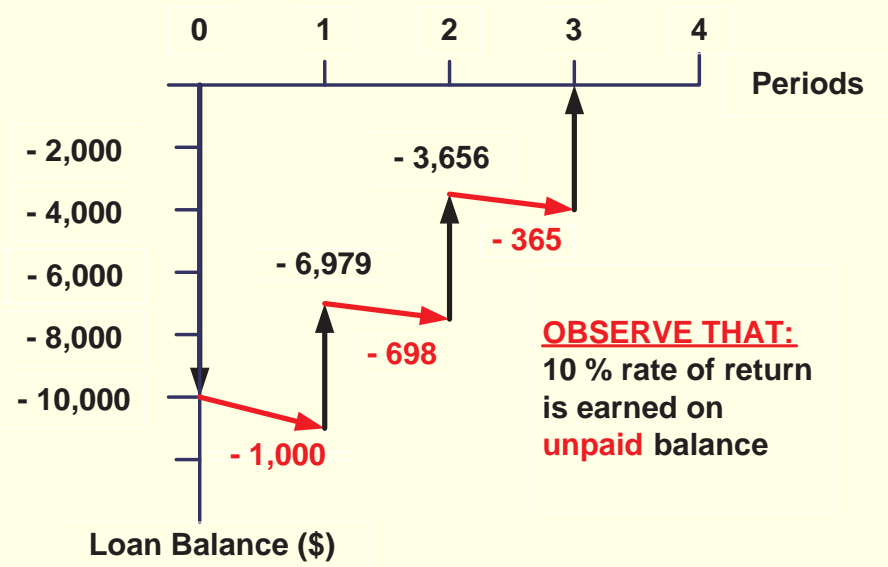
Suppose a bank lends \$10,000, which is repaid in installments of \$4,021 at the end of each year for three years. What is the interest rate the bank is charging for this transaction?

$$\$10,000 = \$4,021(P/A, i, 3) \rightarrow 10\%$$

What is the meaning of 10%?

Year	Unpaid Balance at Beginning	Return on Unpaid Balance	Payment Received	Balance at End
0				-\$10,000
1	-\$10,000	-\$1,000	\$4,021	-\$6,979
2	-\$6,979	-\$698	\$4,021	-\$3,656
3	-\$3,656	-\$365	\$4,021	\$0

OBSERVE THAT: 10% is only calculated over **unpaid balance**. These three annual payments of \$4,021 repay the loan itself and provide a return of 10% **on the amount still outstanding each year**.



If we calculate the PW of the loan transaction at this rate of return of 10%;

$$PW(10\%) = -\$10,000 + \$4,021(P/A, i, 3) = 0$$

which indicates that 10% earned **on the amount still outstanding each year** by the bank is also the break even interest rate (**largest** MARR that can be met by this loan)

Rate of Return: Definition 2

is the break-even interest i^* (**-100% to ∞**) at which the PW of project is zero.

$$PW(i^*) = PW_{\text{Cash Inflows}} - PW_{\text{Cash Outflows}} = 0$$

In detail, the equation $PW(i^*) = 0$ is equivalent to;

$$PW(i^*) = \frac{A_0}{(1+i^*)^0} + \frac{A_1}{(1+i^*)^1} + \cdots + \frac{A_N}{(1+i^*)^N} = 0 \quad (1)$$

Here, we know all A_n values for each period, but not the value of i^* . Since it is the only unknown, we can solve for i^* . But the solution is not always **straight forward** and **unique**.

Note that by multiplying both sides of Equation (1) by $(1+i^*)^N$ and $(A/P, i^*, N)$ we obtain;

$$\begin{aligned} PW(i^*)(1+i^*)^N &= FW(i^*) = 0 \\ PW(i^*)(A/P, i^*, N) &= AW(i^*) = 0 \end{aligned}$$

Rate of Return: Definition 2 (Modified)

is the rate of interest i^* (**-100% to ∞**), that equates the PW , FW , AW of the entire series of cash flows of a project to zero.

$$PW(i^*) = FW(i^*) = AW(i^*) = 0$$

Lastly, we will introduce the concept of rate of return based on the return on invested capital in terms of a project investment. A project's return is referred to as the **internal rate of return (IRR)** promised by an **investment project** over its **useful life**.

Internal Rate of Return: Definition 3

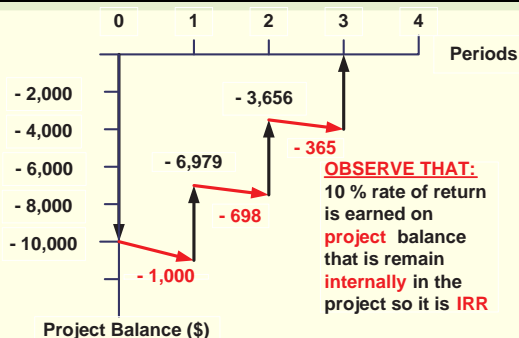
is the interest rate charged on the **unrecovered project balance** of the investment such that, when project terminates, the unrecovered project balance is zero.

Example

Suppose a company invests \$10,000 in a computer with a three-year useful life, which will create equivalent annual labor savings of \$4,021 at the end of each year for three years. Here, we can view the investing firm as the **lender** and the project as the **borrower**.

Example

Year	Beginning Project Balance	Return on Invested Capital	Cash Generated from Project	Project Balance at End
0				-\$10,000
1	-\$10,000	-\$1,000	\$4,021	-\$6,979
2	-\$6,979	-\$698	\$4,021	-\$3,656
3	-\$3,656	-\$365	\$4,021	\$0



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- Firm earns a 10% rate of returns on funds that remain **internally** invested in the project.
- Since it is a return internal to the project it is called IRR
- Computer project brings in enough cash to pay for itself in three years and to provide the firm a return of 10% on invested capital
- If the computer is financed with funds costing 10% annually, the cash generated by this investment will be exactly sufficient to repay the principal and its annual charge on the funds

To facilitate the process of finding the rate of return for an investment project, we will first classify the types of investment cash flows.

Classification of Investments

We classify an investment by counting the number of sign changes in its net cash-flow sequence. A change from either “-” to “+” or “+” to “-” is counted as one sign change. (Ignore the zero cash flows!)

1 Simple (or Conventional) Investment

- Initial cash flows are **negative** and only **one sign change** occurs
- Unique **solution** exists
- Question is: is it **real** or imaginary? If real, is it in $(-100, \infty)$ interval
- A **negative** rate of return implies you **never** recover the initial investment

2 Simple Borrowing

- Initial cash flows are **positive** and only **one sign change** occurs
- Unique **solution** exists but this is not an investment flow

3 Nonsimple (or Nonconventional) Investment

- **more than one sign change** occurs
- **Multiple i^*** can occur

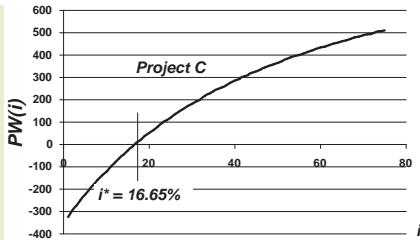
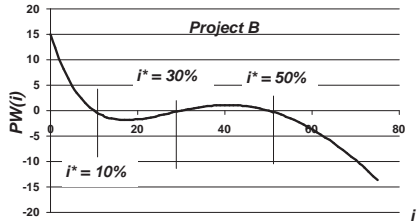
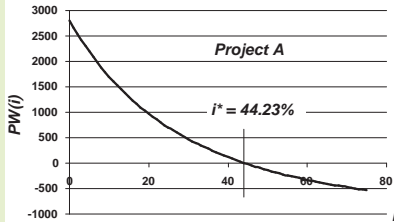
Example

Classify the following three cash flow series;

Period n	Project A	Project B	Project C
0	-\$1,000	-\$1,000	\$1,000
1	-\$500	\$3,900	-\$450
2	\$800	-\$5,030	-\$450
3	\$1,500	\$2,145	-\$450
4	\$2,000	0	0

- Project A is a simple investment (one sign change)
- Project B is a nonsimple investment (three sign change)
- Project C is a simple borrowing (one sign change)

Example



Internal rate of return

Let $(A_0, A_1, A_2, \dots, A_n)$ be a cash flow stream. Then the internal rate of return of this stream is a number i^* satisfying the equation

$$0 = A_0 + \frac{A_1}{1 + i^*} + \frac{A_2}{(1 + i^*)^2} + \dots + \frac{A_n}{(1 + i^*)^n}.$$

Equivalently, it is a number i^* satisfying $x = \frac{1}{1+i^*}$, where x satisfies the polynomial equation

$$0 = A_0 + A_1x + A_2x^2 + \dots + A_nx^n.$$

- IRR is defined without reference to the external financial world (external interest rate)
- It is determined entirely by the cash flow stream
- Equation for IRR is a polynomial equation in x of degree n
- Has **at least one root**, may have as many as n roots, but some or all roots may be complex numbers

$$0 = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n, \text{ where } x = \frac{1}{1 + i^*}$$

A **sign change** in the cash flow (A_i 's) is where successive nonzero terms of A_i 's, written according to descending powers of x , have different signs.

Descartes' Rule

If a polynomial with real coefficients has m **sign changes**, then the number of **positive roots** will be $m - 2k$, where k is a positive integer or zero ($k = 0, 1, 2, 3, \dots$).

number of sign changes m	Number of positive roots of x
0	0 (no IRR exists!)
1	1
2	2 or 0
3	3 or 1
4	4, 2, or 0

$$0 = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n, \text{ where } x = \frac{1}{1+i}$$

Descartes' Rule

If a polynomial with real coefficients has m sign changes, then the number of positive roots will be $m - 2k$, where k is a positive integer or zero ($k = 0, 1, 2, 3, \dots$).

- But this rule is applicable to the roots, x .
- Since, $x = \frac{1}{1+i}$, a positive root x , does not ensure a positive value for i^* .
- There may be as many positive values of i^* (root of ROR equation) as there are sign changes in the cash flow.

Main Theorem of Internal rate of return

Suppose the cash flow stream $(A_0, A_1, A_2, \dots, A_n)$ has $A_0 < 0$ and $A_k \geq 0$ for all $k, k = 1, 2, \dots, n$ with at least one term strictly positive. Then there is a unique positive root to the equation

$$0 = A_0 + A_1x + A_2x^2 + \dots + A_nx^n.$$

Furthermore, if $\sum_{k=0}^n A_k > 0$ (meaning that the total amount returned exceeds the initial investment), then the compounding internal rate of return $i = (1/x) - 1$ is positive.

- The conditions in the theorem ensure a single positive i^* root, for **one sign change**.
- If there is more than one sign change in the net cash flows, it is possible that there will be multiple i^* values.

Proof:

Let $f(x) = A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$. Note that $f(0) = A_0 < 0$. As x increases, the value of $f(x)$ also increases, since at least one of the cash flow terms is strictly positive. Indeed, it increases to infinity. Since the function is continuous, it must cross the axis at some value of x . It cannot cross more than once, since it is strictly increasing. Hence there is a unique positive solution x_0 , for which $f(x_0) = 0$.

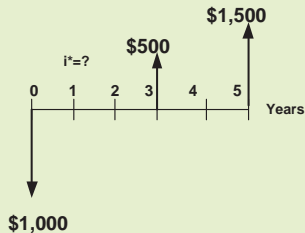
If $\sum_{k=0}^n A_k > 0$, then $f(1) > 0$. Meaning the solution x_0 , for which $f(x_0) = 0$ is less than 1. Therefore, $i^* = (1/x_0) - 1$ is positive.

- To determine the ROR of a cash flow series, set up the ROR equation using either PW or AW relations.
- For PW :
 - Let the present worth of costs or disbursements to be PW_D
 - Let the present worth of incomes or receipts to be PW_R
 - Solve $0 = PW_R - PW_D$ for i
- Similarly for AW :
 - Solve $0 = AW_R - AW_D$ for i
- The i value that makes these values numerically correct is called i^* . It is the **root** of the ROR relation.
- Compare i^* with MARR to decide if the cash flow series is viable:
 - If $i^* \geq MARR$, accept the alternative
 - If $i^* < MARR$, reject the alternative
- But this i^* must be **internal** to the project to be an **IRR**, and we need to check this **even** if it is a unique positive rate.

In the rate of return calculations, objective is to find the interest rate i^* at which the positive and negative cash flows are equivalent.

Example

If you deposit \$1,000 now and promised payments of \$500 three years from now and \$1,500 five years from now, find the rate of return using *PW* factors?



$$0 = -1,000 + 500(P/F, i\%, 3) + 1500(P/F, i\%, 5) \rightarrow i^* = 16,9\%$$

By **Trial and Error**.

Trial and Error

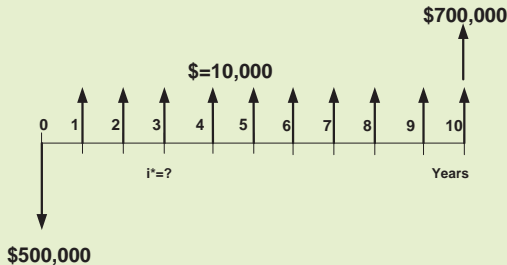
- Draw a cash flow diagram
- Set up the rate of return equation
- Select the i values until the equation is balanced

Choosing the first i as closer to i^* is important. The procedure for obtaining a good initial i is:

- 1 Convert all disbursements into either single amounts or uniform values while neglecting the time value of money
- 2 Convert all receipts into either single amounts or uniform values while neglecting the time value of money
- 3 Combine the receipts and disbursements so that a P/F , P/A , or A/F format applies
- 4 Use interest rate tables to find the approximate interest rate at which P/F , P/A , or A/F value is satisfied. The obtained rate is a good initial rate for trial and error.

Example

A construction company has requested that \$500,000 be spent now during construction on software and hardware to improve the efficiency of environmental control systems for the construction of a skyscraper. This is expected to save \$10,000 per year for 10 years in energy costs and \$700,000 at the end of 10 years in equipment refurbishment costs. Find the rate of return?



- 1 Cash flow diagram
- 2 $0 = -500,000 + 10,000(P/A, i\%, 10) + 700,000(P/F, i\%, 10)$
- 3 Take all income as a single Future amount:

$$500,000 = 800,000(P/F, i, 10) \rightarrow (P/F, i, 10) = 0.625$$

The rate i is between 4% and 5%. Use 5% as the initial one.

$$6,946 = -500,000 + 10,000(P/A, 5\%, 10) + 700,000(P/F, 5\%, 10)$$

indicating the return is more than 5%. Try 6%:

$$\begin{aligned} -35,519 &= -500,000 + 10,000(P/A, 6\%, 10) \\ &+ 700,000(P/F, 6\%, 10) \end{aligned}$$

- 4 Interpolate between 5% and 6%:

$$i^* = 5 + \frac{6946 - 0}{6946 - (-35,519)}(1.0) = 5.16\%$$

When applied correctly, the ROR technique will always result in a correct decision, indeed, the same one as with a *PW* or *AW* analysis.

- However, there are some assumptions and difficulties with ROR analysis while calculating i^* and interpreting its real-world meaning.
 - ❶ Depending on the sequence of net cash flows, there may be more than one real-number root to the ROR equation, resulting in more than one i^* value.
 - ❷ ROR method assumes that any net positive investment are reinvested at rate i^*
 - ❸ Obtaining one or multiple i^* values by trial-and-error solution is not easy
 - ❹ A unique approach must be applied to choose from two or more mutually exclusive alternatives

In Class Work 9

During the last 3 years, the net cash flows for contract payments have varied widely, as shown below:

Year	0	1	2	3
Cash Flow	2,000	-500	-8,100	6,800

- 1 Determine the maximum number of i^* values that may satisfy the ROR relation.
- 2 Write the PW-based ROR relation and approximate the i^* value(s), using 0, 5, 10, 20, 30, 40, and 50%.

- 1 There are two sign changes, therefore it indicates a maximum of two i^* values.

$i\%$	0	5	10	20	30	40	50
PW							

1 Two sign changes, maximum of two i^* values.

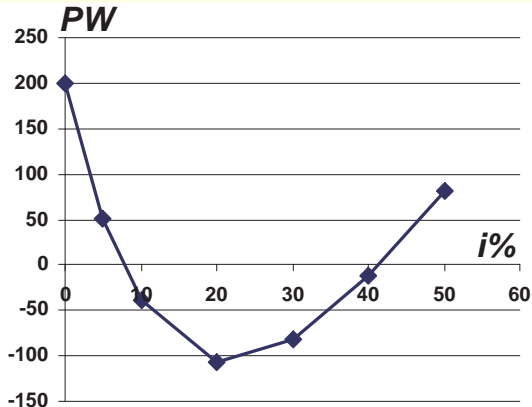
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$$\begin{aligned}
 PW &= 2,000 - 500(P/F, i\%, 1) - 8,100(P/F, i\%, 2) \\
 &+ 6,800(P/F, i\%, 3) \\
 &= 2,000 - \frac{500}{1+i} - \frac{8,100}{(1+i)^2} + \frac{6,800}{(1+i)^3}
 \end{aligned}$$

i%	0	5	10	20	30	40	50
PW	200	50.97	-39.82	-106.48	-82.38	-11.66	81.48

$$i_1^* = 5 + \frac{0 - 50.97}{-39.82 - 50.97}(5.0) = 7.81\%$$

$$i_2^* = 40 + \frac{0 - (-11.66)}{81.48 - (-11.66)}(10) = 41.25\%$$



Are these rates IRR?

By definition; IRR is the rate charged on the **unrecovered project balance** of the investment.

- So the rate is applicable to the **internal** balances not **external**

