MOMENT GENERATING FUNCTIONS

Moments

For each integer k, the k-th moment of X is

$$\mu_k^* = E\left[X^k\right] \rightarrow \textit{ the k-th moment}$$

$$\mu_k = E\left[X - \mu\right]^k \rightarrow \textit{ the k-th central moment}$$

MOMENT GENERATING FUNCTION (mgf)

- Let X be a rv with cdf $F_X(x)$. The moment generating function (mgf) of X, denoted by $M_X(t)$, is

$$M_X(t) = E\left[e^{tX}\right] = \begin{cases} \sum\limits_{all\ x} e^{tx} p(x) & , \textit{if } X \textit{ is discrete} \\ \int\limits_{all\ x} e^{tx} f(x)\,dx, & \textit{if } X \textit{ is continuous} \end{cases}$$

provided that expectation exist for t in some neighborhood of 0. That is, there is $h>0$ such that, for all t in $-h<t<h$, E(e^{tX}) exists.

MOMENT GENERATING FUNCTION (mgf)

- If X has mgf $M_X(t)$, then

$$E\left[X^n\right] = M_X^{(n)}(0)$$

where we define

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0}.$$

That is, the n-th moment is the n-th derivative of $M_X(t)$ evaluated at $t=0$.

MOMENT GENERATING FUNCTION (mgf)

Example: Let X be an rv with pmf

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!};\; x = 0,1,\ldots;\; \lambda > 0$$

Find the mgf of X.

MOMENT GENERATING FUNCTION (mgf)

Example: Let X be an rv with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma^2 > 0$$

Find the mgf of X.

Properties of mgf

a) If an rv X has mgf, $M_X(t)$, then an rv $Y=aX+b$ (where a and b are constants) has an mgf $M_Y(t)=e^{bt}M_X(at)$.

b) The mgf is unique and completely determines the distribution of the rv.

c) If $X_1, X_2, \ldots, X_n$ are independent rvs with mgf $M_{X_i}(t)$, then the mgf of $Y=\sum_{i=1}^{n} X_i$

$$M_Y(t)=\prod_{i=1}^{n} M_{X_i}(t)$$

For a random sample (independent and identically distributed rvs)

$$M_Y(t)=\left[M_X(t)\right]^n$$

Characteristic Function

- Let X be an rv. The complex valued function ϕ defined on $\Re$ by

$$\phi(t) = E\left[e^{itX}\right] = E\left[\cos(tX)\right] + iE\left[\sin(tX)\right], t \in \Re$$

where $i = \sqrt{-1}$ is the imaginary unit, is called the characteristic function (cf) of rv X.

- Unlike an mgf that may not exist for some distributions, a cf always exists, which makes it a much more convenient tool.

SOME DISCRETE PROBABILITY DISTRIBUTIONS

Binomial, Poisson, Hypergeometric, Geometric and Negative Binomial Distributions

The Binomial Distribution

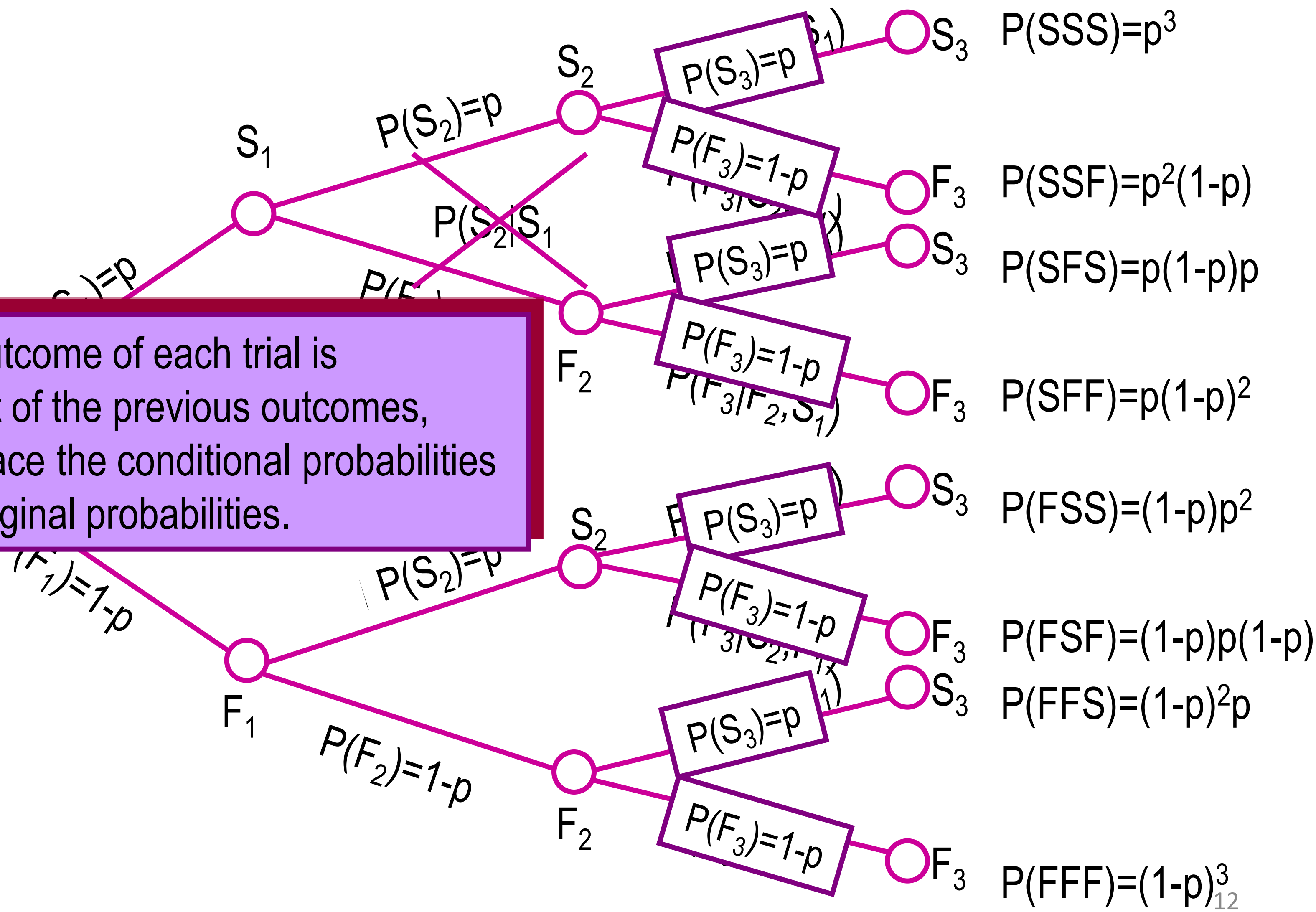
- The binomial experiment can result in only one of two possible outcomes.
- Typical cases where the binomial experiment applies:
  - A coin flipped results in heads or tails
  - An election candidate wins or loses
  - An employee is male or female
  - A car uses 87octane gasoline, or another gasoline.

Binomial Experiment

- There are n trials (n is finite and fixed).
- Each trial can result in a success or a failure.
- The probability p of success is the same for all the trials.
- All the trials of the experiment are independent.

- **Binomial Random Variable**
  - The binomial random variable ***counts*** the number of successes in n trials of the binomial experiment.
  - By definition, this is a discrete random variable.

Developing the Binomial Probability Distribution (n = 3)



Since the outcome of each trial is independent of the previous outcomes, we can replace the conditional probabilities with the marginal probabilities.

Developing the Binomial Probability Distribution (n = 3)

Let X be the number of successes in three trials. Then,

$P(X = 3) = p^3$ ⇐ $X = 3$

$P(X = 2) = 3p^2(1-p)$ ⇐ $X = 2$

$P(X = 1) = 3p(1-p)^2$ ⇐ $X = 1$

$P(X = 0) = (1-p)^3$ ⇐ $X = 0$

$P(SSS) = p^3$

$P(SSF) = p^2(1-p)$

$P(SFS) = p(1-p)p$

$P(SFF) = p(1-p)^2$

$P(FSS) = (1-p)p^2$

$P(FSF) = (1-p)p(1-p)$

$P(FFS) = (1-p)^2p$

$P(FFF) = (1-p)^3$

This multiplier is calculated in the following formula

Calculating the Binomial Probability

In general, The binomial probability is calculated by:

$$P(X = x) = p(x) = C_x^n p^x (1 - p)^{n-x}$$

$$\text{where } C_x^n = \frac{n!}{x!(n-x)!}$$

Calculating the Binomial Probability

- **Example**
  - Pat Statsdud is registered in a statistics course and intends to rely on luck to pass the next quiz.
  - The quiz consists on 10 multiple choice questions with 5 possible choices for each question, only one of which is the correct answer.
  - Pat will guess the answer to each question
  - Find the following probabilities
    - Pat gets no answer correct
    - Pat gets two answer correct?
    - Pat fails the quiz

Calculating the Binomial Probability

- Solution
  - Checking the conditions
    - An answer can be either correct or incorrect.
    - There is a fixed finite number of trials (n=10)
    - Each answer is independent of the others.
    - The probability p of a correct answer (.20) does not change from question to question.

Calculating the Binomial Probability

- Solution – Continued
  - Determining the binomial probabilities:

    Let X = the number of correct answers

$$P(X = 0) = \frac{10!}{0!(10-0)!}(.20)^0(.80)^{10-0} = .1074$$

$$P(X = 2) = \frac{10!}{2!(10-2)!}(.20)^2(.80)^{10-2} = .3020$$

Calculating the Binomial Probability

- Solution – Continued
  - Determining the binomial probabilities:

    Pat fails the test if the number of correct answers is less than 5, which means less than or equal to 4.

$$
\begin{aligned}
P(X \le 4) &= p(0) + p(1) + p(2) + p(3) + p(4) \\
&= .1074 + .2684 + .3020 + .2013 + .0881 \\
&= .9672
\end{aligned}
$$

This is called cumulative probability

Mean and Variance of Binomial Variable

$$E(X) = \mu = np$$
$$V(X) = \sigma^2 = np(1-p)$$

- **Example**
  - If all the students in Pat's class intend to guess the answers to the quiz, what is the mean and the standard deviation of the quiz mark?
- Solution
  - $\mu = np = 10(.2) = 2.$
  - $\sigma = [np(1-p)]^{1/2} = [10(.2)(.8)]^{1/2} = 1.26.$

EXAMPLE

- If the probability is 0.20 that any one person will dislike the taste of a new toothpaste, what is the probability that 5 out of 18 randomly selected person will dislike it?

EXAMPLE

- A food packaging apparatus underfills 10% of the containers. Find the probability that for any particular 5 containers the number of underfilled will be

a) Exactly 3

b) Zero

c) At least one.

BINOMIAL DISTRIBUTION FUNCTION

$$F(x) = P(X \le x) = \sum_{y=0}^{x} C_{y,n} p^{y} (1-p)^{n-y}$$

- Binomial distribution function tables are helpful to find probabilities.
- If n=10 and p=0.3, find P(X≤4).

  Find P(X=4).

  Find P(2 ≤ X ≤4).

EXAMPLE

- Suppose that only 25% of all drivers come to a complete stop at an intersection having flashing red lights in all directions when no other cars are visible. What is the probability that, of 20 randomly chosen drivers coming to an intersection under these conditions,

a) At most 6 will come to a complete stop?

b) Exactly 6 will come to a complete stop?

c) At least 6 will come to a complete stop?

d) How many of the next 20 drivers do you expect to come to a complete stop?

Poisson Distribution

- The Poisson experiment typically fits cases of rare events that occur over a fixed amount of time or within a specified region

- Typical cases
  - The number of errors a typist makes per page
  - The number of customers entering a service station per hour
  - The number of telephone calls received by a switchboard per hour.

Properties of the Poisson Experiment

- The number of successes (events) that occur in a certain time interval is independent of the number of successes that occur in another time interval.
- The probability of a success in a certain time interval is
  - the same for all time intervals of the same size,
  - proportional to the length of the interval.
- The probability that two or more successes will occur in an interval approaches zero as the interval becomes smaller.

The Poisson Variable and Distribution

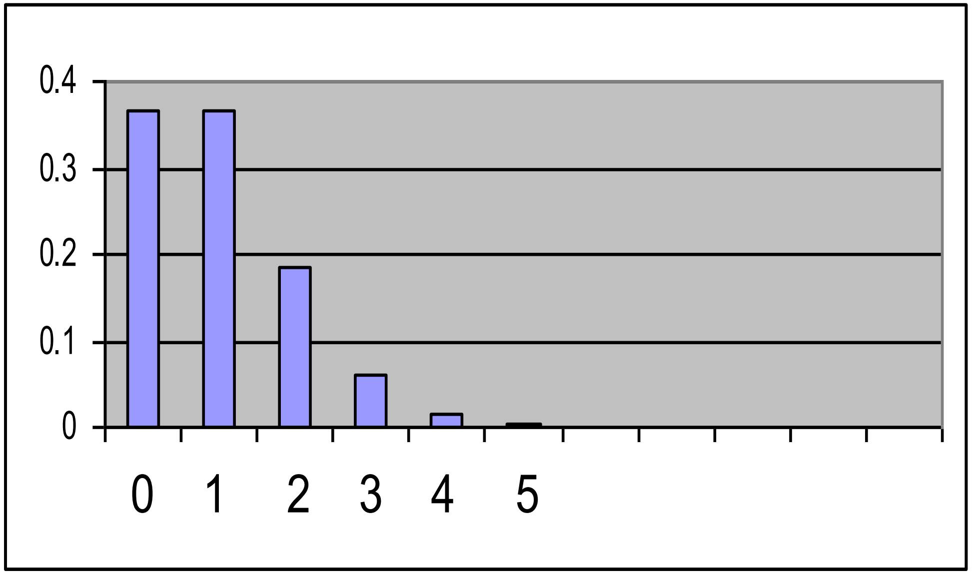
- **The Poisson Random Variable**
  - The Poisson variable indicates the number of successes that occur during a given time interval or in a specific region in a Poisson experiment

- **Probability Distribution of the Poisson Random Variable.**

$$P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2\ldots \text{ and } \lambda > 0$$

$$E(X) = V(X) = \lambda$$

Poisson Distributions (Graphs)



$$P(X=0)=p(0)=\frac{e^{-1}1^{0}}{0!}=e^{-1}=.3678$$

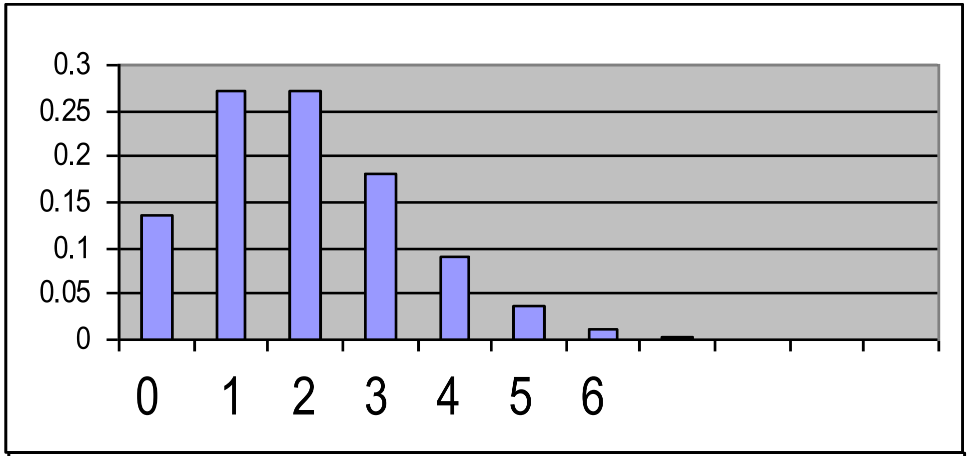
$$P(X=2)=p(2)=\frac{e^{-1}1^{2}}{2!}=\frac{e^{-1}}{2}=.1839$$

$$P(X=1)=p(1)=\frac{e^{-1}1^{1}}{1!}=e^{-1}=.3678$$

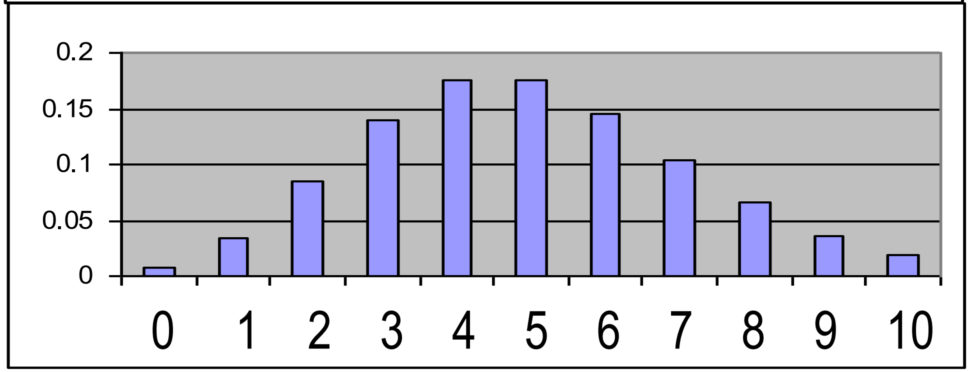
$$P(X=3)=p(3)=\frac{e^{-1}1^{3}}{3!}=\frac{e^{-1}}{6}=.0613$$

Poisson Distributions (Graphs)

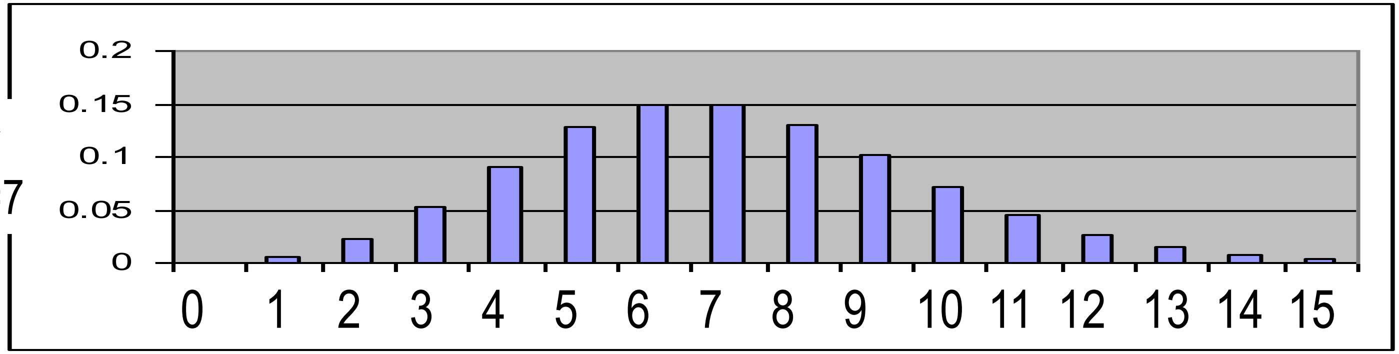
Poisson probability distribution with λ =2



Poisson probability distribution with λ =5



Poisson probability distribution with λ =7



Poisson Distribution

- **Example**
  - The number of Typographical errors in new editions of textbooks is Poisson distributed with a mean of 1.5 per 100 pages.
  - 100 pages of a new book are randomly selected.
  - What is the probability that there are no typos?
- Solution

Poisson Distribution

- **Example**
  - For a 400 page book calculate the following probabilities
    - There are no typos
    - There are five or fewer typos
- Solution
  - P(X=0)= $\dfrac{e^{-\lambda}\lambda^{x}}{x!} = \dfrac{e^{-6}6^{0}}{0!} = e^{-6} = 0.002479$
  - P(X≤5)=<use the formula to find p(0), p(1
    calculate p(0)+p(1)+…+p(5) = .4457

Important!
A mean of 1.5 typos per 100 pages, is equivalent to 6 typos per 400 pages.

EXAMPLE

- Suppose small aircraft arrive at a certain airport according to a Poisson process with rate 8 per hour, so that the number of arrivals during a time period of t hours is a Poisson random variable with parameter $\lambda = 8t$.

a) What is the probability that exactly 6 small aircraft arrive during a 1-hour period? At least 6?

b) What are the expected value and standard deviation of the number of small aircraft that arrive during a 90-min period?

BERNOULLI DISTRIBUTION

- A Bernoulli trial is an experiment with only two outcomes. An r.v. X has Bernoulli(p) distribution if

$$X = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases} ; 0 \leq p \leq 1$$

BINOMIAL DISTRIBUTION

- Define an rv Y by

  Y = total number of successes in n Bernoulli trials.

1. There are n trials (n is finite and fixed).
2. Each trial can result in a success or a failure.
3. The probability p of success is the same for all the trials.
4. All the trials of the experiment are independent.

$$\text{Let } X_i \overset{independent}{\sim} Bin(n_i, p). \ Then,$$

$$\sum_{i=1}^{k} X_i \sim Bin(n_1 + n_2 + \cdots + n_k, p).$$

$$Y = \sum_{i=1}^{n} X_i \sim Bin(n, p) \text{ where } X_i \sim Ber(p).$$

BINOMIAL THEOREM

- For any real numbers x and y and integer $n \geq 0$

$$(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i}.$$

When x=y=1

$$2^n = \sum_{i=1}^{n} \binom{n}{i}.$$

POISSON DISTRIBUTION

- The number of occurrences in a given time interval can be modeled by the Poisson distribution.
- e.g. waiting for bus, waiting for customers to arrive in a bank.

- Another application is in spatial distributions.
- e.g. modeling the distribution of bomb hits in an area or the distribution of fish in a lake.

Relationship between Binomial and Poisson

$X \sim Bin(n, p)$ with mgf $M_X(t) = \left(pe^t + 1 - p\right)^n$

Let $\lambda = np$.

$$\lim_{n\to\infty} M_X(t) = \lim_{n\to\infty}\left(pe^t + 1 - p\right)^n$$

$$= \lim_{n\to\infty}\left(1 + \frac{\lambda\left(e^t - 1\right)}{n}\right)^n = e^{\lambda\left(e^t - 1\right)} = M_Y(t)$$

The mgf of Poisson(λ)

The limiting distribution of Binomial rv is Poisson distribution.

DEGENERATE DISTRIBUTION

- An rv X is degenerate at point k if

$$P(X = x) = \begin{cases} 1, & X = k \\ 0, & o.w. \end{cases}$$

The cdf:

$$F(x) = P(X \le x) = \begin{cases} 0, & X < k \\ 1, & X \ge k \end{cases}$$

The mgf:

$$M_X(t) = e^{kt}, t < \infty$$

NEGATIVE BINOMIAL DISTRIBUTION (PASCAL OR WAITING TIME DISTRIBUTION)

- Distribution of the number of Bernoulli trials required to get a fixed number of successes, such as *r* successes.

$X \sim NB(r,p)$

$$P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}, x=r, r+1, \cdots$$

$$E(X)=\frac{r(1-p)}{p} \text{ and } Var(X)=\frac{r(1-p)}{p^2}$$

GEOMETRIC DISTRIBUTION

- Distribution of the number of Bernoulli trials required to get the first success.
- It is the special case of the Negative Binomial Distribution$\rightarrow$ *r=1*.

X~Geometric(p)

$$P(X=x)=p(1-p)^{x-1}, x=1,2,\cdots$$

$$E(X)=\frac{r(1-p)}{p} \text{ and } Var(X)=\frac{r(1-p)}{p^2}$$

Memoryless Property:

$$P(X>m+n \mid X>m)=P(X\geq n).$$

HYPERGEOMETRIC DISTRIBUTION

- A box contains N marbles. Of these, M are red. Suppose that n marbles are drawn randomly from the box. The distribution of the number of red marbles, x is

$$P(X=x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, x=0,1,\ldots,n$$

X~Hypergeometric($\boldsymbol{N},\boldsymbol{M},\boldsymbol{n}$)

It is dealing with finite population.

SOME CONTINUOUS PROBABILITY DISTRIBUTIONS

Uniform, Normal, Exponential, Gamma, Chi-Square, Student t and F Distributions

Uniform Distribution

– A random variable X is said to be uniformly distributed if its density function is

$$\boxed{f(x) = \frac{1}{b-a} \quad a \le x \le b.}$$

– The expected value and the variance are

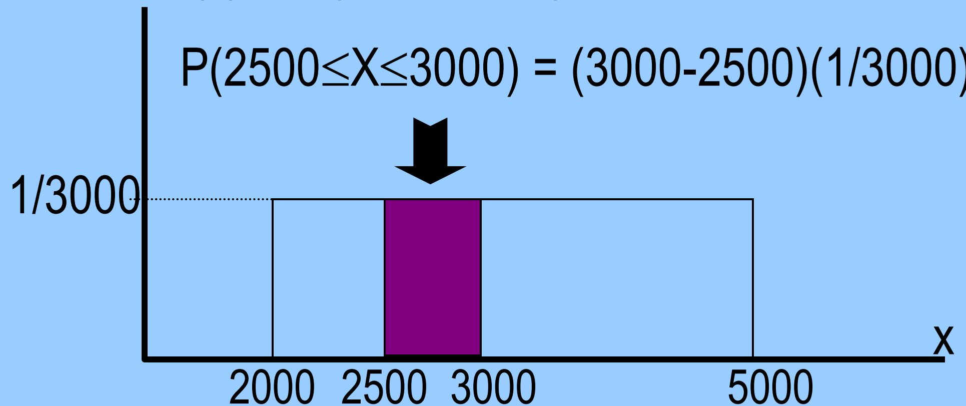
$$E(X) = \frac{a+b}{2} \qquad V(X) = \frac{(b-a)^2}{12}$$

Uniform Distribution

- **Example 1**
  - The daily sale of gasoline is uniformly distributed between 2,000 and 5,000 gallons. Find the probability that sales are:
    - Between 2,500 and 3,000 gallons
    - More than 4,000 gallons
    - Exactly 2,500 gallons

$f(x) = 1/(5000-2000) = 1/3000$ for x: $[2000,5000]$

$P(2500 \leq X \leq 3000) = (3000-2500)(1/3000) = .1667$



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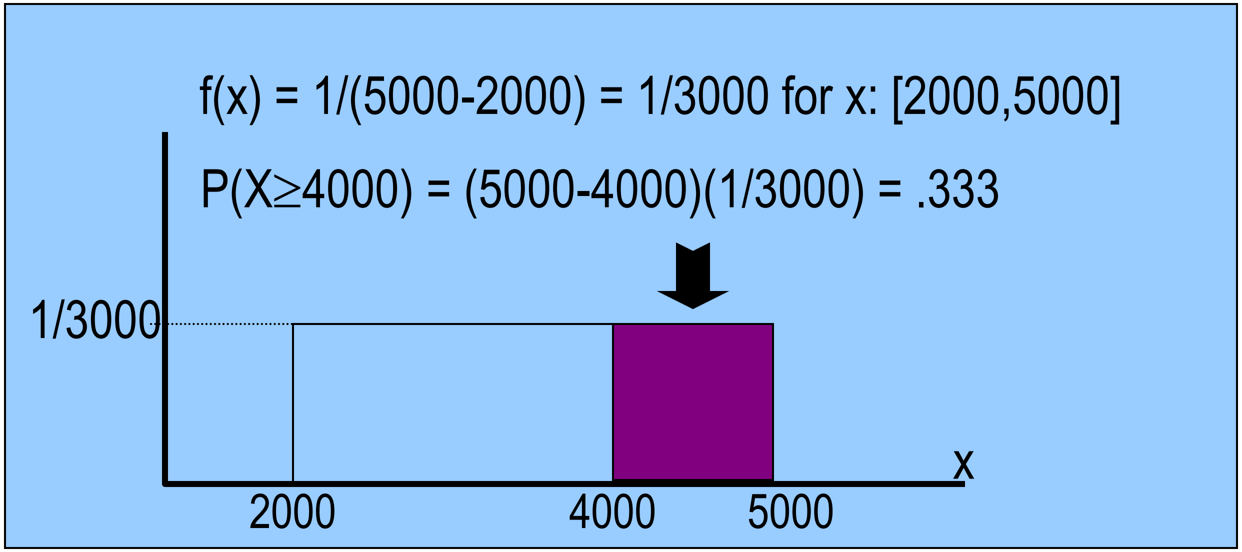

Uniform Distribution

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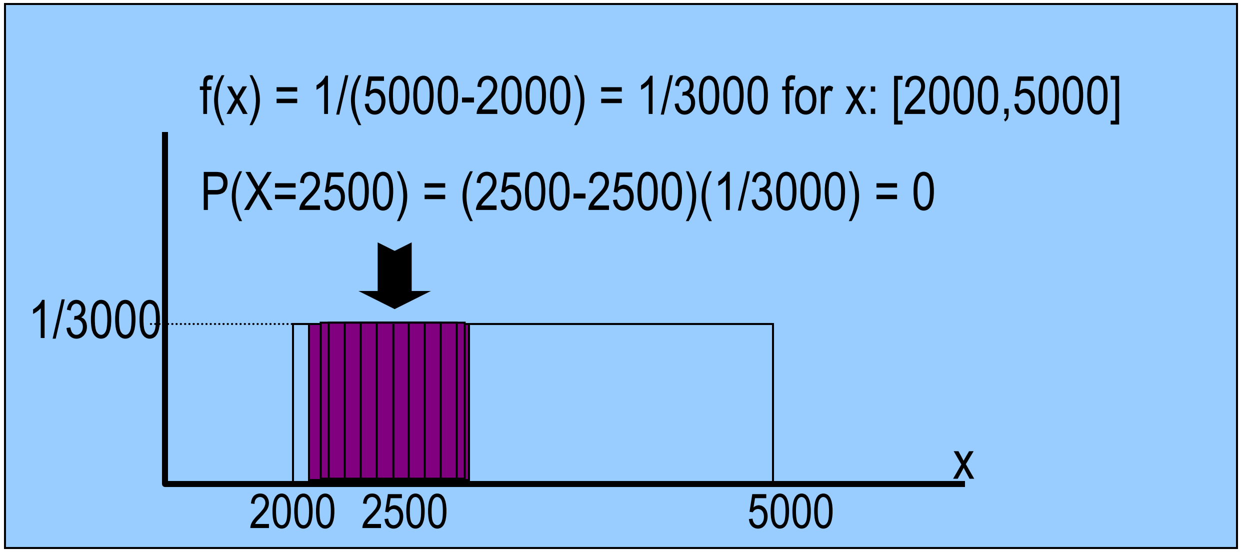


Normal Distribution

- This is the most important continuous distribution.
  - Many distributions can be approximated by a normal distribution.
  - The normal distribution is the cornerstone distribution of statistical inference.

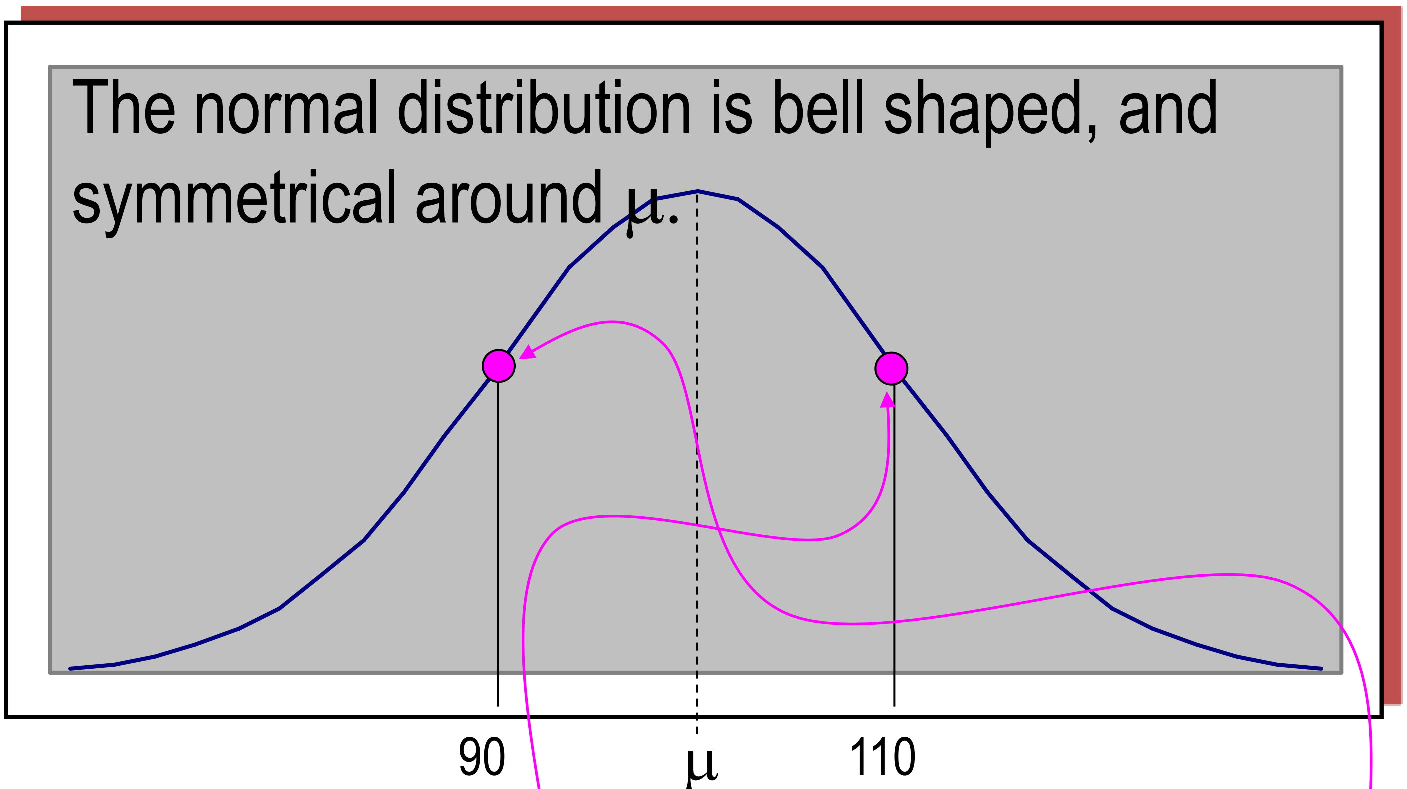
Normal Distribution

- A random variable X with mean μ and variance σ^2 is normally distributed if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \leq x \leq \infty$$

$$\text{where } \pi = 3.14159\ldots \text{ and } e = 2.71828\ldots$$

The Shape of the Normal Distribution

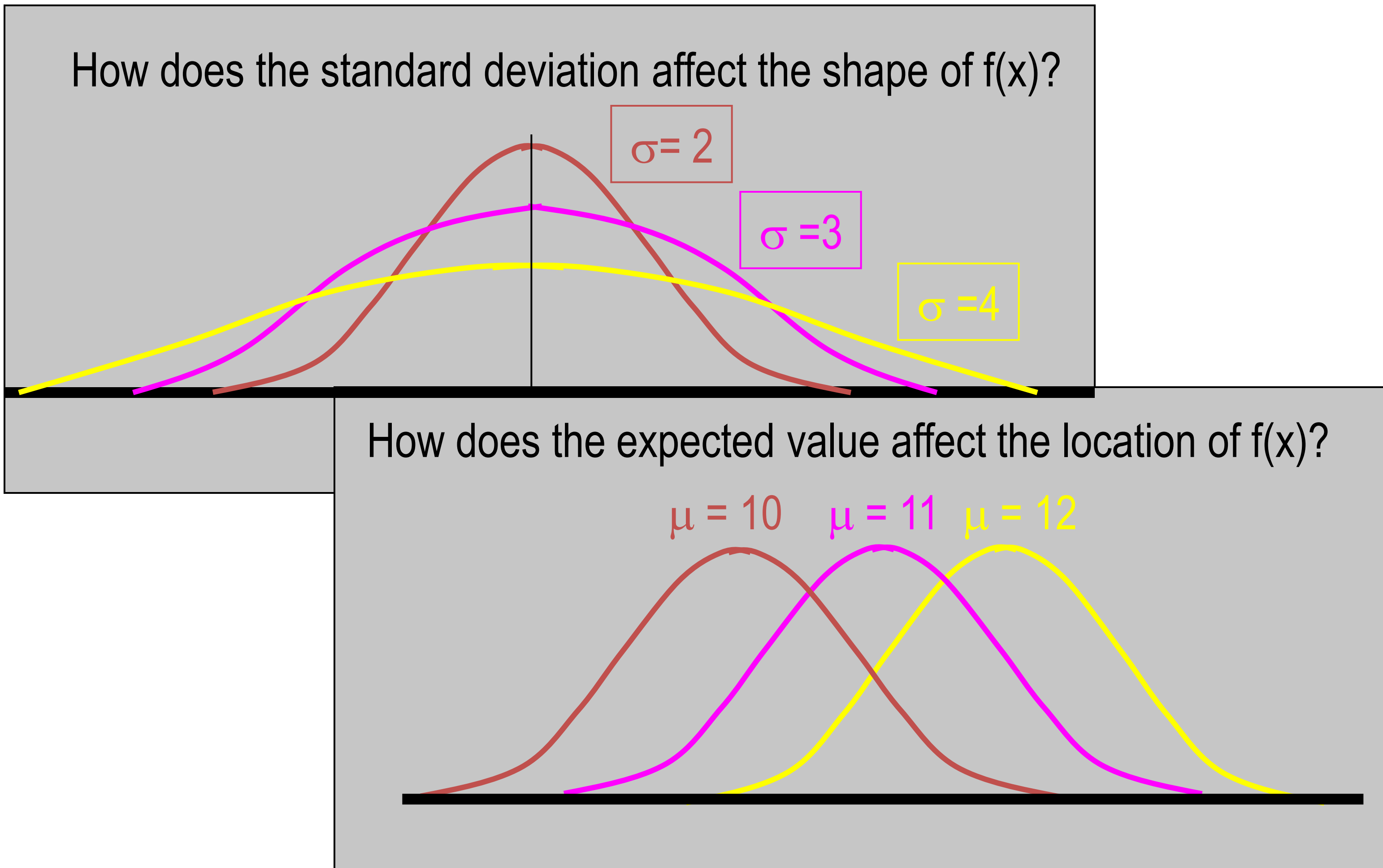


Why symmetrical? Let $\mu = 100$. Suppose $x = 110$. Now suppose $x = 90$

$$f(110) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)\left(\frac{110-100}{\sigma}\right)^2} = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)\left(\frac{10}{\sigma}\right)^2}}$$

$$f(90) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)\left(\frac{90-100}{\sigma}\right)^2} = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)\left(\frac{-10}{\sigma}\right)^2}}$$

The Effects of μ and σ



Finding Normal Probabilities

- Two facts help calculate normal probabilities:
  - The normal distribution is symmetrical.
  - Any normal distribution can be transformed into a specific normal distribution called…

“STANDARD NORMAL DISTRIBUTION”

Example

The amount of time it takes to assemble a computer is normally distributed, with a mean of 50 minutes and a standard deviation of 10 minutes. What is the probability that a computer is assembled in a time between 45 and 60 minutes?

STANDARD NORMAL DISTRIBUTION

- NORMAL DISTRIBUTION WITH MEAN 0 AND VARIANCE 1.
- IF $X \sim N(\mu , \sigma^2)$, THEN

$$Z = \frac{X - \mu}{\sigma} \sim N(0,1)$$

NOTE: Z IS KNOWN AS Z SCORES.

- “ ~ “ MEANS “DISTRIBUTED AS”

Finding Normal Probabilities

- Solution
  - If X denotes the assembly time of a computer, we seek the probability P(45<X<60).
  - This probability can be calculated by creating a new normal variable the **standard normal variable.**

Every normal variable with some μ and σ, can be transformed into this Z.

$$Z = \frac{X - \mu_x}{\sigma_x}$$

Therefore, once probabilities for Z are calculated, probabilities of any normal variable can be found.

$E(Z) = \mu = 0$

$V(Z) = \sigma^2 = 1$

Finding Normal Probabilities

- Example - continued

$$P(45<X<60) = P\left(\frac{45 - 50}{10} < \frac{X-\mu}{\sigma} < \frac{60 - 50}{10}\right)$$

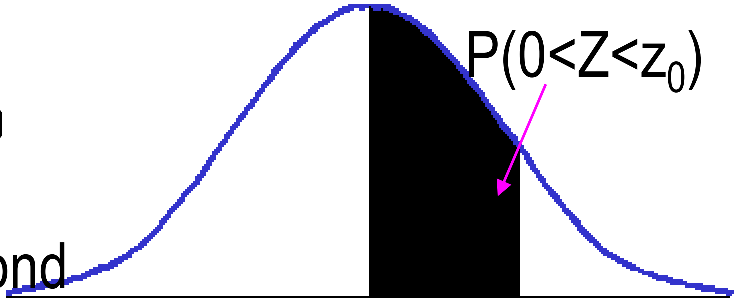
$$= P(-0.5 < Z < 1)$$

To complete the calculation we need to compute the probability under the standard normal distribution

Using the Standard Normal Table

Standard normal probabilities have calculated and are provided in a tab

The tabulated probabilities correspond to the area between Z=0 and some $Z = z_0 > 0$



z	0	0.01	…….	0.05	0.06
0.0	0.0000	0.0040		0.0199	0.0239
0.1	0.0398	0.0438		0.0596	0.0636
.	.	.		.	.
.	.	.		.	.
1.0	0.3413	0.3438		0.3531	0.3554
.	.	.		.	.
.	.	.		.	.
1.2	0.3849	0.3869	…….	0.3944	0.3962
.	.	.	.	.	.
.	.	.	.	.	.

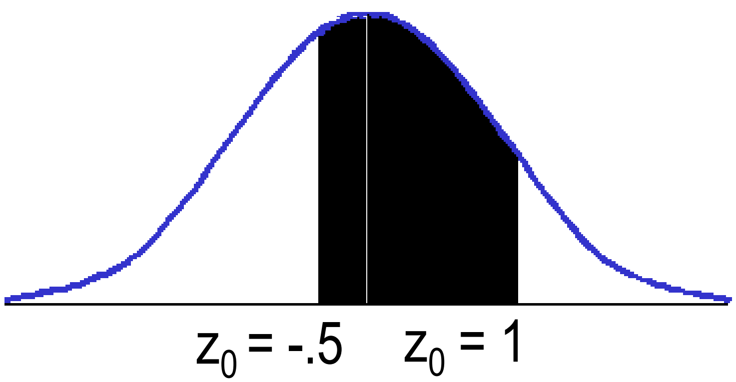
Finding Normal Probabilities

- Example - continued

$$P(45<X<60) = P\left(\frac{45 - 50}{10} < \frac{X - \mu}{\sigma} < \frac{60 - 50}{10}\right)$$

$$= P(-.5 < Z < 1)$$

We need to find the shaded area



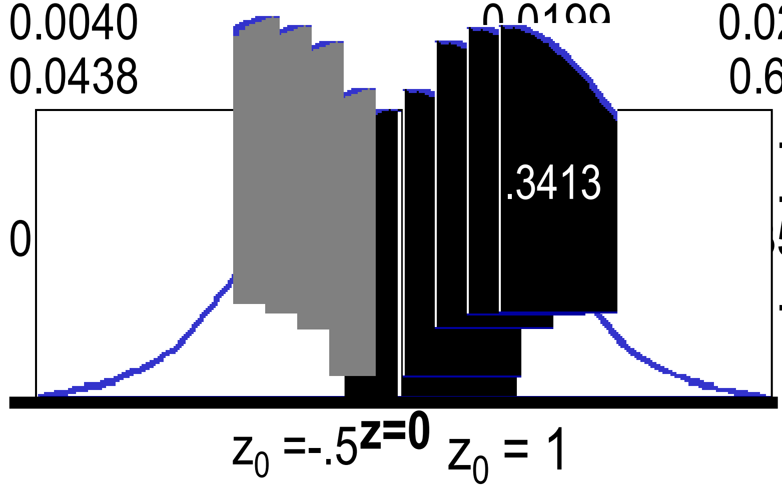
Finding Normal Probabilities

- Example - continued

$$P(45<X<60) = P\left(\frac{45-50}{10} < \frac{X-\mu}{\sigma} < \frac{60-50}{10}\right)$$

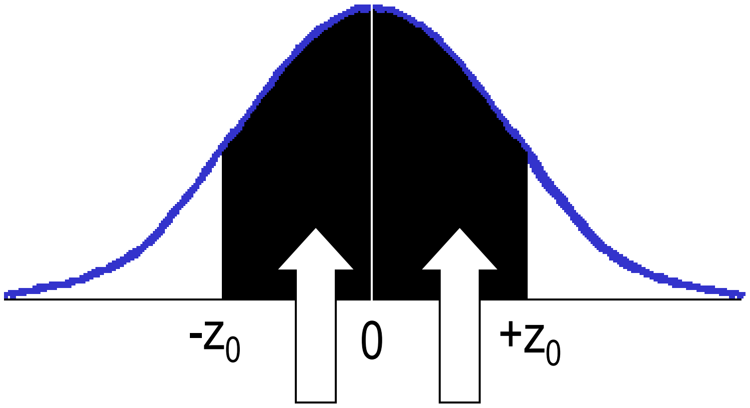
$$= P(-.5<Z<1) = P(-.5<Z<0)+\boxed{P(0<Z<1)}$$

z	0	0.1	....	0.05	0.06
0.0	0.0000	0.0040		0.0199	0.0239
0.1	0.0398	0.0438			0.636
.	.				.
.	.				.
1.0	0.3413	0			554
.	.				.



Finding Normal Probabilities

- The symmetry of the normal distribution makes it possible to calculate probabilities for negative values of Z using the table as follows:

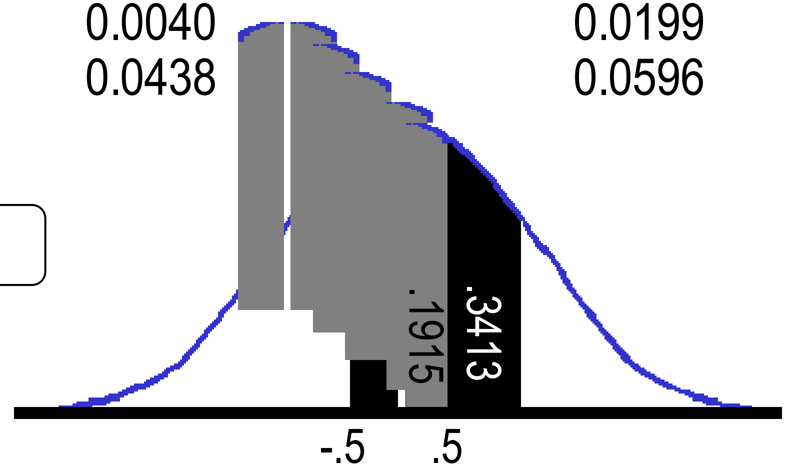


$P(-z_0<Z<0) = P(0<Z<z_0)$

Finding Normal Probabilities

- Example - continued

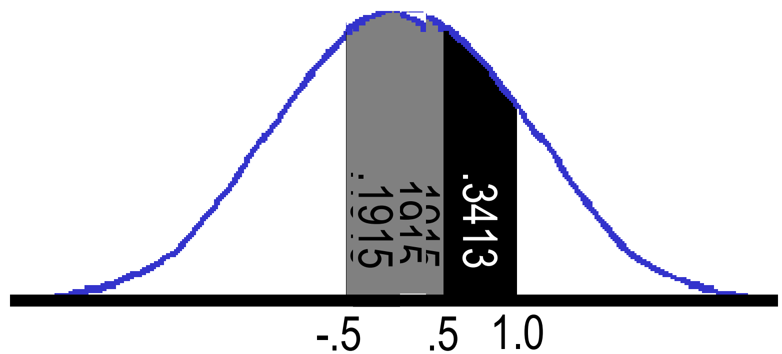
z	0	0.1	…….	0.05	0.06
0.0	0.0000	0.0040		0.0199	0.0239
0.1	0.0398	0.0438		0.0596	0.636
.	.				.
.	.				.
0.5	0.1915				….
.	.				.



Finding Normal Probabilities

- Example - continued

z	0	0.1	....	0.05	0.06
0.0	0.0000	0.0040		0.0199	0.0239
0.1	0.0398	0.0438		0.0596	0.636
.	.				.
.	.				.
0.5	0.1915				....
.	.				.



$$P(-.5<Z<1) = P(-.5<Z<0) + P(0<Z<1) = .1915 + .3413 = .5328$$

Finding Normal Probabilities

- Example - continued

z	0	0.1	…….	0.05	0.06
0.0	0.5000	0.5040		0.5199	0.5239
0.1	0.5398	0.5438		0.5596	0.5636
.	.	.		.	.
.	.	.		.	.
0.5	0.6915	….		….	….
.	.	.		.	.

P(Z<-0.5)=1-P(Z>-0.5)=1-0.6915=0.3085

By Symmetry

P(Z<0.5)

Finding Normal Probabilities

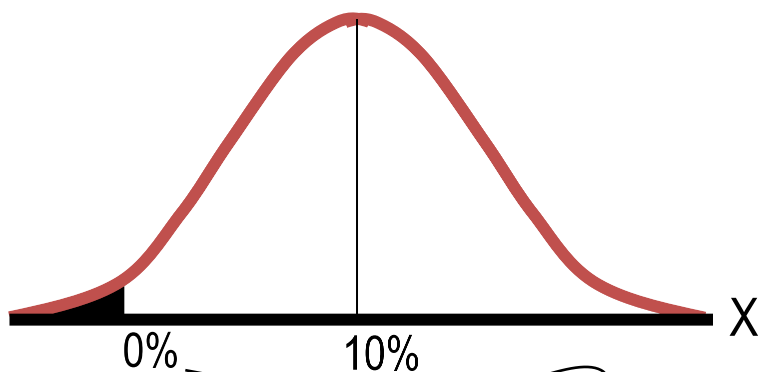
- Example - continued

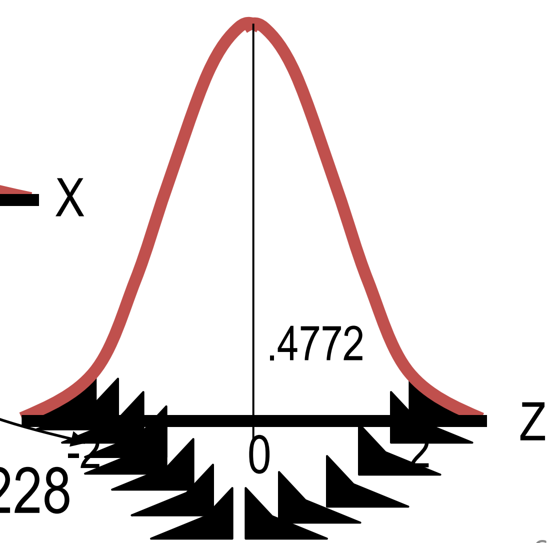
z	0	0.1	…….	0.05	0.06
0.0	0.5000	0.5040		0.5199	0.5239
0.1	0.5398	0.5438		0.5596	0.5636
.	.	.		.	.
.	.	.		.	.
0.5	0.6915	….		….	….
.	.	.		.	.

P(-.5<Z<1) = P(Z<1)-P(Z<-.5) = .8413-.3085 + = .5328

Finding Normal Probabilities

- Example
  - The rate of return (X) on an investment is normally distributed with mean of 10% and standard deviation of (i) 5%, (ii) 10%.
  - What is the probability of losing money?



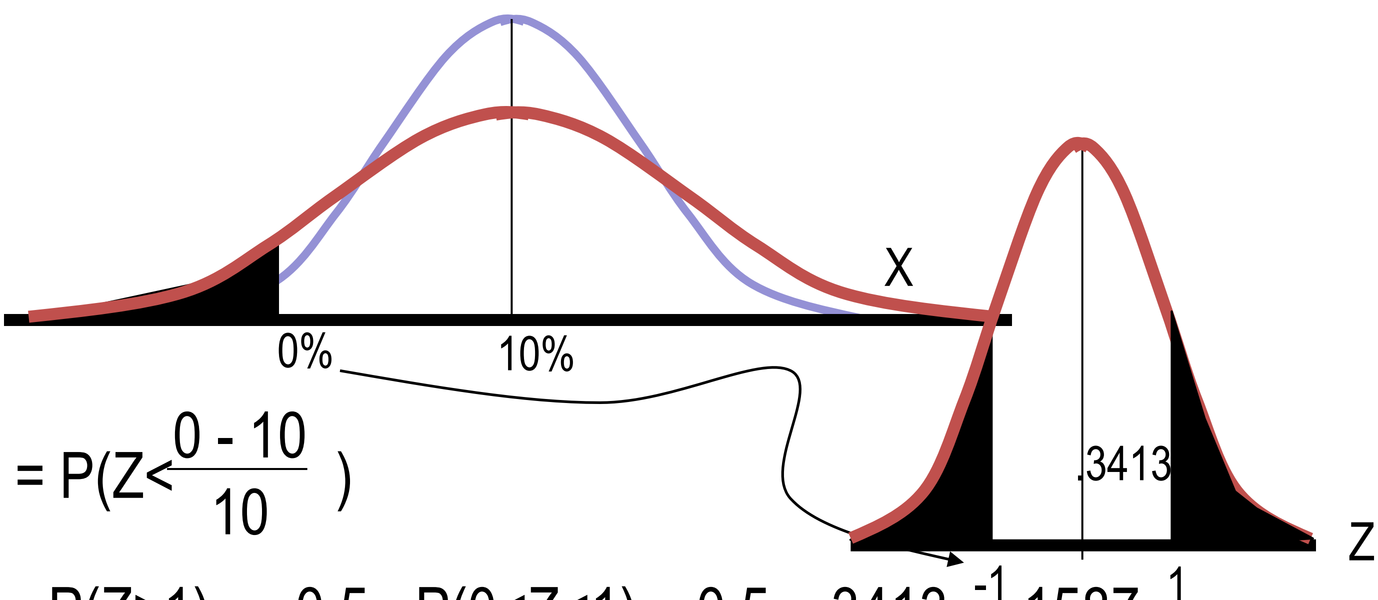


(i) $P(X < 0) = P\left(Z < \dfrac{0 - 10}{5}\right) = P(Z < -2)$

$= P(Z > 2) = 0.5 - P(0 < Z < 2) = 0.5 - .4772 = .0228$

Finding Normal Probabilities

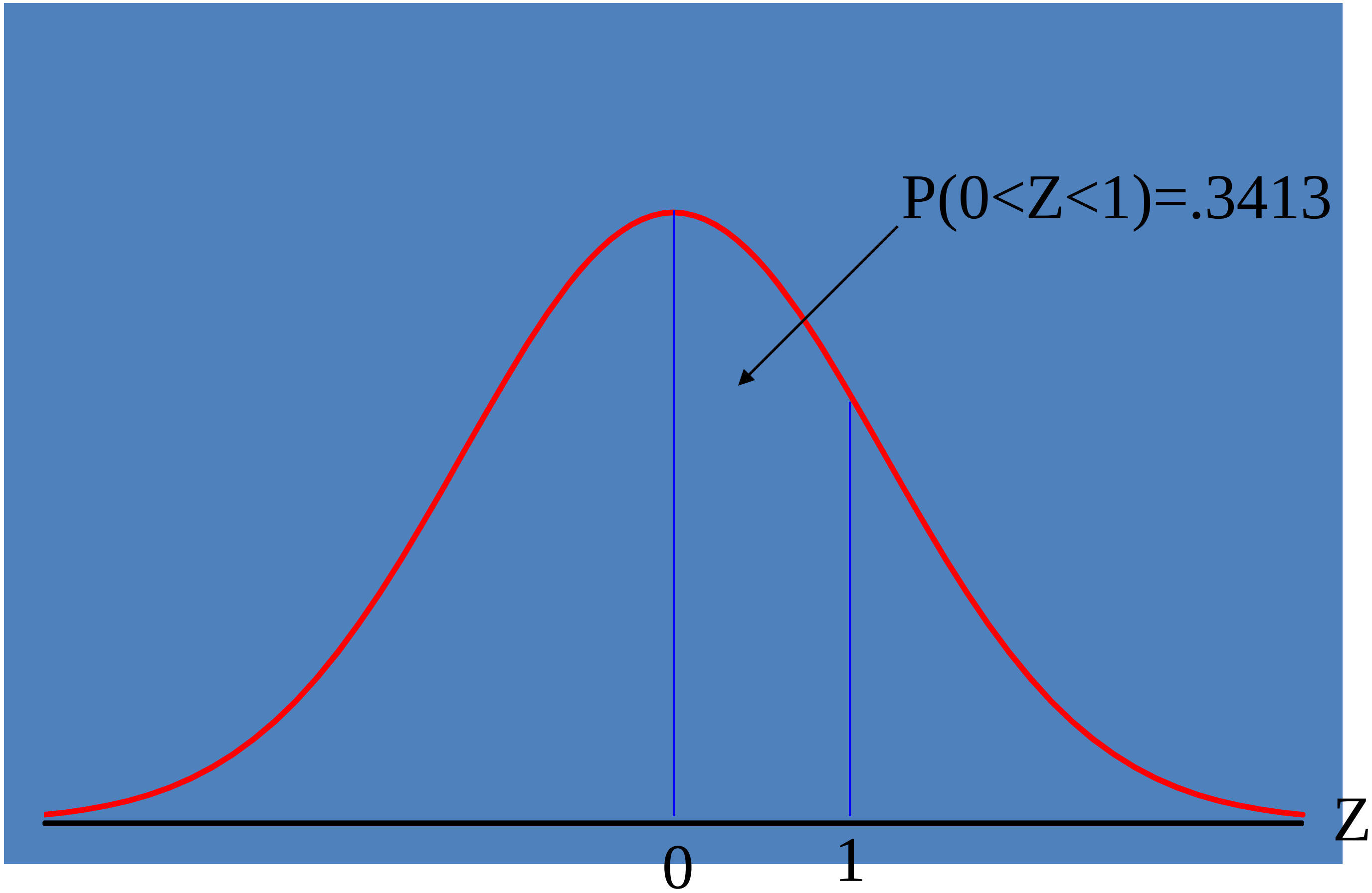
- Example
  - The rate of return (X) on an investment is normally distributed with mean of 10% and standard deviation of (i) 5%, (ii) 10%.
  - What is the probability of losing money?



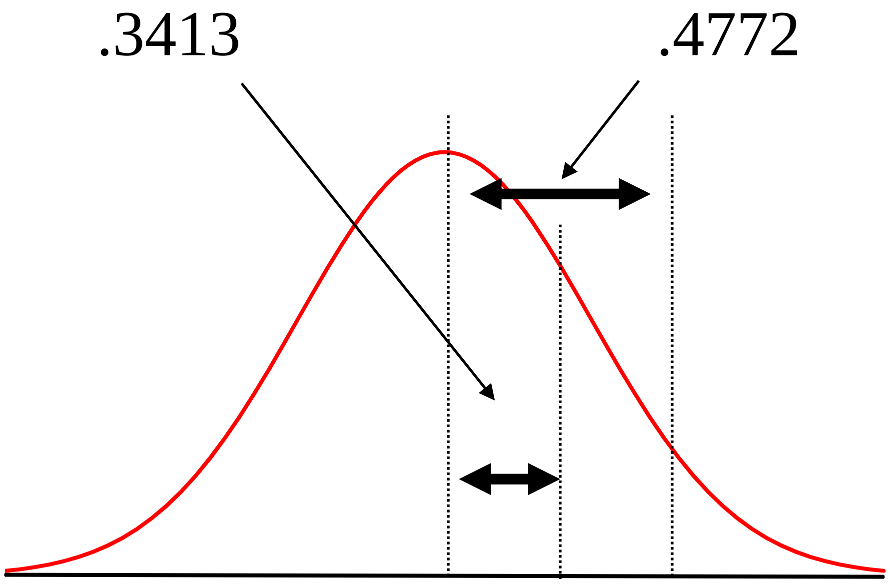
(ii) $P(X < 0) = P\left(Z < \dfrac{0 - 10}{10}\right)$

$= P(Z < -1) = P(Z > 1) = 0.5 - P(0 < Z < 1) = 0.5 - .3413 = .1587$

AREAS UNDER THE STANDARD NORMAL DENSITY



AREAS UNDER THE STANDARD NORMAL DENSITY

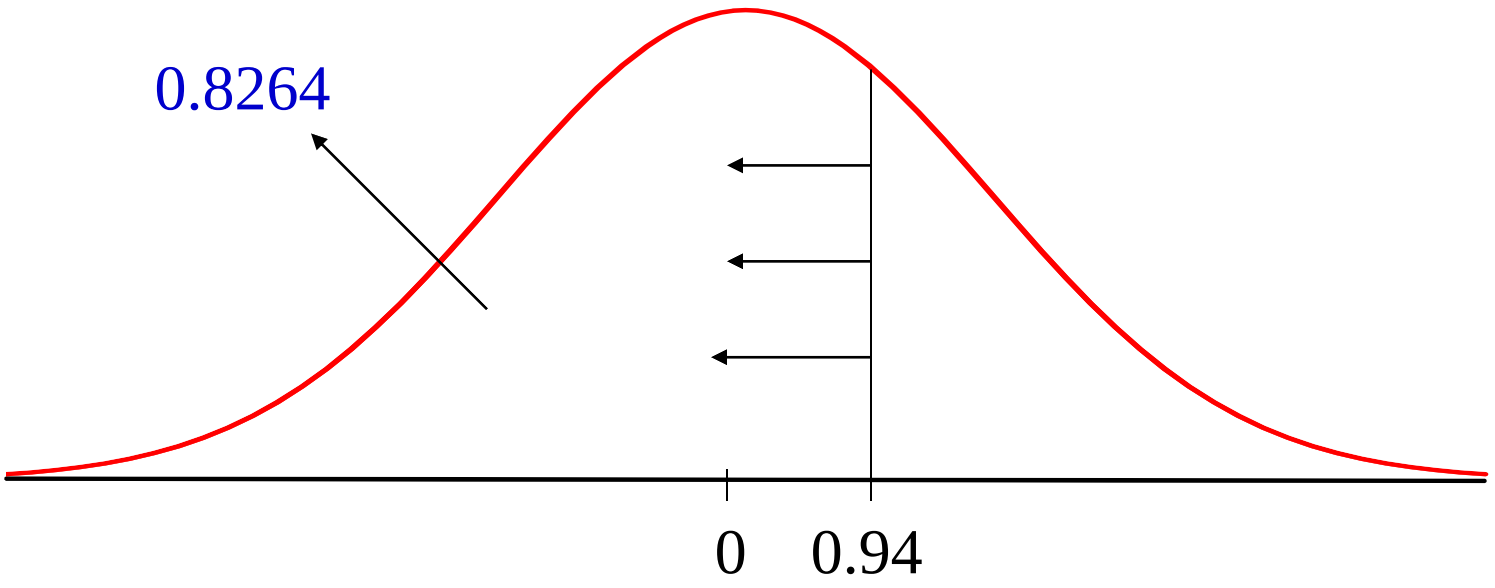


P(1<Z<2)=.4772-.3413=.1359

EXAMPLES

- $P(Z < 0.94) = 0.5 + P(0 < Z < 0.94)$

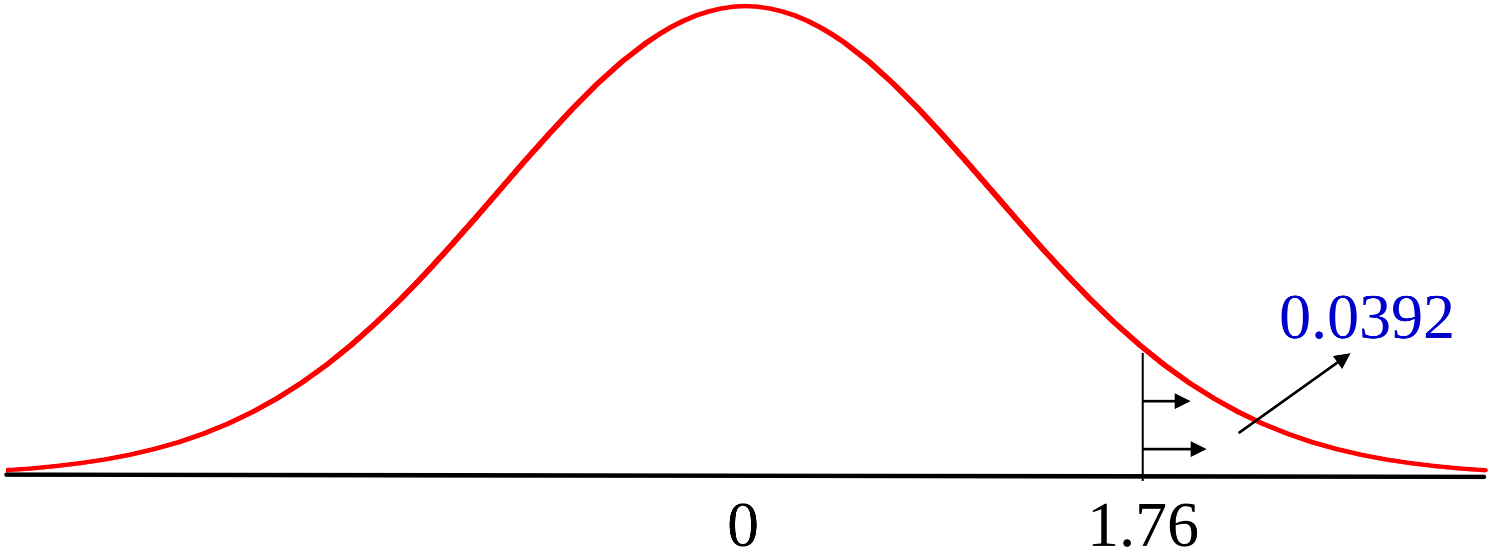
  $= 0.5 + 0.3264 = 0.8264$



EXAMPLES

- $P(\,Z > 1.76\,) = 0.5 - P(\,0 < Z < 1.76\,)$

  $= 0.5 - 0.4608 = 0.0392$



EXAMPLES

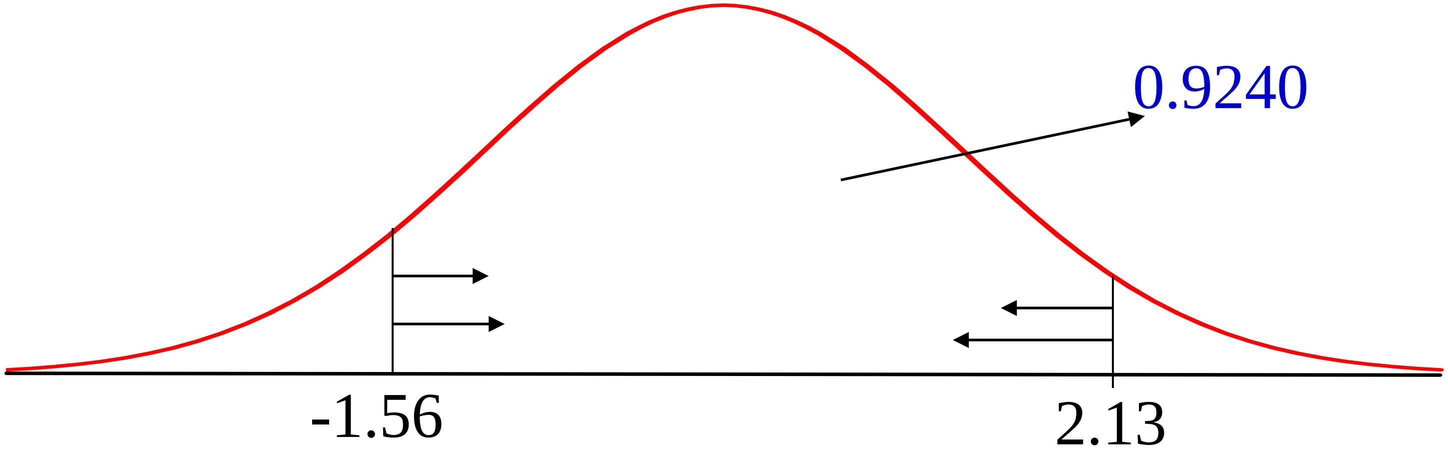
- $P(-1.56 < Z < 2.13) =$

$= P(-1.56 < Z < 0) + P(0 < Z < 2.13)$

Because of symmetry

$P(0 < Z < 1.56)$

$= 0.4406 + 0.4834 = 0.9240$



STANDARDIZATION FORMULA

- If X~N(μ , σ^2), then the standardized value Z of any 'X-score' associated with calculating probabilities for the X distribution is:

$$Z = \frac{X - \mu}{\sigma}$$

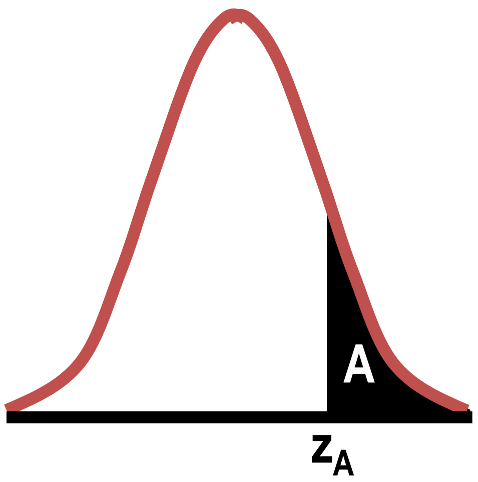
- The standardized value Z of any 'X-score' associated with calculating probabilities for the X distribution is:

(Converse Formula)

$$x = \mu + z.\sigma$$

Finding Values of Z

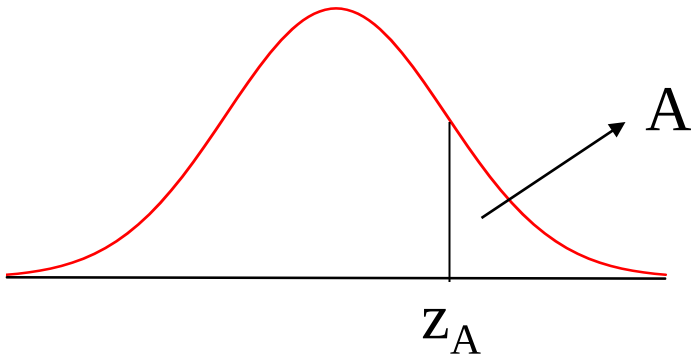
- Sometimes we need to find the value of Z for a given probability
- We use the notation z_A to express a Z value for which $P(Z > z_A) = A$



PERCENTILE

- The p^{th} percentile of a set of measurements is the value for which at most p% of the measurements are less than that value.
- 80^{th} percentile means $P(\ Z < a\ ) = 0.80$
- If $Z \sim N(0,1)$ and A is any probability, then

$$P(\ Z > z_A) = A$$

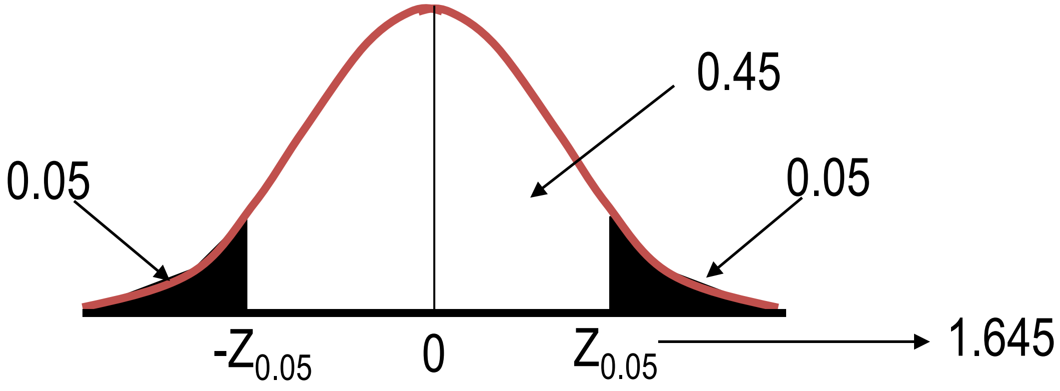


Finding Values of Z

- Example
  - Determine z exceeded by 5% of the population
  - Determine z such that 5% of the population is below

- Solution

  $z_{.05}$ is defined as the z value for which the area on its right under the standard normal curve is .05.



EXAMPLES

- Let X be rate of return on a proposed investment. Mean is 0.30 and standard deviation is 0.1.

a) $P(X>.55)=?$

b) $P(X<.22)=?$

c) $P(.25<X<.35)=?$

Standardization formula

d) 80^{th} Percentile of X is?

e) 30^{th} Percentile of X is?

Converse Formula

ANSWERS

a) $$\mathbf{P(X > 0.55) = P\left\{\frac{X - 0.3}{0.1} = Z > \frac{0.55 - 0.3}{0.1} = 2.5\right\} = 0.5 - 0.4938 = 0.0062}$$

b) $$\mathbf{P(X < 0.22) = P\left\{\frac{X - 0.3}{0.1} = Z < \frac{0.22 - 0.3}{0.1} = -0.8\right\} = 0.5 - 0.2881 = 0.2119}$$

c)

$$\mathbf{P(0.25 < X < 0.35) = P\left\{\frac{0.25 - 0.3}{0.1} = -0.5 < \frac{X - 0.3}{0.1} = Z < \frac{0.35 - 0.3}{0.1} = 0.5\right\}}$$

$$\mathbf{= 2.*(0.1915) = 0.3830}$$

d)

80th Percentile of *X* is $x = \mu + \sigma . z_{0.20} = .3 + (.85)*(.1) = .385$

e)

30th Percentile of *X* is $x = \mu + \sigma . z_{0.70} = .3 + (-.53)*(.1) = .247$

EXAMPLE

- LET X ~ N(10, 81). IF $P(X < a) = 0.95$, FIND a. (95th percentile)

EXAMPLE

– The mean time it took all competitors to run the 100 meters was 12.92 seconds. Assuming a standard deviation of 1.3 seconds and a normal distribution in times, what percentage of the competitors finished the race in under 10.5 seconds?

X:time that takes all competitors to run the 100m.

The Normal Approximation to the Binomial Distribution

- The normal distribution provides a close approximation to the Binomial distribution when n (number of trials) is large and p (success probability) is close to 0.5.
- The approximation is used **only** when

$$np \geq 5$$

$$n(1-p) \geq 5$$

The Normal Approximation to the Binomial Distribution

- If the assumptions are satisfied, the Binomial random variable *X* can be approximated by normal distribution with mean $\mu = np$ and $\sigma^2 = np(1-p)$.
- In probability calculations, the continuity correction must be used. For example, if X is Binomial random variable, then

P($X \leq$ a) approximated by normal X in prob. as P(X<a+0.5) but P(X < a) should be the same. Why?

P($X \geq$ a) approximated by normal X in prob. as P(X>a-0.5) but P(X > a) should be the same. Why?

EXAMPLE

- Probability of getting 16 heads in 40 flips of a balanced coin. Find the approximate probability of getting 16 heads.

EXAMPLE

- Suppose that 10% of all steel shafts produced by a certain process are nonconforming but can be reworked (rather than having to be scrapped). Consider a random sample of 200 shafts, and let X denote the number among these that are nonconforming and can be reworked. What is the (approximate) probability that X is

a) At most 30?

b) Less than 30?

c) Between 15 and 25 (inclusive)?

Exponential Distribution

- The exponential distribution can be used to model
  - the length of time between telephone calls
  - the length of time between arrivals at a service station
  - the lifetime of electronic components.
- When the number of occurrences of an event follows the Poisson distribution, the time between occurrences follows the exponential distribution.

Exponential Distribution

A random variable is exponentially distributed if its probability density function is given by

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda}, x > 0, \lambda > 0$$

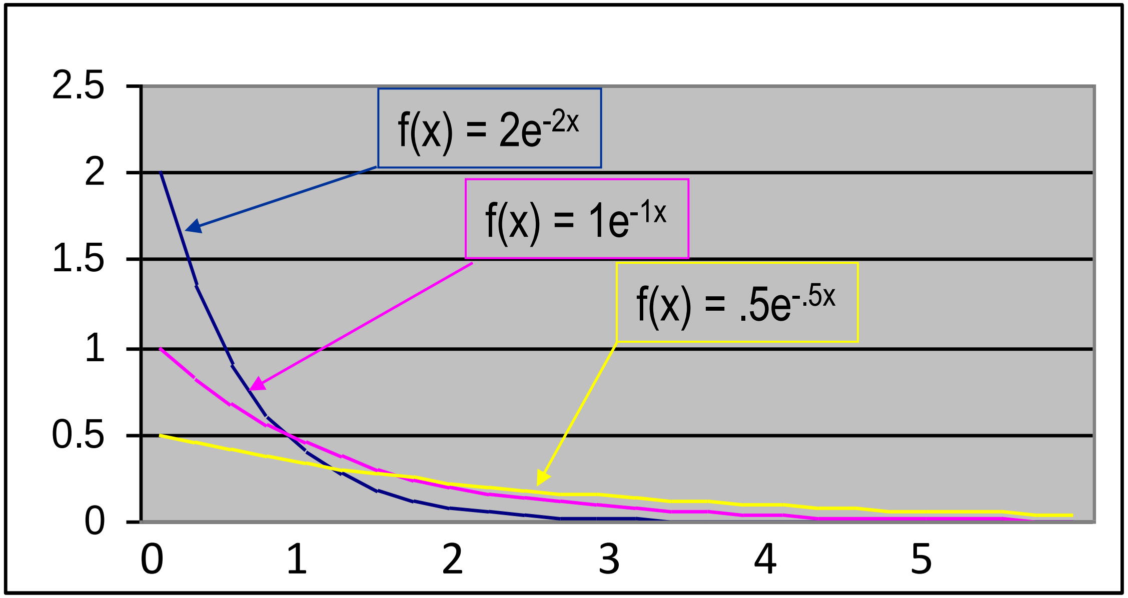
λ is a distribution parameter.

$$E(X) = \lambda \qquad V(X) = \lambda^2$$

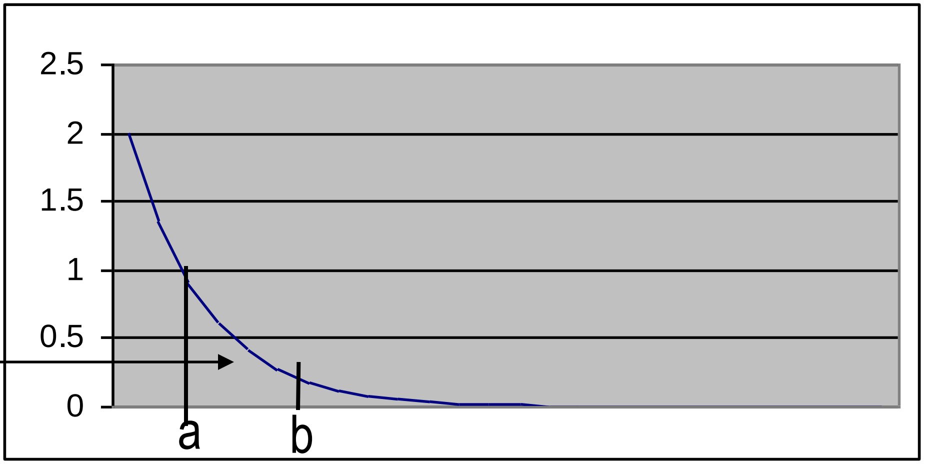
The cumulative distribution function is

$$F(x) = 1 - e^{-x/\lambda}, x \geq 0$$

Exponential distribution for λ^{-1} = .5, 1, 2



$$P(a<X<b) = e^{-a/\lambda} - e^{-b/\lambda}$$



Exponential Distribution

- Finding exponential probabilities is relatively easy:
  - $P(X > a) = e^{-a/\lambda}$.
  - $P(X < a) = 1 - e^{-a/\lambda}$
  - $P(a < X < b) = e^{-a/\lambda} - e^{-b/\lambda}$

Exponential Distribution

- Example
  - The lifetime of an alkaline battery is exponentially distributed with mean 20 hours.
  - Find the following probabilities:
    - The battery will last between 10 and 15 hours.
    - The battery will last for more than 20 hours.

Exponential Distribution

- Solution
  - The mean = standard deviation = $\lambda = 20$ hours.
  - Let X denote the lifetime.
    - $P(10<X<15) = e^{-.05(10)} – e^{-.05(15)} = .1341$
    - $P(X > 20) = e^{-.05(20)} = .3679$

Exponential Distribution

- Example

The service rate at a supermarket checkout is 6 customers per hour.

- If the service time is exponential, find the following probabilities:
  - A service is completed in 5 minutes,
  - A customer leaves the counter more than 10 minutes after arriving
  - A service is completed between 5 and 8 minutes.

Exponential Distribution

- Solution
  - A service rate of 6 per hour = A service rate of .1 per minute (λ^{-1} = .1/minute).
  - $P(X < 5) = 1\text{-}e^{-.lx} = 1 - e^{-.1(5)} = .3935$
  - $P(X > 10) = e^{-.lx} = e^{-.1(10)} = .3679$
  - $P(5 < X < 8) = e^{-.1(5)} - e^{-.1(8)} = .1572$

Exponential Distribution

- The failure time (in years) of an electronic digital display is an exponential random variable with mean 5.
- Find $P(X \leq 4)$.
- Find $P(X>8)$.

Exponential Distribution

- The key property of the exponential random variable is that it is memoryless. That is,

  $P(X > s+t \mid X > t) = P(X > s)$ for all s and $t \geq 0$.

Example: Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000 mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery?

Exponential Distribution

- If pdf of lifetime of fluorescent lamp is exponential with mean 0.10, find the life for 95% reliability?

The reliability function = $R(t) = 1 - F(t) = e^{-t/\lambda}$

$$R(t) = 0.95 \rightarrow t = ?$$

What is the mean time to failure?

GAMMA DISTRIBUTION

- Gamma Function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

where α is a positive integer.

Properties:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \alpha > 0$$

$$\Gamma(n) = (n-1)! \text{ for any integer } n > 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

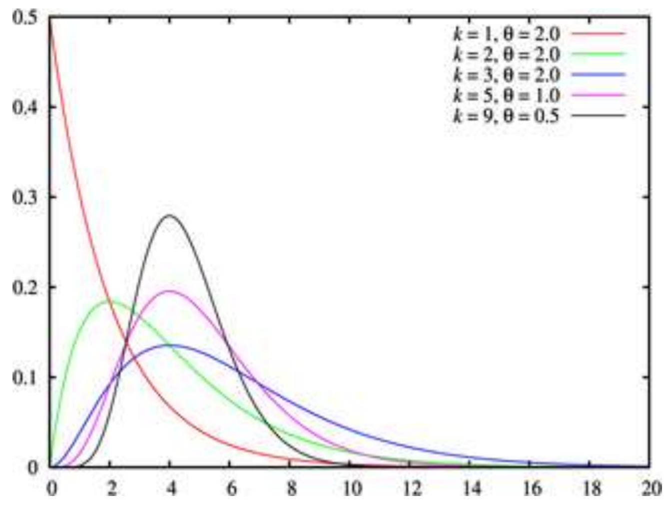
GAMMA DISTRIBUTION

- $X \sim Gamma(\alpha,\beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \; x > 0, \; \alpha > 0, \; \beta > 0$$

$$E(X) = \alpha\beta \text{ and } Var(X) = \alpha\beta^{2}$$

$$M(t) = (1-\beta t)^{-\alpha}, \; t < \frac{1}{\beta}$$



GAMMA DISTRIBUTION

- Let $X_1, X_2, \ldots, X_n$ be independent rvs with $X_i \sim Gamma(\alpha_i, \beta)$. Then,

$$\sum_{i=1}^{n} X_i \sim Gamma\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$$

- Let X be an rv with $X \sim Gamma(\alpha, \beta)$. Then,

$$cX \sim Gamma(\alpha, c\beta) \text{ where } c \text{ is positive constant.}$$

- Let $X_1, X_2, \ldots, X_n$ be a random sample with $X_i \sim Gamma(\alpha, \beta)$. Then,

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \sim Gamma\left(n\alpha, \frac{\beta}{n}\right)$$

CHI-SQUARE DISTRIBUTION

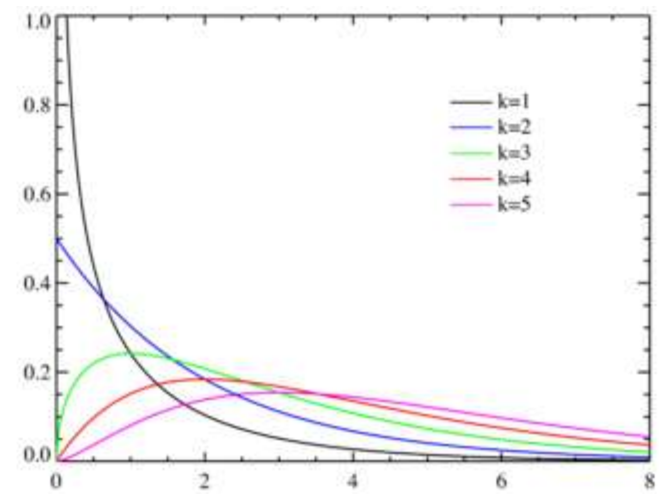
Chi-square with α degrees of freedom

- $X \sim \chi^2(\alpha) = Gamma(\alpha/2, 2)$

$$f(x) = \frac{1}{\Gamma(\alpha/2)2^{\alpha/2}} x^{\alpha/2-1} e^{-x/2}, \; x > 0, \; \alpha > 0$$

$$E(X) = \alpha \text{ and } Var(X) = 2\alpha$$

$$M(t) = (1-2t)^{-\alpha/2}, \; t < \frac{1}{2}$$



DEGREES OF FREEDOM

- In statistics, the phrase ***degrees of freedom*** is used to describe the number of values in the final calculation of a statistic that are free to vary.

- The number of independent pieces of information that go into the estimate of a parameter is called the degrees of freedom (df) .

- Mathematically, df is the dimension of the domain of a random vector, or essentially the number of 'free' components: how many components need to be known before the vector is fully determined?

CHI-SQUARE DISTRIBUTION

- If rv X has *Gamma*(α,β) *distribution, then* $Y=2X/\beta$ *has Gamma*$(\alpha,2)$ *distribution. If* 2α *is positive integer, then* Y *has* $\chi^2_{2\alpha}$ *distribution.*
  - Let X be an rv with $X \sim N(0, 1)$. Then,

$$X^2 \sim \chi^2_1$$

- Let $X_1, X_2, \ldots, X_n$ be a r.s. with $X_i \sim N(0,1)$. Then,

$$\sum_{i=1}^{n} X_i^2 \sim \chi^2_n$$

WEIBULL DISTRIBUTION

- To model the failure time data or hazard functions.

- If $X \sim Exp(\theta)$, then $Y = X^{1/\gamma}$ has $Weibull(\gamma, \theta)$ distribution.

$$f_Y(y) = \frac{\gamma}{\theta} y^{\gamma-1} e^{-y^{\gamma}/\theta}, \; y > 0, \; \gamma > 0, \; \theta > 0$$

BETA DISTRIBUTION

- The Beta family of distributions is a continuous family on (0,1) and often used to model proportions.

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1},\ 0<x<1,\ \alpha>0,\ \beta>0.$$

where

$$B(\alpha,\beta) = \int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$E(X) = \frac{\alpha}{\alpha+\beta} \text{ and } Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

CAUCHY DISTRIBUTION

- It is a symmetric and bell-shaped distribution on $(-\infty,\infty)$ with pdf

$$f(x)=\frac{1}{\pi}\frac{1}{1+(x-\theta)^2}, -\infty<x<\infty, -\infty<\theta<\infty,$$

Since $E|X|=\infty$, the mean does not exist.

- The mgf does not exist.
- θ measures the center of the distribution and it is the median.

- If X and Y have N(0,1) distribution, then Z=X/Y has a Cauchy distribution with $\theta=0$.

CAUCHY DISTRIBUTION

- The Cauchy distribution is important as an example of a pathological case. When studying hypothesis tests that assume normality, seeing how the tests perform on data from a Cauchy distribution is a good indicator of how sensitive the tests are to heavy-tail departures from normality. Likewise, it is a good check for robust techniques that are designed to work well under a wide variety of distributional assumptions.
- Its importance in physics is due to it being the solution to the differential equation describing forced resonance.
- In spectroscopy, it is the description of the line shape of spectral lines which are subject to homogeneous broadening in which all atoms interact in the same way with the frequency range contained in the line shape.

LOG-NORMAL DISTRIBUTION

- An rv X is said to have the ***lognormal distribution***, with parameters μ and σ^2, if $Y=ln(X)$ has the N(μ, σ^2) distribution.

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}x^{-1}e^{-(\ln x-\mu)^2/\left(2\sigma^2\right)},-\infty<x<\infty,-\infty<\mu<\infty,\sigma^2>0$$

- The lognormal distribution is used to model continuous random quantities when the distribution is believed to be skewed, such as certain income and lifetime variables.

STUDENT'S T DISTRIBUTION

- This distribution will arise in the study of a standardized version of the sample mean when the underlying distribution is normal.
- Let Z be a standard normal rv and let U be a chi-square distributed rv independent of Z, with ν degrees of freedom. Then,

$$X = \frac{Z}{\sqrt{U/\nu}} \sim t_{\nu}$$

When $n \to \infty$, $X \to N(0,1)$.

F DISTRIBUTION

- This distribution arises from ratios of sums of squares when sampling from a normal distribution.
- Let U and V be independent rvs with chi-square distributions with ν_1 and ν_2 degrees of freedom. Then,

$$X = \frac{U/\nu_1}{V/\nu_2} \sim F_{\nu_1,\nu_2}$$