LIMITING DISTRIBUTIONS

CONVERGENCE IN DISTRIBUTION

- Consider that $X_1, X_2, \ldots, X_n$ is a sequence of rvs and $Y_n = u(X_1, X_2, \ldots, X_n)$ be a function of rvs with cdfs $F_n(y)$ so that for each $n = 1, 2, \ldots$

$$F_n(y) = P(Y_n \le y),$$

$$\lim_{n\to\infty} F_n(y) = F(y) \text{ for all } y$$

where $F(y)$ is continuous. Then, the sequence $X_1, X_2, \ldots, X_n$ is said to converge in distribution.

$$Y_n \xrightarrow{d} Y$$

CONVERGENCE IN DISTRIBUTION

- Theorem: If $\lim_{n\to\infty} F_n(y) = F(y)$ for every point y at which $F(y)$ is continuous, then Y_n *is said to have a* limiting distribution with cdf $F(y)$.

- Definition of convergence in distribution requires only that limiting function agrees with cdf at ***its points of continuity***.

EXAMPLES

1. Let $\{X_n\}$ be a sequence of rvs with pmf

$$f_n(x) = P(X = x) = \begin{cases} 1, \text{ if } x = 2 + \dfrac{1}{n} \\ 0, \text{ o.w.} \end{cases}$$

Find the limiting distribution of X_n.

EXAMPLES

2. Let X_n have the pmf

$$f_n(x) = 1 \text{ if } x = n.$$

Find the limiting distribution of X_n.

EXAMPLES

3. Let $X_n \sim N(\mu, \sigma^2)$ be a sequence of Normal rvs. Let $\bar{X}_n$ be the sample mean. Find the limiting distribution of $\bar{X}_n$.

CONVERGENCE IN PROBABILITY (STOCHASTIC CONVERGENCE)

- A rv Y_n convergence in probability to a rv Y if

$$\lim_{n\to\infty} P\left(\left|Y_n - Y\right| < \varepsilon\right) = 1$$

for every $\varepsilon > 0$.

Special case: $Y=c$ where c is a constant not depending on n.

The limiting distribution of Y_n is degenerate at point c.

CHEBYSHEV'S INEQUALITY

- Let X be an rv with $E(X)=\mu$ and $V(X)=\sigma^2$.

$$P\left(\left|X-\mu\right|<\varepsilon\right)\geq 1-\frac{\sigma^2}{\varepsilon^2},\varepsilon>0$$

- The Chebyshev's Inequality can be used to prove stochastic convergence in many cases.

CONVERGENCE IN PROBABILITY (STOCHASTIC CONVERGENCE)

- The Chebyshev's Inequality proves the convergence in probability if the following three conditions are satisfied.

1. $E(Y_n)=\mu_n$ where $\lim_{n\to\infty} \mu_n = \mu.$

2. $V\left(Y_n\right) = \sigma_n^2 < \infty$ for all $n.$

3. $\lim_{n\to\infty} \sigma_n^2 = 0.$

EXAMPLES

1. Let X be an rv with $E(X)=\mu$ and $V(X)=\sigma^2<\infty$. For a r.s. of size n, $\bar{X}_n$ is the sample mean. Is $\bar{X}_n \xrightarrow{p} \mu$?

EXAMPLES

- Let Z_n be χ_n^2 and let $W_n = Z_n / n^2$.

  Show that the limiting distribution of W_n is degenerate at 0.

WEAK LAW OF LARGE NUMBERS

- Let $X_1, X_2,\ldots,X_n$ be iid rvs with $E(X_i)=\mu$ and $V(X_i)=\sigma^2<\infty$. Define $\bar{X}_n = 1/n\sum_{i=1}^{n} X_i$. Then, for every $\varepsilon>0$,

$$\lim_{n\to\infty} P\left(\left|\bar{X}_n - \mu < \varepsilon\right|\right) = 1,$$

that is, $\bar{X}_n$ converges in probability to μ.

STRONG LAW OF LARGE NUMBERS

- Let $X_1, X_2, \ldots, X_n$ be iid rvs with $E(X_i)=\mu$ and $V(X_i)=\sigma^2<\infty$. Define $\bar{X}_n = 1/n\sum_{i=1}^{n} X_i$. Then, for every ε>0,

$$P\left(\lim_{n\to\infty}\left|\bar{X}_n-\mu\right|<\varepsilon\right)=1$$

that is, $\bar{X}_n$ converges almost sure to μ.

LIMITING MOMENT GENERATING FUNCTIONS

- Let rv Y_n have an mgf $M_n(t)$ that exists for all n. If

$$\lim_{n\to\infty} M_n(t) = M(t),$$

then Y_n has a limiting distribution which is defined by $M(t)$.

EXAMPLES

1. Let $X_n \sim Gamma(n, \beta)$ where β does not depend on n. Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

EXAMPLES

2. Let $X_n \sim Exp(1)$ *and* $\bar{X}_n$ *be the sample mean of r.s. of size* n*. Find the limiting distribution of*

$$Y_n = \sqrt{n}\left(\bar{X}_n - 1\right).$$

THE CENTRAL LIMIT THEOREM

- Let $X_1, X_2, \ldots, X_n$ be a sequence of iid rvs whose mgf exist in a neighborhood of 0. Let $E(X_i)=\mu$ and $V(X_i)=\sigma^2>0$. Define $\bar{X}_n = 1/n\sum_{i=1}^{n} X_i$. Then,

$$Z = \frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sigma} \xrightarrow{d} N(0,1)$$

or

$$Z = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{d} N(0,1).$$

EXAMPLES

1. Let X_n~ $Exp(1)$ *and* $\bar{X}_n$ *be the sample mean of r.s. of size* n*.* Find the limiting distribution of

$$Y_n = \sqrt{n}\left(\bar{X}_n - 1\right).$$

EXAMPLES

2. Let $\overline{X}_n$ be the sample mean from a r.s. of size $n=100$ from χ^2_{50}. Compute approximate value of $P\left(49 < \overline{X} < 51\right)$.

SLUTKY'S THEOREM

- If $X_n \to X$ in distribution and $Y_n \to a$, a constant, in probability, then

  a) $Y_n X_n \to aX$ in distribution.

  b) $X_n + Y_n \to X + a$ in distribution.

SOME THEOREMS ON LIMITING DISTRIBUTIONS

- If $X_n \to c > 0$ in probability,

$$\sqrt{X_n} \xrightarrow{p} \sqrt{c}.$$

- If $X_n \to c$ in probability and $Y_n \to c$ in probability, then
  - $aX_n + bY_n \to ac + bd$ in probability.
  - $X_n Y_n \to cd$ in probability
  - $1/X_n \to 1/c$ in probability for all $c \neq 0$.

EXAMPLES

1. $X \sim Gamma(\mu, 1)$. Show that

$$\frac{\sqrt{n}\left(\bar{X}_n - \mu\right)}{\sqrt{\bar{X}_n}} \xrightarrow{d} N\left(0,1.\right)$$

EXAMPLES

- $X \sim Gamma(1,n)$. Let

$$Z_n = \frac{X_n - n}{\sqrt{n}}$$

Let $Z_n \to N(0,1)$ in distribution and $Y_n \to c$ in probability. Find the limiting distribution of the following

$a) W_n = Y_n Z_n$.

$b) U_n = Z_n / n$.

$c) V_n = Z_n + Y_n$.

ORDER STATISTICS

- Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from a distribution of continuous type having pdf $f(x)$, $a<x<b$. Let $X_{(1)}$ be the smallest of X_i, $X_{(2)}$ be the second smallest of X_i,…, and $X_{(n)}$ be the largest of X_i.

$$a < X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)} < b$$

  - $X_{(i)}$ is the i-th order statistic.

$$X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$$

$$X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$$

ORDER STATISTICS

- If $X_1, X_2,\ldots,X_n$ be a r.s. of size n from a population with continuous pdf $f(x)$, then the joint pdf of the order statistics

  $X_{(1)}, X_{(2)},\ldots,X_{(n)}$ is

$$g\left(x_{(1)}, x_{(2)}, \cdots, x_{(n)}\right) = n!\, f\left(x_{(1)}\right) f\left(x_{(2)}\right) \cdots f\left(x_{(n)}\right)$$

ORDER STATISTICS

- The Maximum Order Statistic: $X_{(n)}$

$$G_{X_{(n)}}(y) = P\left(X_{(n)} \leq y\right)$$

$$g_{X_{(n)}}(y) = \frac{\partial}{\partial y} G_{X_{(n)}}(y)$$

ORDER STATISTICS

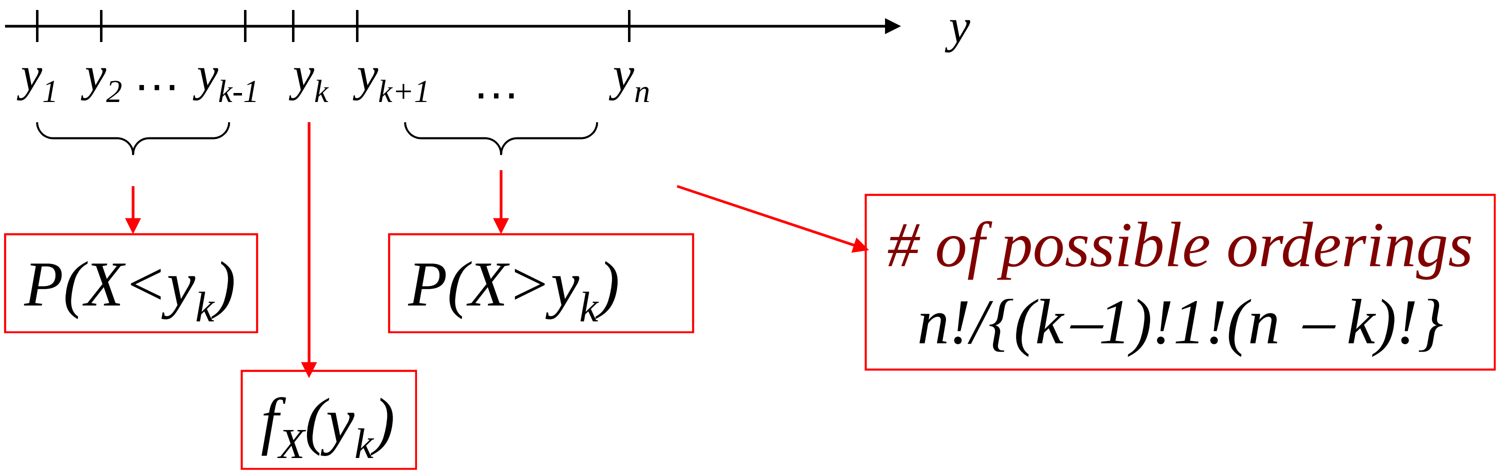
- The Minimum Order Statistic: $X_{(1)}$

$$G_{X_{(1)}}(y) = P\left(X_{(1)} \le y\right)$$

$$g_{X_{(1)}}(y) = \frac{\partial}{\partial y} G_{X_{(1)}}(y)$$

ORDER STATISTICS

- *k*-th Order Statistic



$$g_{X_{(k)}(y)} = \frac{n!}{(k-1)!(n-k)!}\left[F_X(y)\right]^{k-1} f_X(y)\left[1-F_X(y)\right]^{n-k}, a<y<b$$

EXAMPLE

- $X \sim Uniform(0,\theta)$. A r.s. of size n is taken. $X_{(n)}$ is the largest order statistic. Then,

a) Find the limiting distribution of $X_{(n)}$.

b) Find the limiting distribution of $Z_n = n(\theta - X_{(n)})$.