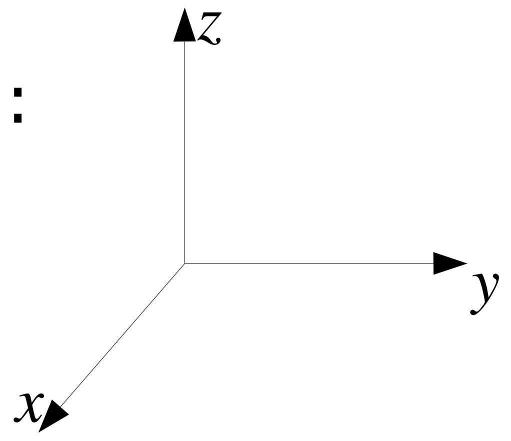
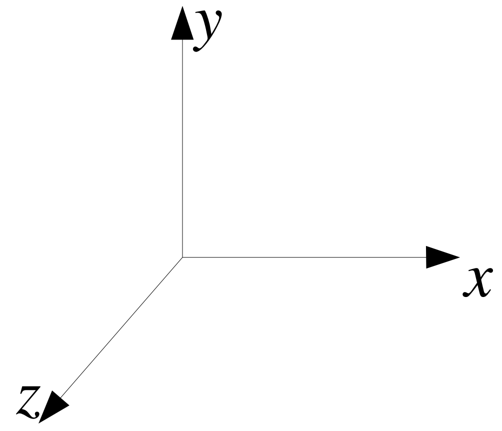
3 DIMENSIONAL TRANSFORMATIONS

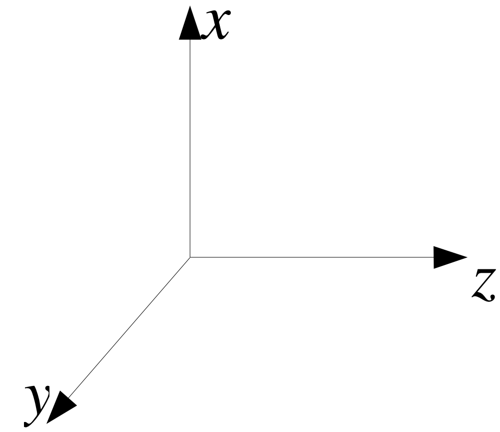
Ceng 477 Computer Engineering
Lecture notes
METU

3D Transformations



- x,y,z coordinates. Usual notation: Right handed coordinate system
- Similer to 2D. 4 dimensions in homogenous coordinates.
- Basic transformations:
  - Translation
  - Rotation
  - Scaling



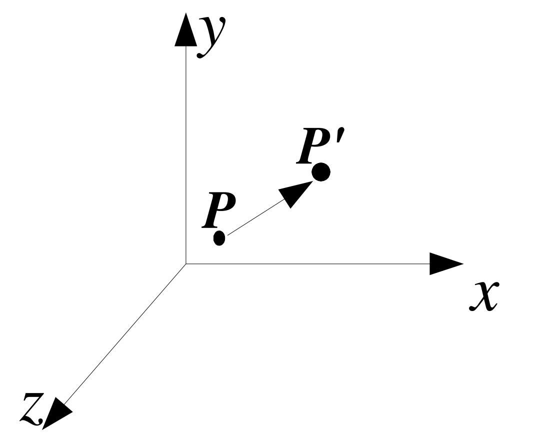


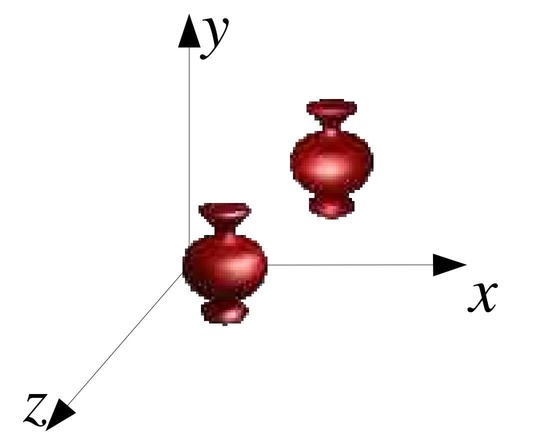
Translation

- move the object to a relative position.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

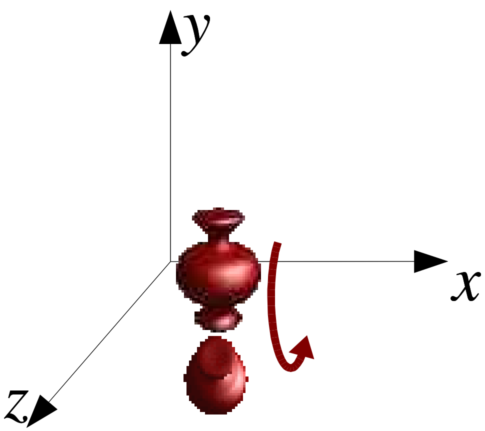
$$\boldsymbol{P}' = \boldsymbol{T} \cdot \boldsymbol{P}$$



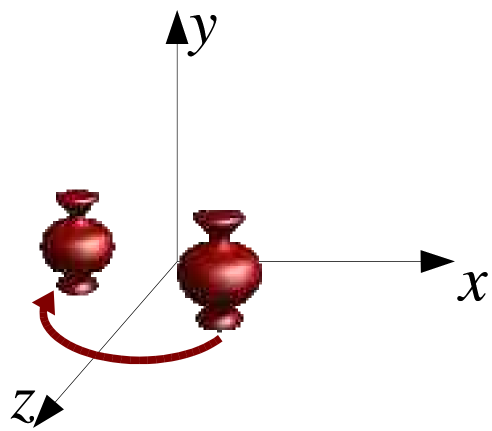


Rotation

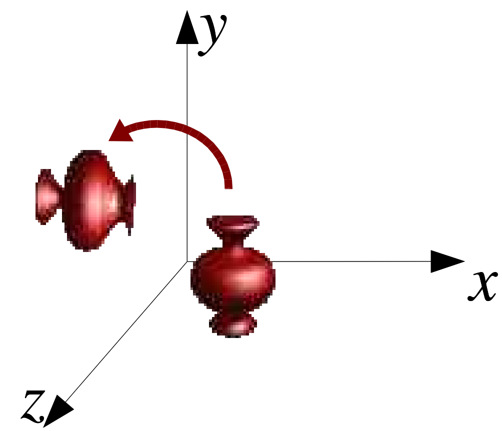
- Rotation arround the coordinate axes



x axis



y axis



z axis

- Arround *x*

$$\boldsymbol{R_x}(\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \boldsymbol{P}'=\boldsymbol{R_x}(\theta)\cdot\boldsymbol{P}$$

- Arround *y*

$$\boldsymbol{R_y}(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta & 0\\ 0 & 1 & 0 & 0\\ \sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \boldsymbol{P}'=\boldsymbol{R_y}(\theta)\cdot\boldsymbol{P}$$

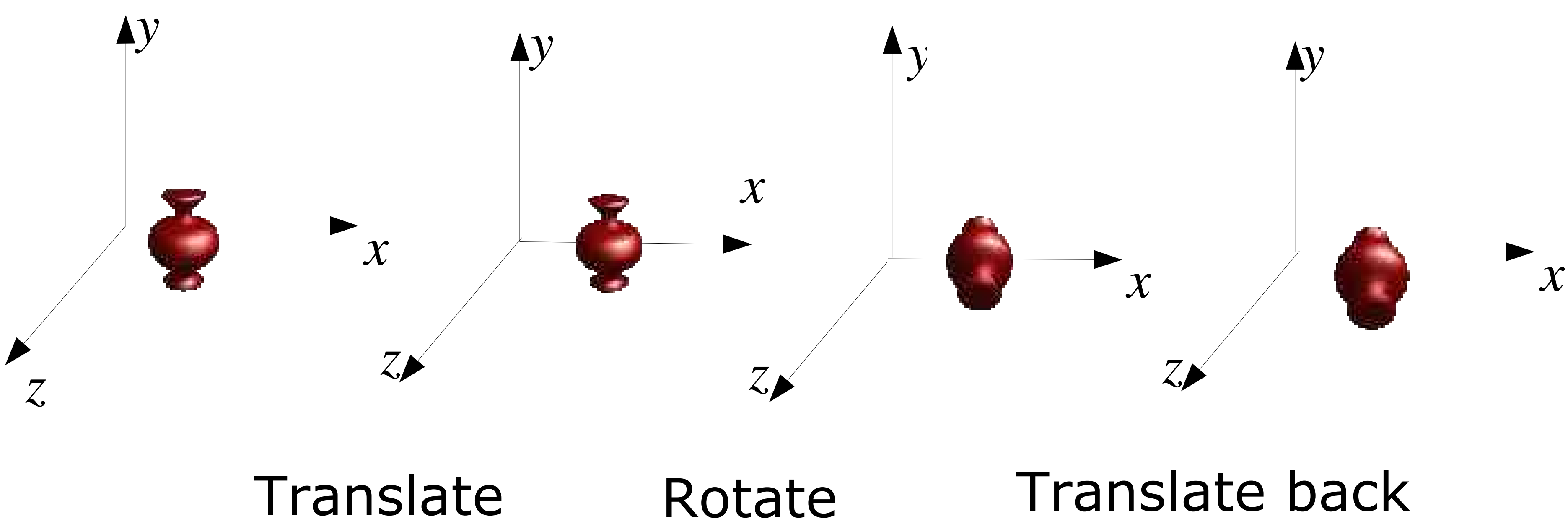
- Arround *z*

$$\boldsymbol{R_z}(\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \boldsymbol{P}'=\boldsymbol{R_z}(\theta)\cdot\boldsymbol{P}$$

Rotation Arround a Parallel Axis

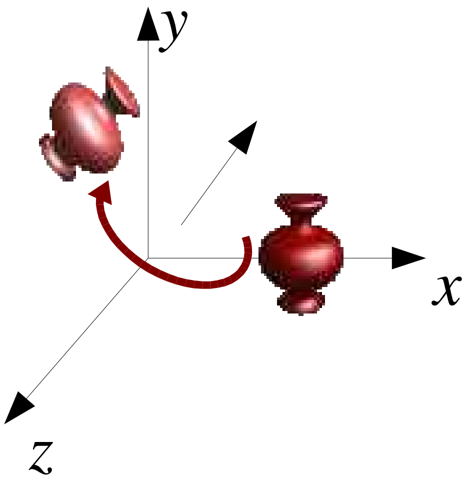
- Rotating the object on an line parallel to one of the axes. Translate to axis, rotate, translate back.

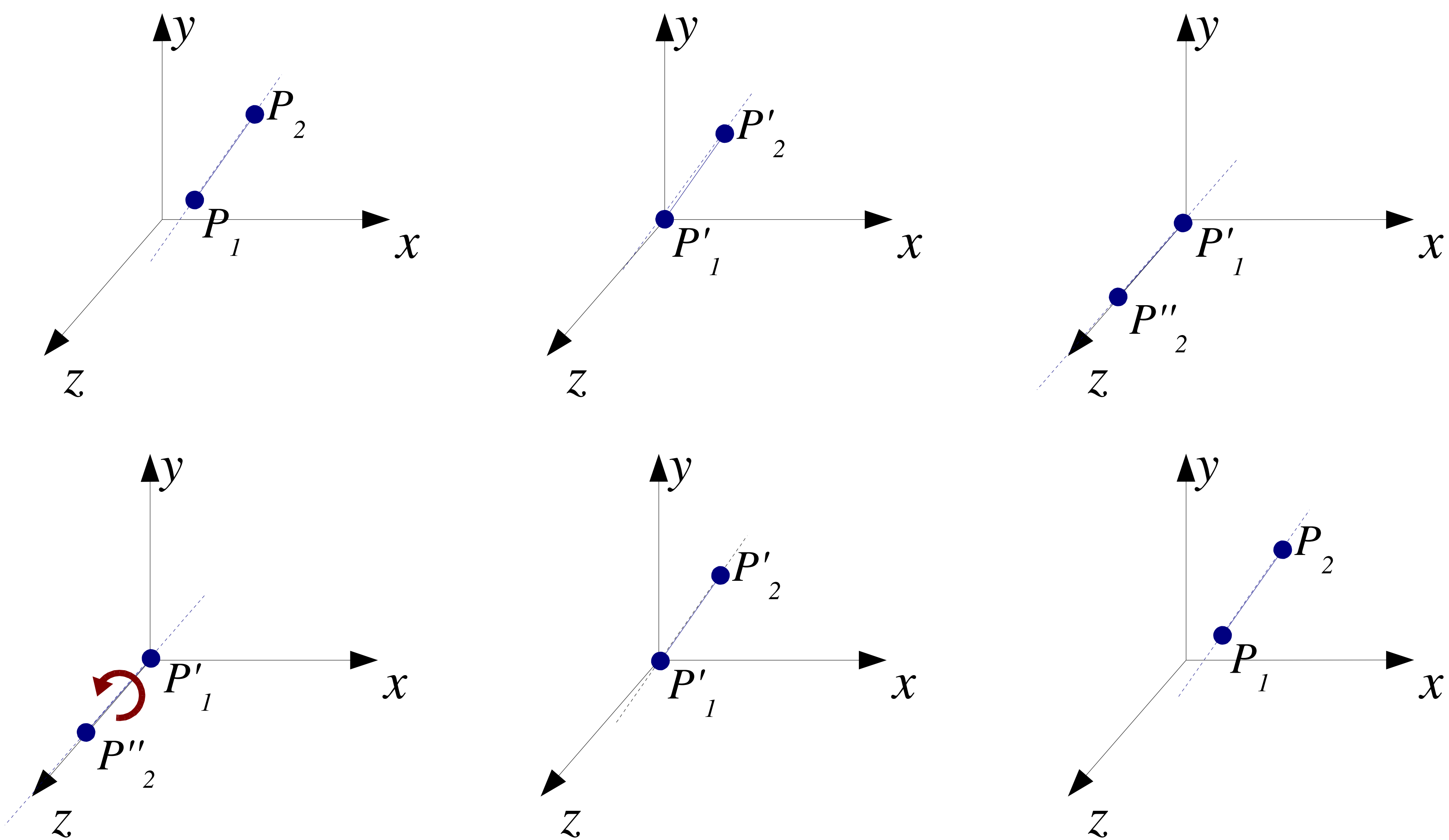
$$\boldsymbol{P}' = \boldsymbol{T}(0, y_p, z_p) \cdot \boldsymbol{R_x}(\theta) \cdot \boldsymbol{T}(0, -y_p, -z_p) \cdot \boldsymbol{P}$$



Rotation Arround an Arbitrary Axis

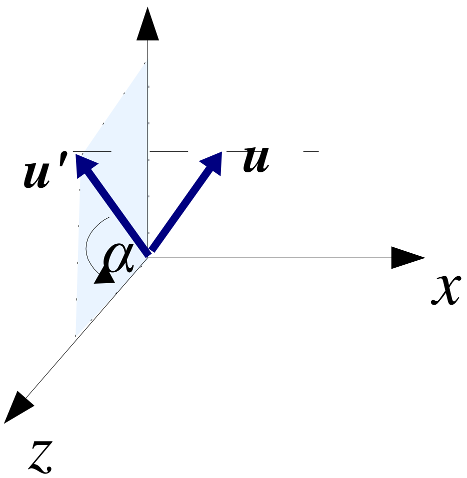
- Translate start to origin
- Rotate axis so that it is aligned with one of the axis
- Make the rotation
- Reverse the axis rotation
- Translate back




y
x
z
P 2
P 1
P' 2
P' 1
P'' 2

$$\boldsymbol{V} = P_2 - P_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$\boldsymbol{u}$, unit vector along $\boldsymbol{V}$: $\quad \boldsymbol{u} = \frac{\boldsymbol{V}}{|\boldsymbol{V}|} = (a, b, c)$



$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{u}' = (0, b, c)$$

We need cosine and sine of α for rotation:

$$\cos\alpha = \frac{\boldsymbol{u}' \cdot \boldsymbol{u}_z}{|\boldsymbol{u}'| \cdot |\boldsymbol{u}_z|} = \frac{c}{d} \qquad d = \sqrt{b^2 + c^2}$$

$$\boldsymbol{u}' \times \boldsymbol{u} = \boldsymbol{u}_x |\boldsymbol{u}'| |\boldsymbol{u}_z| \sin\alpha = \boldsymbol{u}_x b \qquad b = d \sin\alpha$$

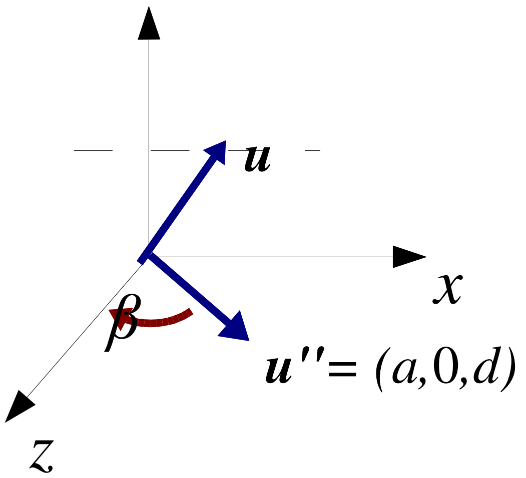
$$\cos\alpha = \frac{c}{d} \qquad \sin\alpha = \frac{b}{d} \qquad \boldsymbol{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cos\beta = \frac{\boldsymbol{u}''\cdot\boldsymbol{u}_z}{|\boldsymbol{u}''|\cdot|\boldsymbol{u}_z|} = d$$

$$\boldsymbol{u}''\times\boldsymbol{u} = \boldsymbol{u}_y|\boldsymbol{u}''||\boldsymbol{u}_z|\sin\beta = \boldsymbol{u}_y(-a)$$

$$\cos\beta = d \qquad \sin\beta = -a \qquad \sin(-\beta) = a$$

$$\boldsymbol{R}_y(-\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



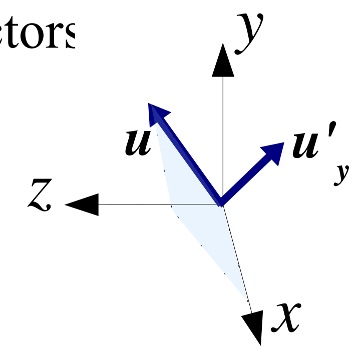
$$\boldsymbol{T}(x_1, y_1, z_1)\cdot\boldsymbol{R}_x(-\alpha)\cdot\boldsymbol{R}_y(\beta)\cdot\boldsymbol{R}_z(\theta)\cdot\boldsymbol{R}_y(-\beta)\cdot\boldsymbol{R}_x(\alpha)\cdot T(-x_1, -y_1, -z_1)$$

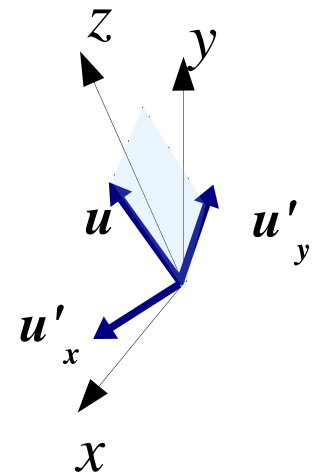
Rotation, ... Alternative Method

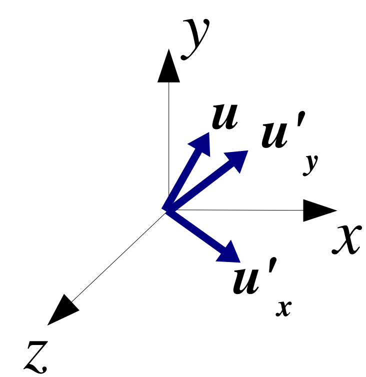
Any rotation on origin can be represented by 3 orthogonal unit vectors

$$\boldsymbol{R}=\begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{32} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boldsymbol{r}_{1*}, \boldsymbol{r}_{2*}, \boldsymbol{r}_{3*}$ are orthogonal unit vectors.







Purpose: align $\boldsymbol{u}'_{\{x,y,z\}}$ to $\boldsymbol{u}_{\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\}}$

Start: align $\boldsymbol{u}'_z$ to $\boldsymbol{u}$

$$\boldsymbol{u}'_z=\boldsymbol{u}$$

$\boldsymbol{u}'_y$ is orthogonal to $\boldsymbol{u}'_z$ and $\boldsymbol{u}_{\boldsymbol{x}}$

$$\boldsymbol{u}'_y=\frac{\boldsymbol{u}\times\boldsymbol{u}_{\boldsymbol{x}}}{|\boldsymbol{u}\times\boldsymbol{u}_{\boldsymbol{x}}|}$$

$\boldsymbol{u}'_x$ is orthogonal to $\boldsymbol{u}'_z$ and $\boldsymbol{u}'_y$

$$\boldsymbol{u}'_x=\boldsymbol{u}'_y\times\boldsymbol{u}_z$$

$$\boldsymbol{u}'_z = (a, b, c)$$

$$\boldsymbol{u}'_z \times \boldsymbol{u_x} = \begin{vmatrix} u_x & u_y & u_z \\ a & b & c \\ 1 & 0 & 0 \end{vmatrix} = (0, c, -b) \qquad \boldsymbol{u}'_y = \frac{(0, c, -b)}{\sqrt{b^2 + c^2}} = (0, c/d, -b/d)$$

$$\boldsymbol{u}'_y \times \boldsymbol{u_z} = \begin{vmatrix} u_x & u_y & u_z \\ 0 & c/d & -b/d \\ a & b & c \end{vmatrix} = u_x \frac{b^2 + c^2}{d} - u_y \frac{a \cdot b}{d} - u_z \frac{a \cdot c}{d}$$

$$\boldsymbol{u}'_x = (d^2/d, -a \cdot b/d, -a \cdot c/d) = (d. -a \cdot b/d, -a \cdot c/d)$$

$$\boldsymbol{R} = \begin{bmatrix} d & \frac{-a \cdot b}{d} & \frac{-a \cdot c}{d} & 0 \\ 0 & \frac{c}{d} & \frac{-b}{d} & 0 \\ a & b & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

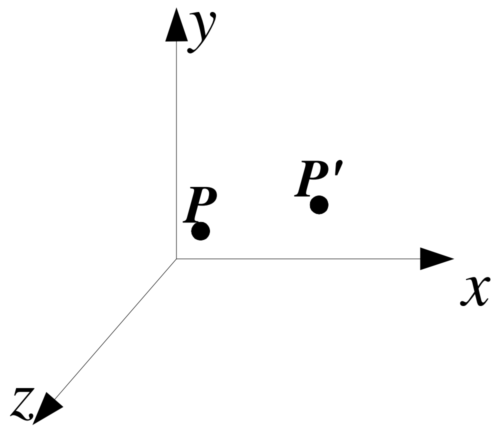
Check if equal to $\boldsymbol{R_y}(-\beta) \cdot \boldsymbol{R_x}(\alpha)$

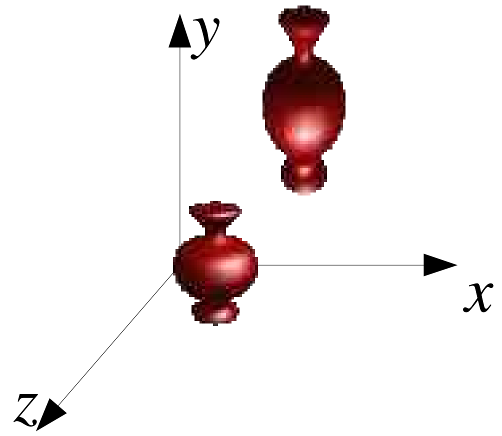
Scaling

- Change the coordinates of the object by scaling factors.

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boldsymbol{P}' = \boldsymbol{S} \cdot \boldsymbol{P}$$

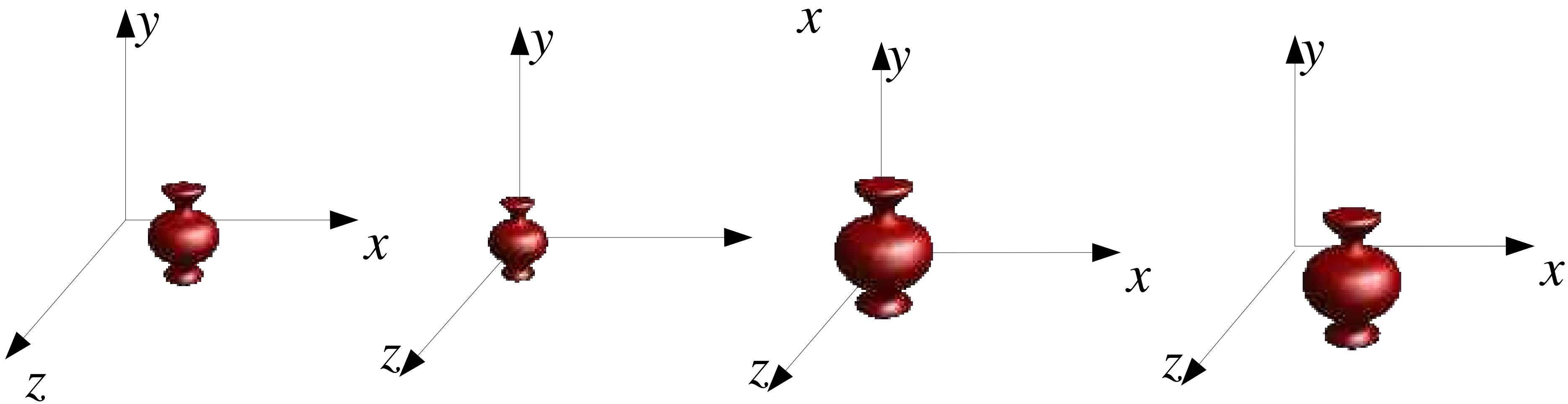




Scaling on a Fixed Point

- Translate to origin, scale, translate back

$$\boldsymbol{P}' = \boldsymbol{T}(x_f, y_f, z_f) \cdot \boldsymbol{S} \cdot \boldsymbol{T}(-x_f, -y_f, -z_f) \cdot \boldsymbol{P}$$



Translate Scale Translate back

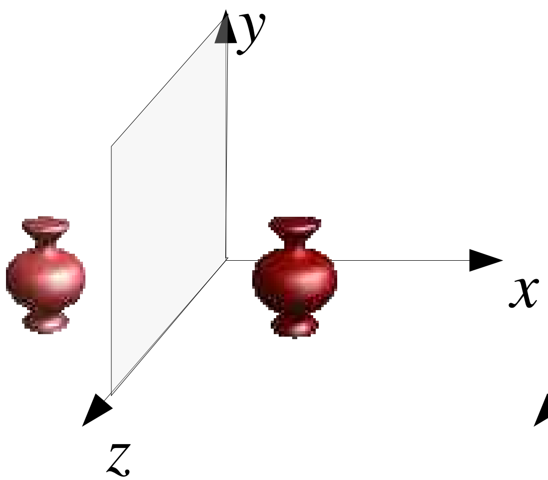
$$\boldsymbol{T}(x_f, y_f, z_f)\cdot\boldsymbol{S}\cdot\boldsymbol{T}(-x_f, -y_f, -z_f)=\begin{bmatrix}1 & 0 & 0 & x_f\\ 0 & 1 & 0 & y_f\\ 0 & 0 & 1 & z_f\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}s_x & 0 & 0 & 0\\ 0 & s_y & 0 & 0\\ 0 & 0 & s_z & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0 & -x_f\\ 0 & 1 & 0 & -y_f\\ 0 & 0 & 1 & -z_f\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0 & x_f\\ 0 & 1 & 0 & y_f\\ 0 & 0 & 1 & z_f\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}s_x & 0 & 0 & -s_x x_f\\ 0 & s_y & 0 & -s_y y_f\\ 0 & 0 & s_z & -s_z z_f\\ 0 & 0 & 0 & 1\end{bmatrix}$$

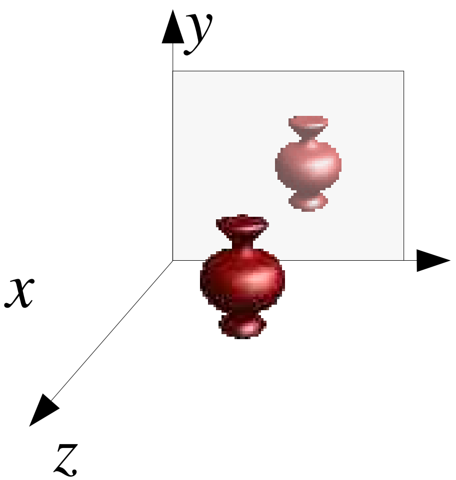
$$\boldsymbol{T}(x_f, y_f, z_f)\cdot\boldsymbol{S}\cdot\boldsymbol{T}(-x_f, -y_f, -z_f)=\begin{bmatrix}s_x & 0 & 0 & x_f(1-s_x)\\ 0 & s_y & 0 & y_f(1-s_y)\\ 0 & 0 & s_z & z_f(1-s_z)\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Reflection

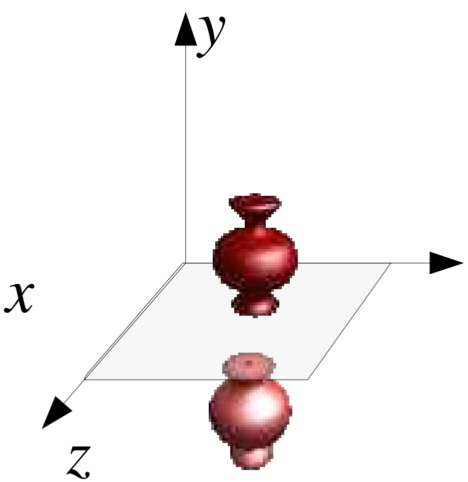
- Reflection over planes, lines or points



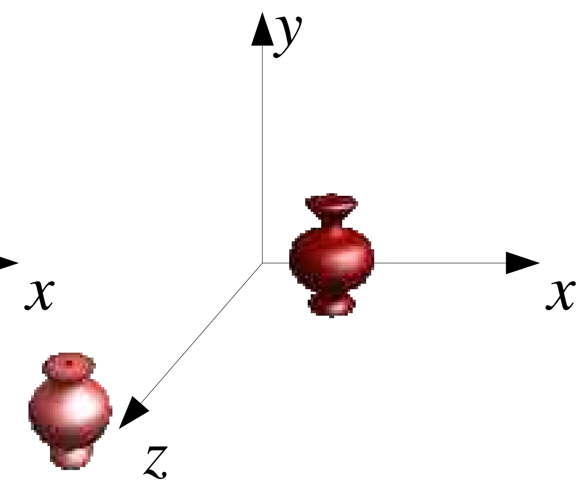
$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shear

- Deform the shape depending on another dimension

$$SH_z = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

x and y value depends on z value of the shape

$$SH_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

y and z value depends on x value of the shape