3D OBJECT REPRESENTATIONS

CEng 477
Computer Graphics
METU, 2004

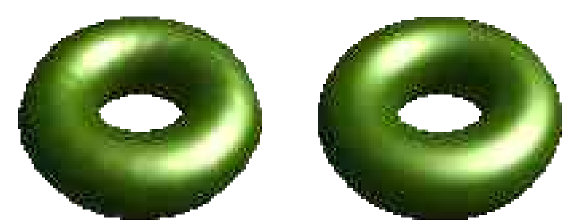
Object Representations

- Types of objects:
  geometrical shapes, trees, terrains, clouds, rocks, glass, hair, furniture, human body, etc.
- Not possible to have a single representation for all
  - Polygon surfaces
  - Spline surfaces
  - Procedural methods
  - Physical models
  - Solid object models
  - .....

Polygon Surfaces

- Set of adjacent polygons representing the object exteriors.
- All operations linear, so fast.
- Non-polyhedron shapes can be approximated by polygon meshes.
- Smoothness is provided either by increasing the number of polygons or interpolated shading methods.

Levels of detail



Interpolated shading

Data Structures

- Data structures for representing polygon surfaces:
  - Efficiency
    - Intersection calculations
    - Normal calculations
    - Access to adjacent polygons
  - Flexibility
    - Interactive systems
    - Adding, changing, removing vertices, polygons
  - Integrity

Polygon Tables

- Vertices

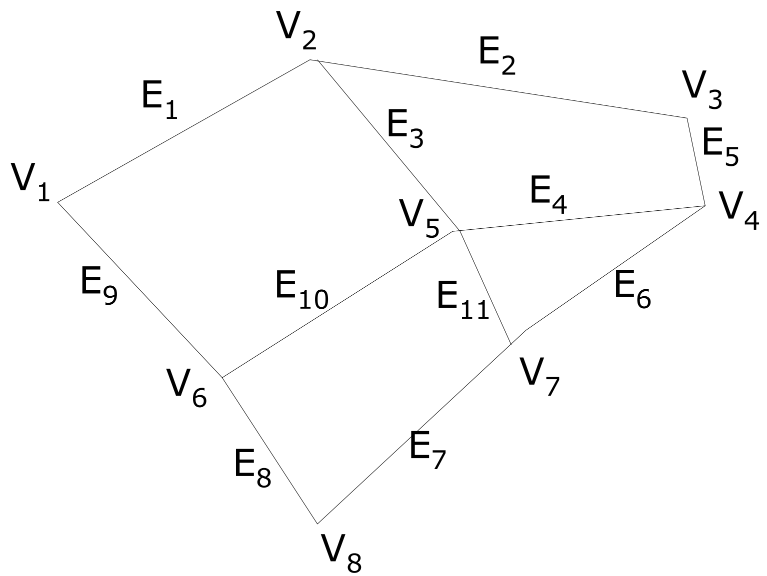
V_1:(x_1,y_1,z_1)
V_2:(x_2,y_2,z_2)
V_3:(x_3,y_3,z_3)
V_4:(x_4,y_4,z_4)
V_5:(x_5,y_5,z_5)
V_6:(x_6,y_6,z_6)
V_7:(x_7,y_7,z_7)
V_8:(x_8,y_8,z_8)

Edges

E_1: V_1,V_2
E_2: V_2,V_3
E_3: V_2,V_5
E_4: V_4,V_5
E_5: V_3,V_4
E_6: V_4,V_7
E_7: V_7,V_8
E_8: V_6,V_8
E_9: V_1,V_6
E_{10}: V_5,V_6
E_{11}: V_5,V_7

Polygons

S_1: E_1,E_3,E_{10},E_9
S_2: E_2,E_5,E_4,E_3
S_3: E_{10},E_{11},E_7,E_8
S_4: E_4,E_6,E_{11}



- Forward pointers: i.e. to access adjacent surfaces edges

V_1: E_1,E_9	V_2: E_1,E_2,E_3
V_3: E_2,E_5	V_4: E_4,E_5,E_6
V_5: E_3,$E_4$$E_{10}$,$E_{11}$	V_6: E_8,E_9,E_{10}
V_7: E_6,E_7,E_{11}	V_8: E_7,E_8

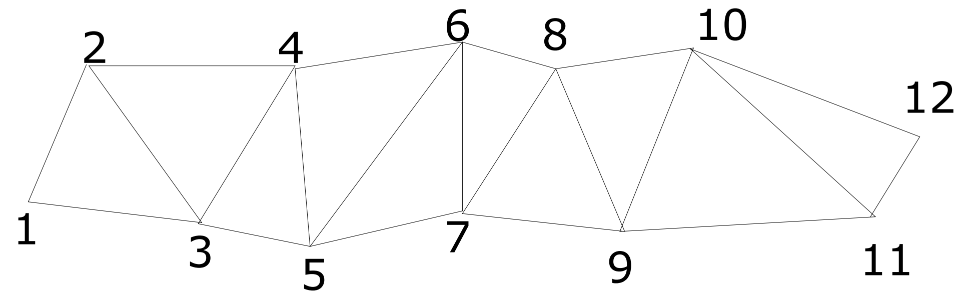
E_1: S_1	E_2: S_2
E_3: S_1,S_2	E_4: S_2,S_4
E_5: S_2	E_6: S_4
E_7: S_3	E_8: S_3
E_9: S_1	E_{10}: S_1,S_3
E_{11}: S_3,S_4	

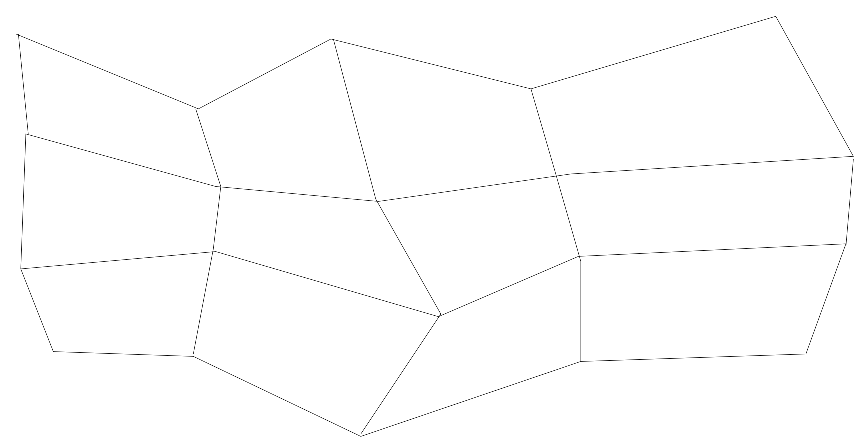
- Additional geometric properties:
  - Slope of edges
  - Normals
  - Extends (bounding box)
- Integrity checks
  - $\forall V, \ \exists E_a, E_b \text{ such that } V \in E_a, V \in E_b$
  - $\forall E, \ \exists S \text{ such that } E \in S$
  - $\forall S, \ S \text{ is closed}$
  - $\forall S_1, \ \exists S_2 \text{ such that } S_1 \cap S_2 \neq \emptyset$
  - $S_k \text{ is listed in } E_m \Leftrightarrow E_m \text{ is listed in } S_k$

Polygon Meshes

- Triangle strips:
  123, 234, 345, ..., 10 11 12

  1 2 3 4 5 6 7 8 9 10 11 12
- Quadrilateral meshes:
  $n \times m$ array of vertices





Plane Equations

- Equation of a polygon surface:

  $$A\,x+B\,y+C\,z+D=0$$

  Linear set of equations:

  $$(A/D)\,x_k+(B/D)\,y_k+(C/D)\,z_k=-1, \qquad k=1,2,3$$

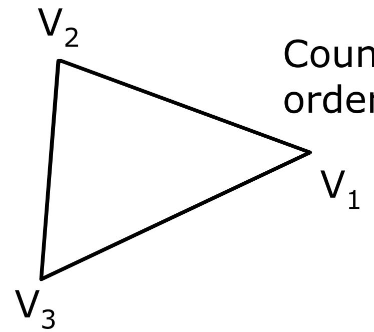
  $$A=y_1(z_2-z_3)+y_2(z_3-z_1)+y_3(z_1-z_2)$$
  $$B=z_1(x_2-x_3)+z_2(x_3-x_1)+z_3(x_1-x_2)$$
  $$C=x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)$$
  $$D=-x_1(y_2 z_3-y_3 z_2)-x_2(y_3 z_1-y_1 z_3)-x_3(y_1 z_2-y_2 z_1)$$

- Surface Normal:

  $$\boldsymbol{N}=(A,B,C)$$

  extracting normal from vertices:

  $$\boldsymbol{N}=(V_2-V_1)\times(V_3-V_1)$$



Counterclockwise order.

- Find plane equation from normal

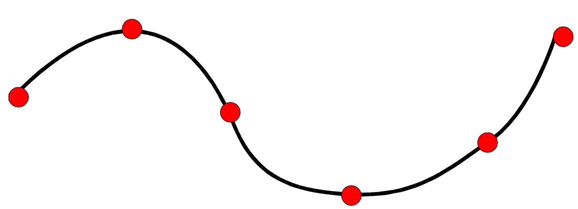
$$(A, B, C) = \boldsymbol{N}$$
$$\boldsymbol{N} \cdot (x, y, z) + D = 0$$
$\boldsymbol{P}$ is a point in the surface (i.e. a vertex)
$$D = -\boldsymbol{N} \cdot \boldsymbol{P}$$

- Inside outside tests of the surface:

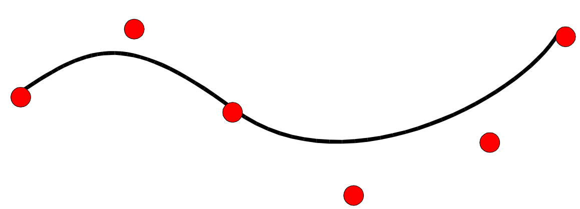
$$A x + B y + C z + D < 0, \quad \text{point is inside the surface}$$
$$A x + B y + C z + D > 0, \quad \text{point is outside the surface}$$

Spline Representations

- Spline curve: Curve consisting of continous curve segments approximated or interpolated on polygon control points.
- Spline surface: a set of two spline curves matched on a smooth surface.
- Interpolated: curve passes through control points
- Approximated: guided by control points but not necessarily passes through them.

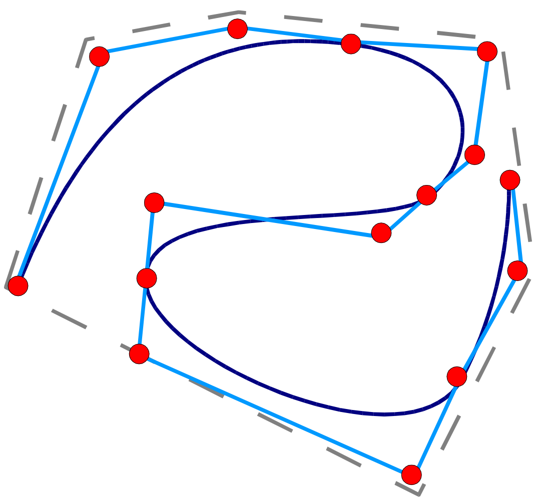


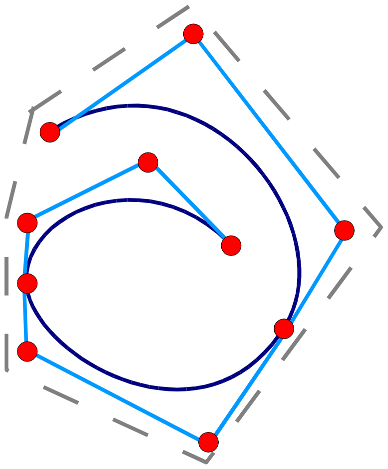
Interpolated



Approximated

- Convex hull of a spline curve: smallest polygon including all control points.
- Characteristic polygon, control path: vertices along the control points in the same order.

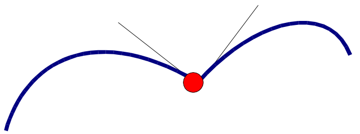




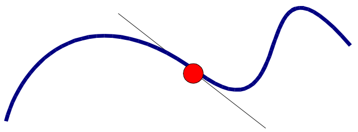
- Parametric equations:

$$x = x(u), \quad y = y(u), \quad z = z(u), \quad u_1 \leqslant u \leqslant u_2$$

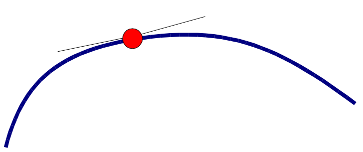
- Parametric continuity: Continuity properties of curve segments.
  - Zero order: Curves intersects at one end-point: C^0



  - First order: C^0 and curves has same tangent at intersection: C^1



  - Second order: C^0 , C^1 and curves has same second order derivative: C^2



- Geometric continuity:
  Similar to parametric continuity but only the direction of derivatives are significant. For example derivative (1,2) and (3,6) are considered equal.
- G^0, G^1, G^2 : zero order, first order, and second order geometric continuity.

Spline Equations

- Cubic curve equations:

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$
$$y(u) = a_y u^3 + b_y u^2 + c_y u + d_y \qquad 0 \leqslant u \leqslant 1$$
$$z(u) = a_z u^3 + b_z u^2 + c_z u + d_z$$

$$x(u) = \left[u^3\, u^2\, u\, 1\right] \begin{bmatrix} a_x \\ b_x \\ c_x \\ d_x \end{bmatrix} = \boldsymbol{U} \cdot \boldsymbol{C}$$

- General form: $x(u) = \boldsymbol{U} \cdot \boldsymbol{M_s} \cdot \boldsymbol{M_g}$
  M_g: geometric constraints (control points)
  M_s: spline transformation (blending functions)

Natural Cubic Splines

- Interpolation of n+1 control points. n curve segments. 4n coefficients to determine
- Second order continuity. 4 equation for each of n-1 common points:

$$x_k(1)=p_k, \quad x_{k+1}(0)=p_k, \quad x_k'(1)=x_{k+1}'(0) \quad x_k''(1)=x_{k+1}''(0)$$

  4n equations required, 4n-4 so far.

- Starting point condition, end point condition.

$$x_1(0)=p_0, \quad x_n(1)=p_n$$

- Assume second derivative 0 at end-points or add phantom control points p_{-1}, p_{n+1}.

$$x_1''(0)=0, \quad x_n''(1)=0$$

- Write $4n$ equations for $4n$ unknown coefficients and solve.

- Changes are not local. A control point effects all equations.
- Expensive. Solve $4n$ system of equations for changes.

Hermite Interpolation

- End point constraints for each segment is given as:

$$\boldsymbol{P}(0)=\boldsymbol{p}_k, \quad \boldsymbol{P}(1)=\boldsymbol{p}_{k+1}, \quad \boldsymbol{P}'(0)=\boldsymbol{Dp}_k, \quad \boldsymbol{P}'(1)=\boldsymbol{Dp}_{k+1}$$

- Control point positions and first derivatives are given as constraints for each end-point.

$$\boldsymbol{P}(u)=\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} \qquad \boldsymbol{P}'(u)=\begin{bmatrix} 3u^2 & 2u & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{p}_{k+1} \\ \boldsymbol{Dp}_k \\ \boldsymbol{Dp}_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} \qquad \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{p}_{k+1} \\ \boldsymbol{Dp}_k \\ \boldsymbol{Dp}_{k+1} \end{bmatrix}$$

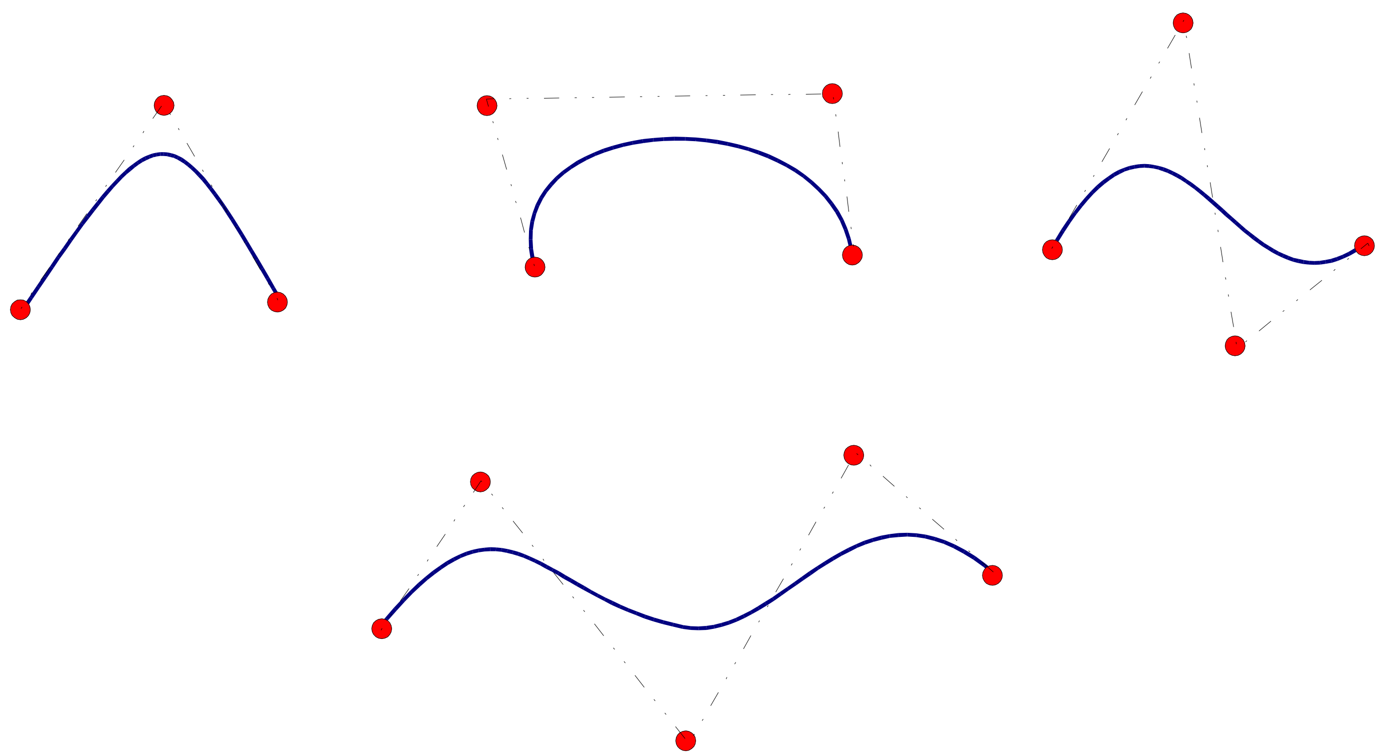
$$\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{p}_{k+1} \\ \boldsymbol{Dp}_k \\ \boldsymbol{Dp}_{k+1} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{p}_{k+1} \\ \boldsymbol{Dp}_k \\ \boldsymbol{Dp}_{k+1} \end{bmatrix} = \boldsymbol{M}_H \cdot \begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{p}_{k+1} \\ \boldsymbol{Dp}_k \\ \boldsymbol{Dp}_{k+1} \end{bmatrix}$$

$$\boldsymbol{P}(u) = p_k(2u^3 - 3u^2 + 1) + \boldsymbol{p}_{k+1}(-2u^3 + 3u^2) + \boldsymbol{Dp}_k(u^3 - 2u^2 + u) + \boldsymbol{Dp}_{k+1}(u^3 - u^2)$$

- Segments are local. First order continuity
- Slopes at control points are required.
- Cardinal splines and Kochanek-Bartel splines approximate slopes from neighbor control points.

Bézier Curves

- A Bézier curve approximates any number of control points for a curve section (degree of the Bézier curve)



$$\boldsymbol{P}(u) = \sum_{k=0}^{n} \boldsymbol{p_k} BEZ_{k,n}(u), \qquad 0 \leqslant u \leqslant 1$$

$$BEZ_{k,n}(u) = \binom{n}{k} u^k (1-u)^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Polynomial degree of a Bézier curve is one less than the number of control points.
  3 points : parabola
  4 points : cubic curve
  5 points : fourth order curve

- Properties of Bézier curves:
  - Passes through start and end points

    $$\boldsymbol{P}(0)=\boldsymbol{p}_{\boldsymbol{0}}, \quad \boldsymbol{P}(1)=\boldsymbol{p}_{\boldsymbol{n}}$$

  - First derivates at start and end are:

    $$\boldsymbol{P}'(0)=-n\,\boldsymbol{p}_{\boldsymbol{0}}+n\,\boldsymbol{p}_{\boldsymbol{1}}, \quad \boldsymbol{P}'(1)=-n\,\boldsymbol{p}_{\boldsymbol{n-1}}+n\,\boldsymbol{p}_{\boldsymbol{n}}$$

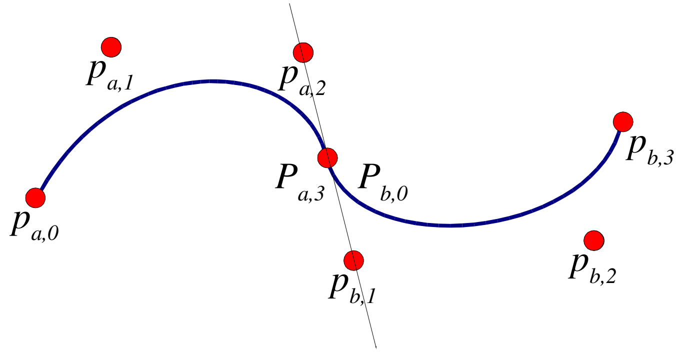
  - Lies in the convex hull

- Joining Bézier curves:
  - Start and end points are same (C^0)
  - Choose adjacent points to start and end in the same line (C^1)

$$\boldsymbol{p}_{a,n} = \boldsymbol{p}_{b,0}, \qquad \boldsymbol{p}_{b,1} = \boldsymbol{p}_{a,n} + (\boldsymbol{p}_{a,n} - \boldsymbol{p}_{a,n-1})$$

  - For second order (C^2) choose the next point in terms of the previous 2 of the other segment.

$$\boldsymbol{p}_{b,2} = \boldsymbol{p}_{a,n-2} + 4(\boldsymbol{p}_{a,n} - \boldsymbol{p}_{a,n-1})$$



Cubic Bézier Curves

- Most graphics packages provide Cubic Béziers.
- $BEZ_{0,3}(u) = (1-u)^3 \qquad BEZ_{1,3}(u) = 3u(1-u)^2$

  $BEZ_{2,3}(u) = 3u^2(1-u) \quad BEZ_{3,3}(u) = u^3$

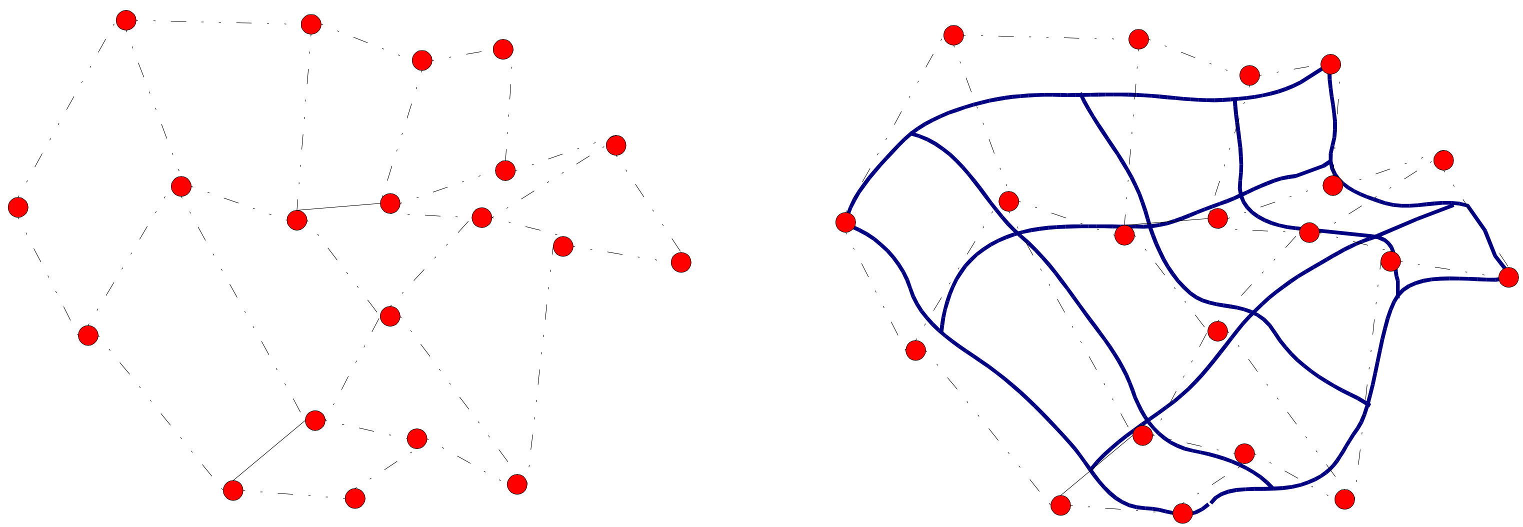
$$\boldsymbol{P}(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \cdot \boldsymbol{M}_{Bez} \cdot \begin{bmatrix} \boldsymbol{p_0} \\ \boldsymbol{p_1} \\ \boldsymbol{p_2} \\ \boldsymbol{p_3} \end{bmatrix}$$

$$\boldsymbol{M}_{Bez} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bézier Surfaces

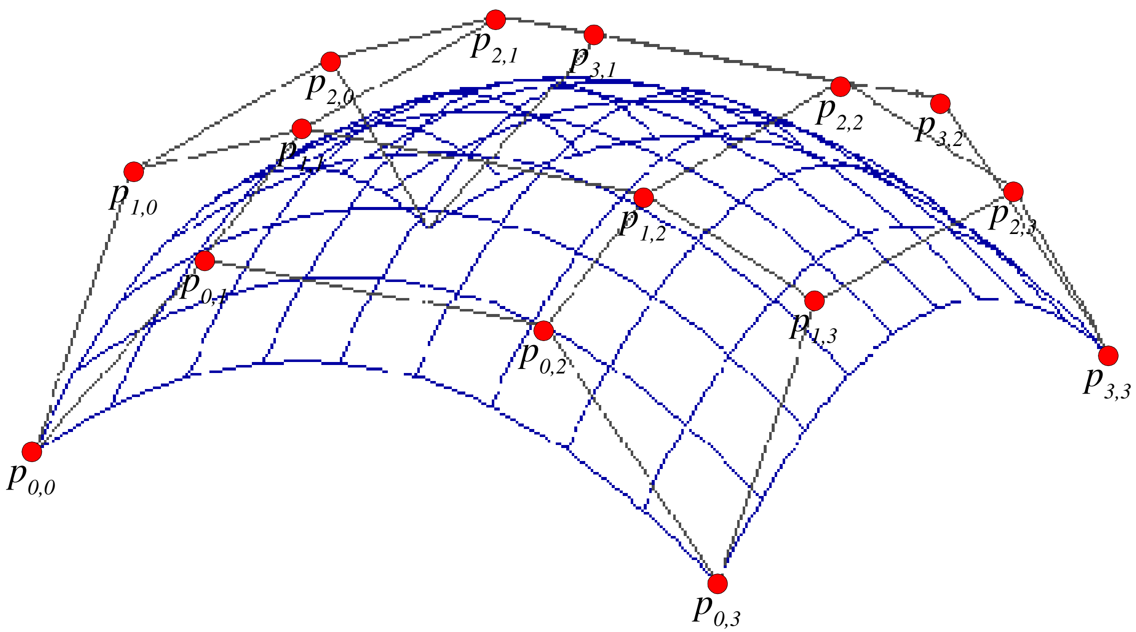
- Cartesian product of Bézier blending functions:

$$\boldsymbol{P}(u,v)=\sum_{j=0}^{m}\sum_{k=0}^{n}\boldsymbol{p}_{\boldsymbol{j},\boldsymbol{k}}\,BEZ_{j,m}(v)\,BEZ_{k,n}(u) \qquad 0\leqslant u,v\leqslant 1$$



Bézier Patches

- A common form of approximating larger surfaces by tiling with cubic Bézier patches. $m=n=3$
- 4 by 4 = 16 control points.



- Matrix form

$$\boldsymbol{P}(u,v)=\boldsymbol{U}\cdot\boldsymbol{M}_{Bez}\cdot\boldsymbol{P}\cdot\boldsymbol{M}_{Bez}^{T}\cdot\boldsymbol{T}^{T}=$$

$$\begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}\cdot\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} \boldsymbol{p}_{0,0} & \boldsymbol{p}_{0,1} & \boldsymbol{p}_{0,2} & \boldsymbol{p}_{0,3} \\ \boldsymbol{p}_{1,0} & \boldsymbol{p}_{1,1} & \boldsymbol{p}_{1,2} & \boldsymbol{p}_{1,3} \\ \boldsymbol{p}_{2,0} & \boldsymbol{p}_{2,1} & \boldsymbol{p}_{2,2} & \boldsymbol{p}_{2,3} \\ \boldsymbol{p}_{3,0} & \boldsymbol{p}_{3,1} & \boldsymbol{p}_{3,2} & \boldsymbol{p}_{3,3} \end{bmatrix}\cdot\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

- Joining patches:
  similar to curves. C^0, C^1 and C^2 can be established by choosing control points accordingly.

Displaying Curves and Surfaces

- Horner's rule: less number of operations for calculating polynoms.

$$x(u) = a_x u^3 + b_x u^2 + c_x u + d_x$$
$$x(u) = ((a_x u + b_x) u + c_x) u + d_x$$

- Forward differences calculations: Incremental calculation of the next value.
  - Linear case:

$$u_{k+1} = u_k + \delta, \qquad k = 0, 1, 2 \ldots \qquad u_0 = 0$$
$$x_k = a_x u_k + b_x \qquad x_{k+1} = a_x (u_k + \delta) + b_x$$
$$x_{k+1} = x_k + \Delta x \qquad \boxed{\Delta x = a_x \delta}$$

- Cubic equations

$$x_k = a_x u_k^3 + b_x u_k^2 + c_x u_k + d_x \qquad x_{k+1} = a_x (u_k + \delta)^3 + b_x (u_k + \delta)^2 + c_x (u_k + \delta) + d_x$$

$$\Delta x_k = 3 a_x \delta u_k^2 + (3 a_x \delta^2 + 2 b_x \delta) u_k + (a_x \delta^3 + b_x \delta^2 + c_x \delta)$$

$$\boxed{\begin{aligned} \Delta x_{k+1} &= \Delta x_k + \Delta^2 x_k \\ \Delta^2 x_{k+1} &= \Delta^2 x_k + \Delta^3 x_k \end{aligned}} \qquad \Delta^2 x_k = 6 a_x \delta^2 u_k + 6 a_x \delta^3 + 2 b_x \delta^2$$

$$\boxed{\Delta^3 x_k = 6 a_x \delta^3}$$

$$x_0 = d_x$$

$$\Delta x_0 = a_x \delta^3 + b_x \delta^2 + c_x \delta$$

$$\Delta^2 x_0 = 6 a_x \delta^3 + 2 b_x \delta^2$$

$$\Delta^3 x_k = 6 a_x \delta^3$$
$$x_0 = d_x$$
$$\Delta x_0 = a_x \delta^3 + b_x \delta^2 + c_x \delta$$
$$\Delta^2 x_0 = 6 a_x \delta^3 + 2 b_x \delta^2$$

- Example:
  $(a_x, b_x, c_x, d_x) = (1,2,3,4)$, δ = 0.1

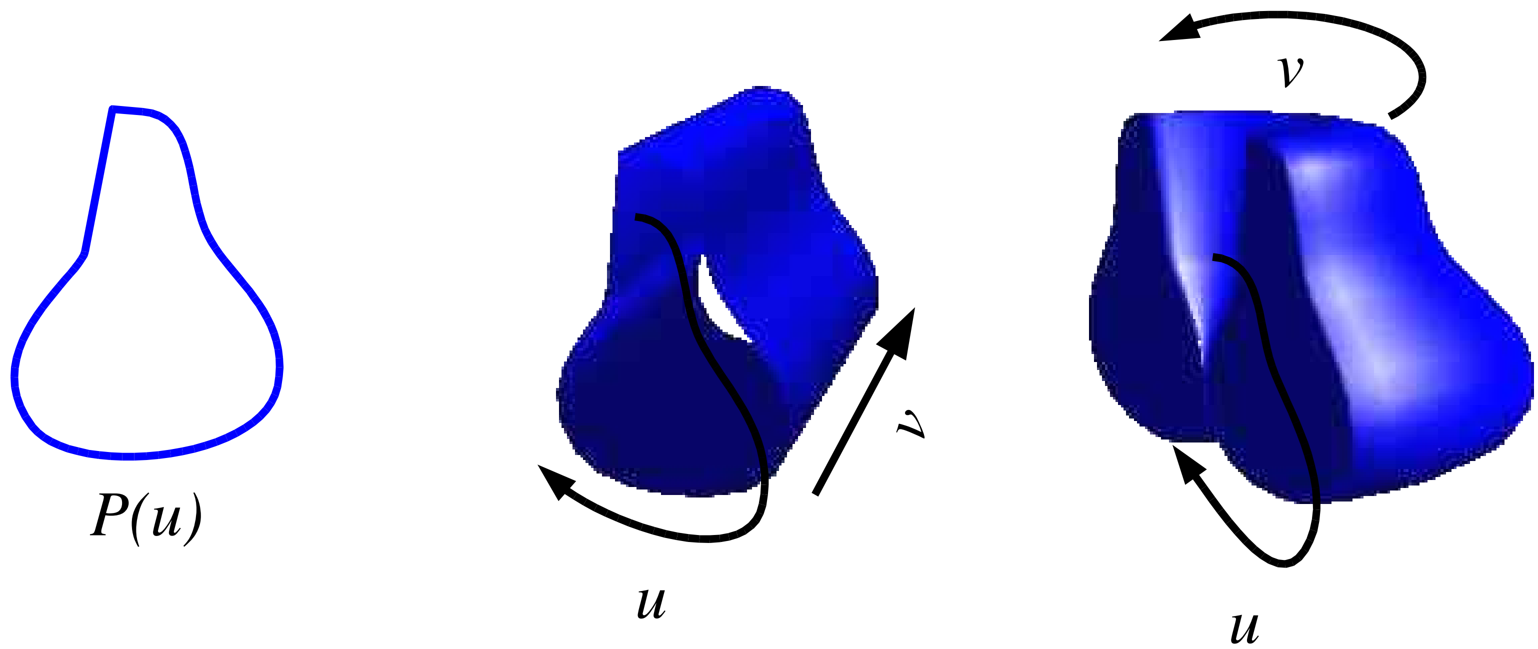
$$\Delta^3 x_k = 6 \delta^3 = 0.006$$

x	Δx	$\Delta^2 x$
4.000	0.321	0.046
4.321	0.367	0.052
4.688	0.419	0.058
5.107	0.477	0.064
5.584	0.541	0.070
6.125	0.611	0.076
6.736	0.687	0.082
7.423	0.769	0.088
8.192	0.857	0.094
9.049	0.951	0.100

$\Delta^3 x_k$ (difference between 0.046 and 0.052)

Sweep Representations

- Use reflections, translations and rotations to construct new shapes.

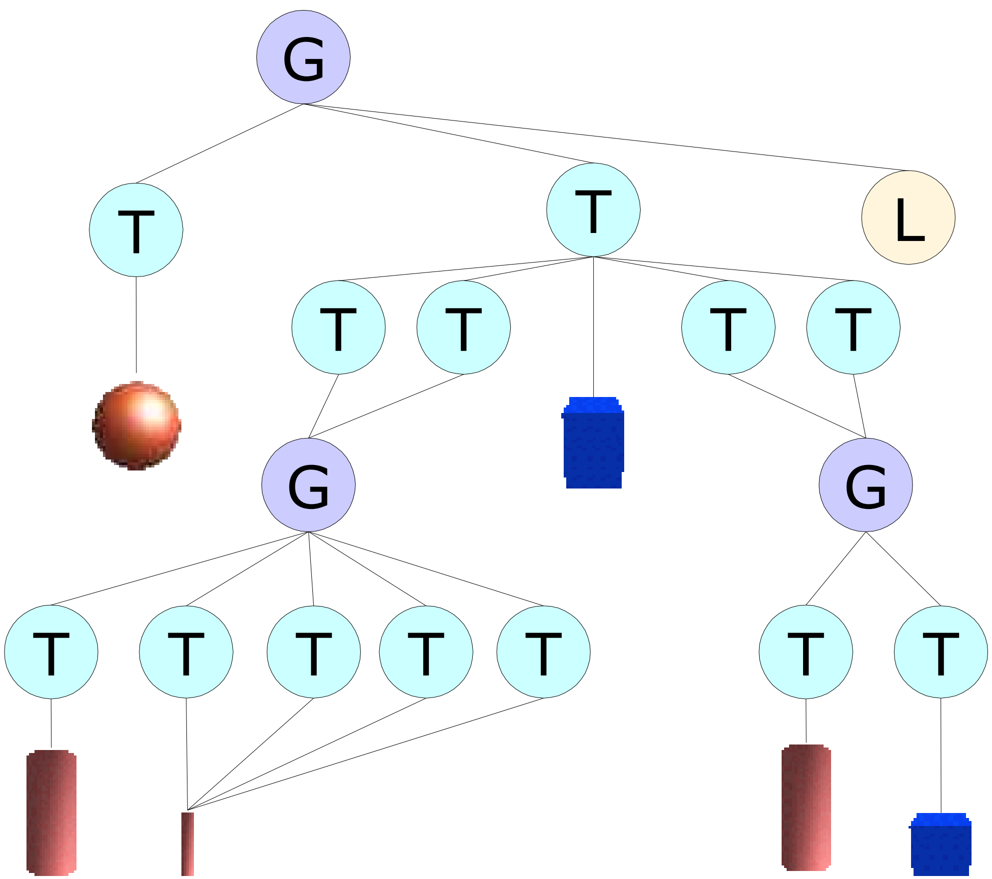


Hierarchical Models

- Combine smaller/simpler shapes to construct complex objects and scenes.
- Stored in trees or similar data structures
- Operations are based on traversal of the tree
- Keeping information like bounding boxes in tree nodes accelarate the operations.

Scene Graphs

- DAG's (Directed Acyclic Graphs) to represent scenes and complex objects.
- Nodes: Grouping nodes, Transform nodes, Level Of Detail nodes, Light Source nodes, Attribute nodes, State nodes.
  Leaves: Object geometric descriptions.
- Why not tree but DAG?
- Available libraries: i.e. www.openscenegraph.org
- Efficient display of objects, picking objects, state change and animations.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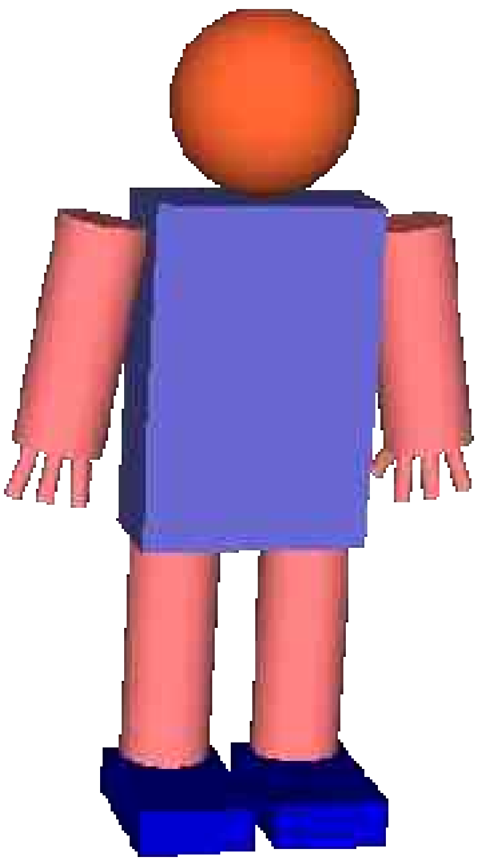
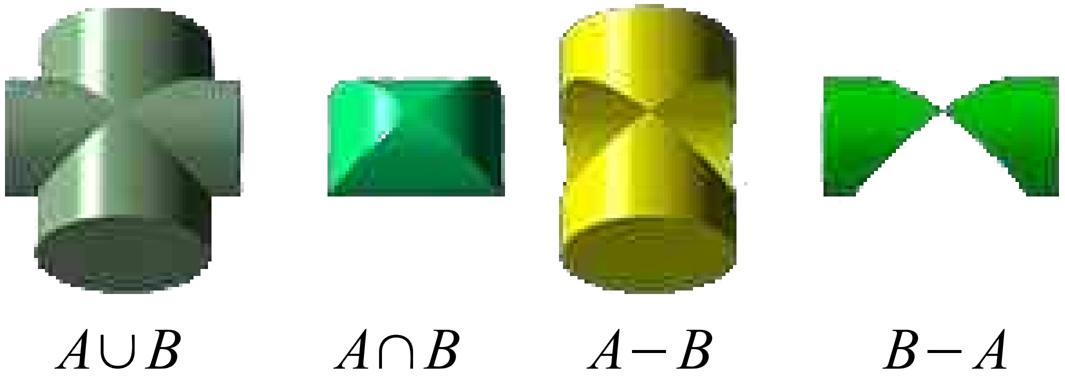
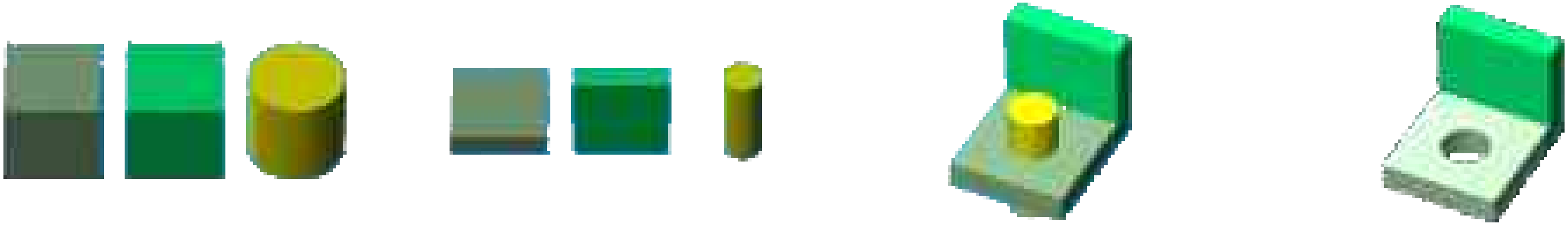


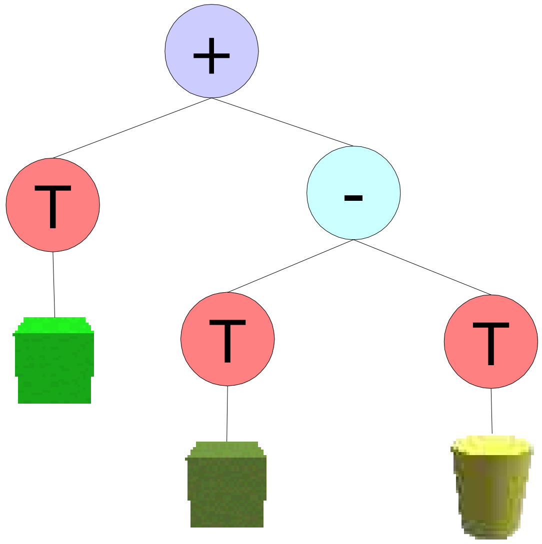
Constructive Solid Geometry

- Combine multiple shapes with set operations (intersection, union, deletion) to construct new shapes.
- 

- Set operations and transformations combined:

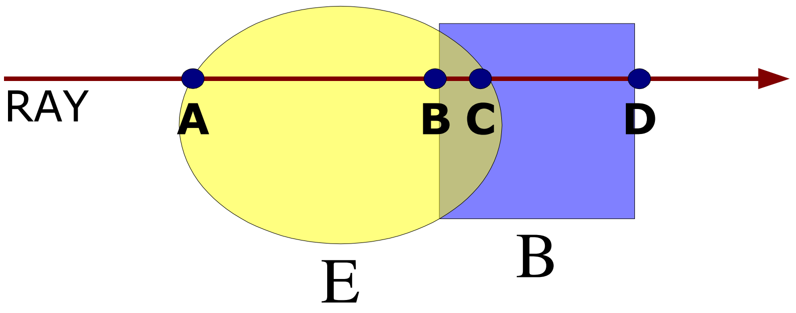


- `union(transA(box),diff(transB(box),transC(cylinder)))`



- Ray casting methods are used for rendering and finding properties of volumes constructed with this method.
- Simply +1 for outside inside
  -1 for inside outside transition.
  Positives are solid.

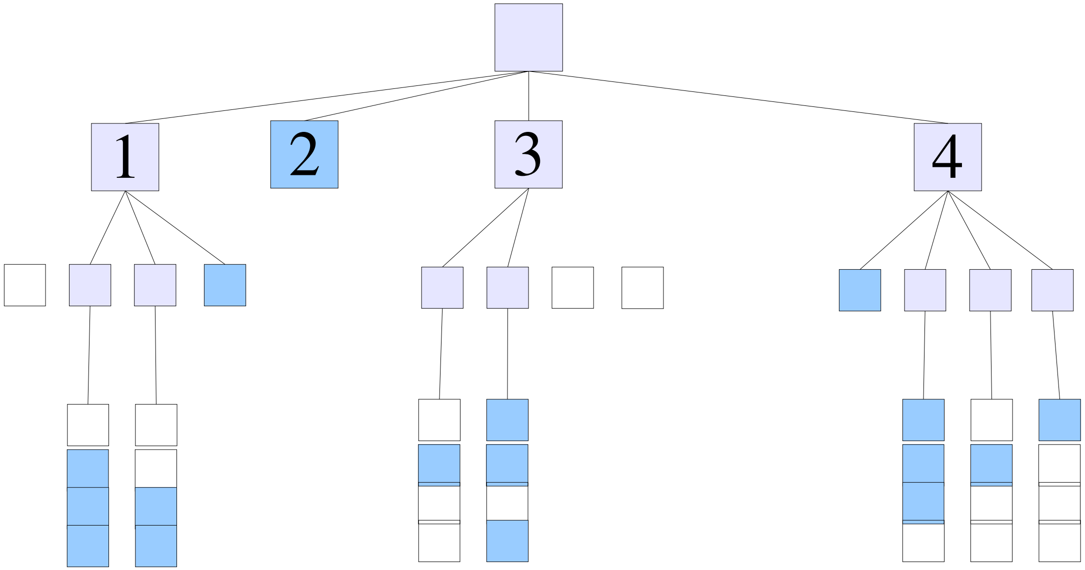
	A	B	C	D
E ∪ B	1	2	1	0
E ∩ B	1	2	1	0
E - B	1	0	-1	0
B - E	-1	0	1	0

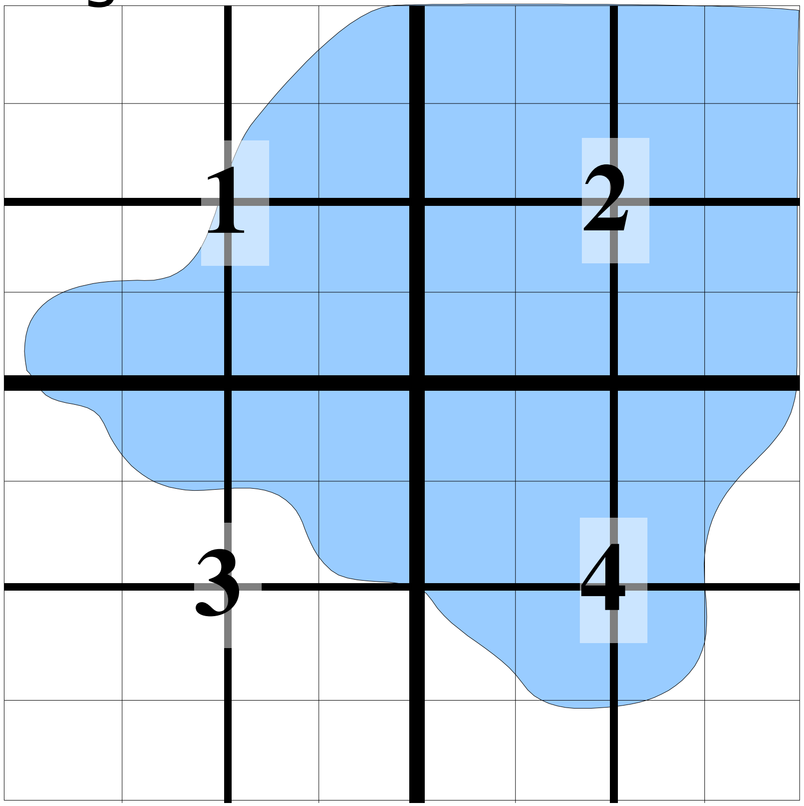


- Similarly find unit cubes interior to calculate mass, center of mass etc.

Octrees

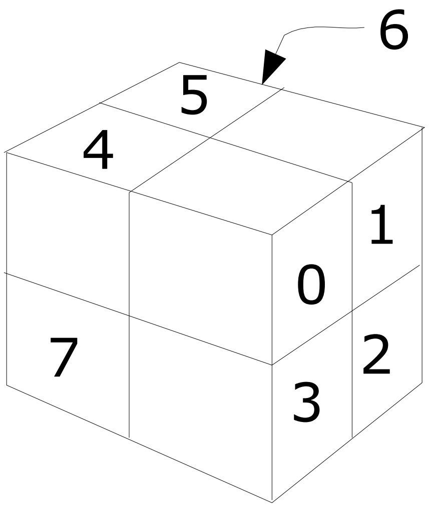
- Divide a volume in equal binary partitions in all dimensions recursively to represent solid object volumes. Combining leaf cubes give the volume.
- 2D: quadtree





- 2D: quadtree; 3D: octree
- Volume data: Medical data like Magnetic Resonance. Geographical info (minerals etc.)
- 2D: Pixel ; 3D: voxel.
- Volumes consisting of large continous subvolumes with properties. Volumes with many wholes, spaces. Surface information is not sufficient or tracktable.
- Keeping all volume in terms of voxels, too expensive: space and processor.

- 8 elements at each node.
- If volume completely resides in a cube, it is not further divided: leaf node
- Otherwise nodes recursively subdivided.



- Extends of a tree node is the extend of the cube it defines.
- Surfaces can be extracted by traversing the leaves with geometrical adjacency.

Fractal Geometry Methods

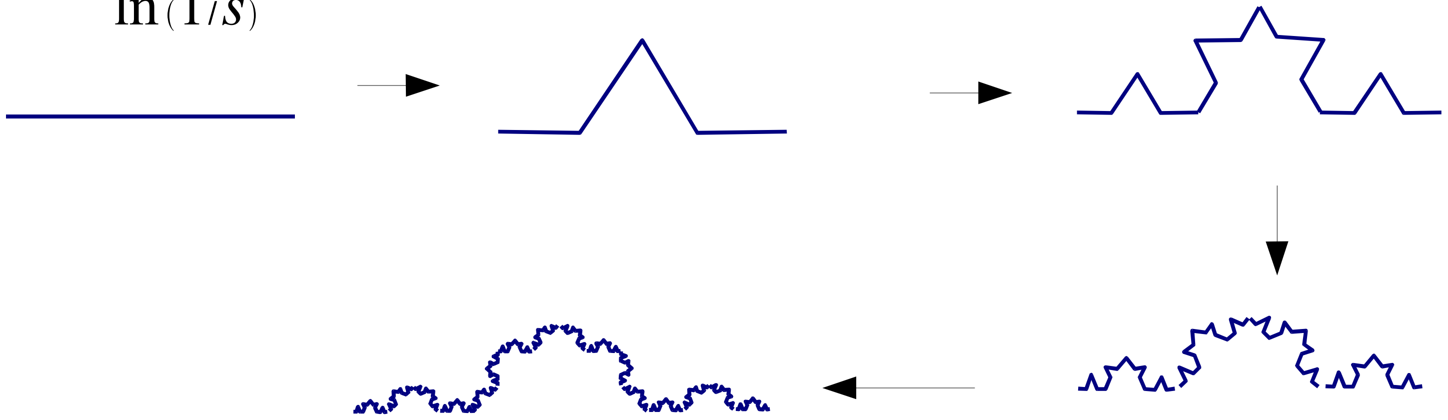
- Synthetic objects: regular, known dimension
- Natural objects: recursive (self repeating), the higher the precision, the higher the details you get.
- Example: tree branches, terrains, textures.
- Classification:
  - Self-similar: scaled-down shape is similar to original
  - Self-affine: self similar with different scaling parameters and transformations. Statistical when random parameters are involved.
  - Invariant: non-linear transformations, i.e. Complex space.

- Fractal dimension:
  - Detail variation of a self similar object. Denoted as *D*.
  - Fragmentation of the object.

$$n\,s^{D}=1$$

$$D=\frac{\ln n}{\ln(1/s)}$$

n : number of pieces s : scaling factor



- $n=4$ $s=1/3$ $D=\dfrac{\ln 4}{\ln 1/(1/3)}=1.2619$

Random Mid-point Variation

- Find the midpoint of an edge A-B. Add a random factor and divide the edge in two as: A-M, M-A at each step.
- Usefull for height maps, clouds, plants.
- 2D: $x_m = (x_A + x_B)/2$

  $y_m = (y_A + y_B)/2 + r$, r: a random value in $0 - c$

  $c \leftarrow c \times f$, f a fraction in $0 - 1$

- 3D: For corners of a square: A, B, C, D

$$z_{AB} = (z_A + z_B)/2 + r, \quad z_{BC} = (z_B + z_C)/2 + r,$$
$$z_{CD} = (z_C + z_D)/2 + r, \quad z_{DA} = (z_D + z_A)/2 + r$$
$$z_M = (z_{AB} + z_{BC} + z_{CD} + z_{DA})/4 + r$$