Stress Equilibrium

Consider stress equilibrium of an infinitesimal volume element and assume that each component of stress is continuously differentiable function of \( x, y \) and \( z \). Continuously differentiable function \( f \) of \( x \)

\[ f(x + \Delta x) \approx f(x) + \frac{\partial f}{\partial x} \Delta x \]

The components of stresses on faces of an infinitesimal cube are in figure

![Diagram of stress components on a cubic infinitesimal element](image)

Normal and shear stresses on cubic infinitesimal element
The subscript notation used for the nine stress components have the following meaning:

\[ \boldsymbol{\sigma}_{\xi\eta} : \text{Stress on the } \xi \text{ plane along } \eta \text{ direction} \]

Direction of stress component

Direction of the normal upon which the stress acts

In 2D, applying Newton's first law of motion, the equilibrium of forces in the \( x \) direction is

\[ \sum F_x = 0 \]

\[ (\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx) dy dz - \sigma_{xx} dy dz + (\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy) dx dz - \tau_{xy} dx dz + B_x dx dy dz = 0 \]

Which is equal to

\[ \frac{\partial \sigma_{xx}}{\partial x} dx dy dz + \frac{\partial \tau_{xy}}{\partial y} dx dy dz + B_x dx dy dz = 0 \]

Cancelling out \( dx dy dz \) terms yields

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0 \]

In 3D, the equation of equilibrium expands to, accordingly;

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0 \]

Similarly \( \sum F_y = 0 \) and \( \sum F_z = 0 \) give

\[ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0 \]

\[ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0 \]
In summary, the equations of equilibrium in 3D:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + B_x = 0
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + B_y = 0
\]

\[
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0
\]

where \(B_x\), \(B_y\) and \(B_z\) are body forces per unit volume.

In the case of two dimensional stresses, the terms including “z” subscript dropping out, the above equations reduce to,

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0
\]

To remember the equations of equilibrium, following hint may be helpful;

**Hint for 3-D case:**

\[
\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial j}
\]

\[
1 \equiv x \quad 2 \equiv y \quad 3 \equiv z
\]

\[
\sigma_{ij,j} + B_i = 0
\]
APPLICATION

So what is Bridge Builder? The name says it all: you build bridges. The trick is that you need to build *structurally sound* bridges with limited resources. The goal of each level is to cross a span and then run a hefty train over your freshly-built trestle. Proper engineering is crucial: An understanding of triangles and support structures will go a long way.

*Bridge Builder* also does a great job communicating the status of your structure during the simulation/test mode. Bridge segments are color-coded by the amount of stress they’re under. It’s an elegant way to show you which areas of your bridge are robust and which are likely to fail.

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