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COMPONENTS AND INDEX NOTATION

Range Convention

- Indicies in Latin alphabet: i, j,k… range varies from 1, 2, 3
- Indicies in Greek alphabet: α, β ... range varies from 1, 2

Example:

1) $x_i = y_i$ ⟹ $x_1 = y_1$
$x_2 = y_2$
$x_3 = y_3$ (3eqns)

2) $A_{ij} = B_{ij}$ ⟹ $A_{11}=B_{11}$, $A_{12}=B_{12}$, $A_{13}=B_{13}$,
$A_{21}=B_{21}$, $A_{22}=B_{22}$, $A_{23}=B_{23}$,
$A_{31}=B_{31}$, $A_{32}=B_{32}$, $A_{33}=B_{33}$ (9 eqns)

3) $A_{ijk}=B_{ijk}$ ⟹ $A_{111}=B_{111}$, $A_{112}=B_{112}$, $A_{113}=B_{113}$
$A_{211}=B_{211}$, $A_{212}=B_{212}$, $A_{213}=B_{213}$
⋮ (27 eqns)

Summation Convention

1) A repeated index in a term implies summation over that index:

Example

i) $x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3$
$(x_\alpha y_\alpha = x_1 y_1 + x_2 y_2)$

ii) $A_{ij} x_i = A_{i1} x_1 \neq A_{i2} x_j + A_{i3} x_3 \quad i=1,2,3$

iii) $A_{ij}\, A_{ij} = A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13}$
$+A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23}$
$+A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}$
↑
$(A_{ij})^2=(A_{ij})^2$

2) An index cannot appear more than twice in a term .

Non-example
$x_i y_i z_i$ (WRONG!)

3) A repeated index is called “dummy” index. The letter does not matter provided that coice does not violate other rules.

Example

$A_{mm} = A_{ii} = A_{jj}$

$A_{ij}x_i = A_{im}x_m \neq A_{ii}x_i$ → WRONG!

4) An index which is not repeated is called “free” index.

Note: The free index must appear on both sides of an equality. (But not the summed ones)

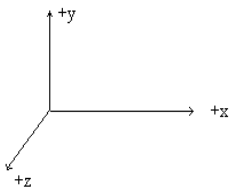
Example

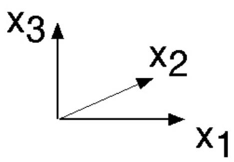
$y_i = A_{ij}x_j = A_{im}x_m$ (i’s have to match , you can change dummy indicies.)

- $x_j \longrightarrow$ vector

Force

$$[F_i] = \vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad F_i \quad i=1, 2, 3 \qquad F_{ij} = \text{tensor} = \text{matrix}$$





Example

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$Ax = b$

$A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \Longrightarrow A_{1J}X_J = b_1$

$A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \Longrightarrow A_{2J}X_J = b_2 \quad \Rightarrow \quad A_{ij}x_j = b_i$

$A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_{23} \Longrightarrow A_{2J}X_J = b_3$

5) The summation convention may be suspended by writing “No Sum” to the right of the equation.

Example

$x_i + y_i z_i = \lambda_i$ (no sum on i)

$x_1 + y_1 z_1 = \lambda_1$

$x_2 + y_2 z_2 = \lambda_2$

$x_3 + y_3 z_3 = \lambda_3$

previous example

$A_{ij}x_j = b_i$ (no sum on j)

Indexed Variables in Equations

1) Any free index must agree in all terms.

Exception: constants

Examples:

- $u_i + v_i = w_i$
- $A_{ij}x_j = z_i + B_{kk}u_p v_p x_i$ where i is a free index and j, p and k are dummy indices.
- $A_{ij}x_j = 0$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(1) $A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = 0$ $\quad A_{1j}x_j = 0$

(2) $A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = 0$ $\quad A_{2j}x_j = 0$

(3) $A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = 0$ $\quad A_{3j}x_j = 0$

Non-examples:

- $u_i = v_j$
- $z_i + A_{pj}w_r = 0$

2) You can multiply both sides of an equation by an indexed variable provides you do not violate any other rules.

Example:

$x_i + y_i = z_i$

$x_i\lambda_j + y_i \lambda_j = z_i\lambda_j$

$x_i\lambda_i + y_i \lambda_i = z_i\lambda_i$

$v_j x_i\lambda_i + v_j y_i \lambda_i = v_j z_i\lambda_i$ (cannot multiply by v_i)

Non-example:

$x_i x_i + y_r y_r = z_i z_i$

$y_r x_i x_i + y_r y_r y_r = y_r z_i z_i$

3) It is illegal to divide through by an indexed variable where index already appears in the equation.

Example:

$x_i x_i = 1 \quad \not\Rightarrow x_i = 1/x_i$

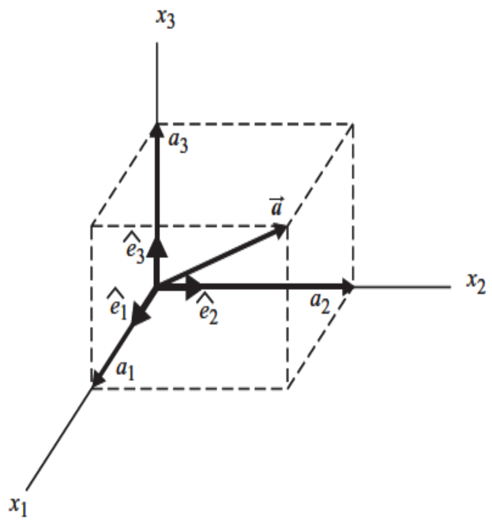
Example:

Index notation is an extremely useful tool for performing vector algebra. In coordinate system, instead of using the typical axis labels x, y and z, x_1, x_2 and x_3 are used.

x_i i= 1,2,3

The corresponding unit basis vectors are $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$.

$\hat{e}_i$ i= 1,2,3



$\vec{\alpha} = \alpha_1\, \hat{e}_1 + \alpha_2\, \hat{e}_2 + \alpha_3\, \hat{e}_3$

Kronecker Delta

1) Kronecker Delta is represented by 9 numbers defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\delta_{11} = 1 \quad \delta_{12} = 0 \quad \delta_{13} = 0$

$\delta_{21} = 0 \quad \delta_{22} = 1 \quad \delta_{23} = 0$

$\delta_{31} = 0 \quad \delta_{32} = 0 \quad \delta_{33} = 1$

2) $\delta_{ij} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

3) δ_{ij} can be used to "sum on " indices with a term it multiplies.

Example:

$\delta_{ij}u_j = u_i$

$\delta_{ij}u_j = \delta_{i1}u_1 + \delta_{i2}u_2 + \delta_{i3}u_3$

i=1 $\quad \delta_{11}u_1 + \delta_{12}u_2 + \delta_{13}u_3 = u_1$

i=2 $\quad \delta_{21}u_1 + \delta_{22}u_2 + \delta_{23}u_3 = u_2$

i=3 $\quad \delta_{31}u_1 + \delta_{32}u_2 + \delta_{33}u_3 = u_3$

4) In an orthonormal basis $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$\hat{e}_i . \hat{e}_j = \delta_{ij}$

$\hat{e}_1 . \hat{e}_1 = \hat{e}_2 . \hat{e}_2 = \hat{e}_3 . \hat{e}_3 = 1$

$\hat{e}_1 . \hat{e}_2 = \hat{e}_2 . \hat{e}_3 = \hat{e}_1 . \hat{e}_3 = 0$

Permutation or Alternator Symbol

ε_{ijk} is represented by 27 numbers defined by

- $\epsilon_{ijk} = 1$ if (ijk) is an even (cyclic) permutation of (123), i.e. $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$
- $\epsilon_{ijk} = -1$ if (ijk) is an odd (noncyclic) permutationof (123), i.e. $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$
- $\epsilon_{ijk} = 0$ if two or more subscripts are the same, i.e. $\epsilon_{111} = \epsilon_{112} = \epsilon_{313} = 0$ etc.

Properties

1) Switching any 2 indices changes the sign.

Examples

$\epsilon_{312} = -\epsilon_{321}$

$\epsilon_{ijk} = -\epsilon_{ikj}$

$\epsilon_{kji} = -\epsilon_{ijk}$

$\epsilon_{iij} = 0$

$\epsilon_{ijk}\epsilon_{ijk} = 6$

identities

$\epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}$

$\epsilon_{ijk}\epsilon_{ijr} = 2\delta_{kr}$

$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{kk} = 6 \ \ (k = r)$

2)Cross product

$(\vec{u} \times \vec{v})_i = \epsilon_{ijk}\, u_k v_j$

3) Partial differentiation denoted by "comma"

Examples:

- $\frac{\partial f}{\partial x} = f_{,x}$ $\quad \frac{\partial f}{\partial x_1} = f_{,1}$ $\quad \frac{\partial u_1}{\partial x_2} = u_{1,2}$

- x, y, z $\Rightarrow$ σ_{xx} σ_{yy} σ_{zz} τ_{xy} τ_{xz} τ_{yz}

  x_1, x_2, x_3 $\Rightarrow$ σ_{11} σ_{22} σ_{33} τ_{12} τ_{13} τ_{23}

Stress equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} + B_x = 0$$

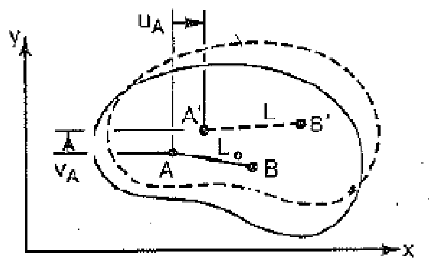
$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + B_1 = 0$$

$$\boxed{\sigma_{ij,j} + B_i = 0}$$

STRAIN

Deformation

Consider a body subjected to external loading that causes it to deform;



Displacement due to 1) Deformation (strain)

2) Rigid body motion (translation and rotation)

Displacement at any point within the body: u = u(x,y,z)

v = v(x,y,z)

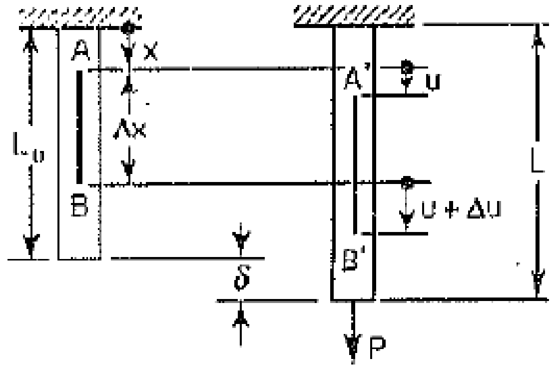
w = w(x,y,z)

Assumption: 1) Small deformations (h.o.t. neglected)

2) The principle of superposition can be used.

Definition of Strain

Consider an axially loaded member



Normal strain, unit change in length, is defined

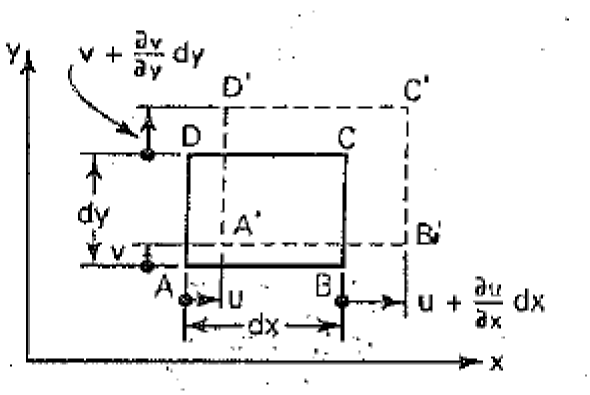
$$\varepsilon_x = \lim_{\Delta x \to 0} \frac{u + \Delta u - u}{\Delta x}$$

If deformation is uniformly distributed over the original length,

$$\varepsilon_x = \frac{L - L_0}{L_0} = \frac{\delta}{L_0}$$

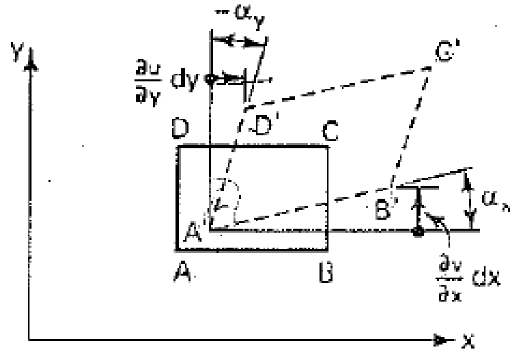
2-D Plane Strain Case

Normal strain



$$\varepsilon_{xx} = \varepsilon_x = \frac{\left(u + \frac{\partial u}{\partial x}dx\right) - u}{\partial x} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \varepsilon_y = \frac{\partial v}{\partial y}$$

Shear strain



Small angles $\alpha_x \cong \tan\alpha_x = \frac{\frac{\partial v}{\partial x}dx}{dx} = \frac{\partial v}{\partial x}$

Similarly, $\alpha_y \cong \tan\alpha_y = \frac{\frac{\partial u}{\partial y}dy}{dy} = \frac{\partial u}{\partial y}$

Total angular change

$$\gamma_{xy} = \alpha_x + \alpha_y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

↗ Shear strain

3-D PLANE STRAIN

$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

i = j = 1 $\epsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$

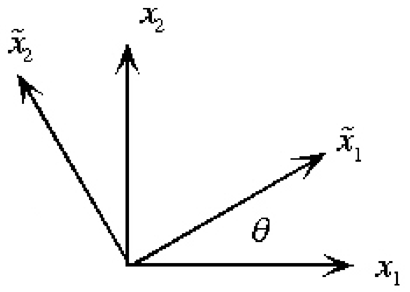
i =1, j=2 $\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\gamma_{12}$

Strain Tensor at a Point

$$\left[\epsilon_{ij}\right] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{32} & \epsilon_{33} \end{bmatrix}$$

The nine quantities ϵ_{ij} $(i, j = 1,2,3)$ form a symmetric matrix. They are called the components of strain of the particle with respect to the system of axes x_1, x_2 and x_3. The quantities $\epsilon_{11}, \epsilon_{22}$ *and* ϵ_{33} are called the normal components of strain of the particle in the directions of the axes x_1, x_2 and x_3, respectively, while $\epsilon_{12} = \epsilon_{21}, \epsilon_{13} = \epsilon_{31}$ *and* $\epsilon_{23} = \epsilon_{32}$ are called shearing components of strain in the directions of the axes x_1, x_2 and x_1, x_3 and x_2, x_3, respectively.

STRAIN TRANSFORMATION



$\epsilon'_{11} = \frac{\epsilon_{11}+\epsilon_{22}}{2} + \frac{\epsilon_{11}-\epsilon_{22}}{2}\cos 2\theta + \epsilon_{12}\sin 2\theta$ (where $\epsilon_{12} = \frac{1}{2}\gamma_{12}$)

The normal stress ϵ'_{22} is determined by replacing θ *by* $\theta + \pi/2$.

$\epsilon'_{12} = \frac{1}{2}\gamma'_{12} = \frac{(\epsilon_{22}-\epsilon_{11})}{2}\sin 2\theta + \frac{\epsilon_{12}}{2}\cos 2\theta$

REFERENCES

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3) Lecture notes