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Lecture note-8

Generalized Hook's Law

Stres-Strain Relation

Generalized Hooke's Law

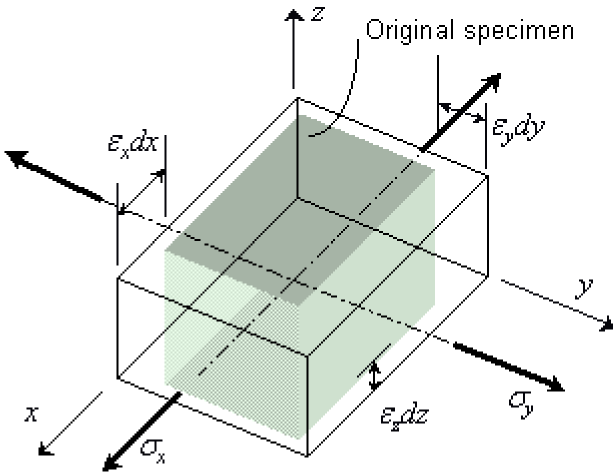
The generalized Hooke's Law can be used to predict the deformations caused in a given material by an arbitrary combination of stresses.

The linear relationship between stress and strain applies for $0 \le \sigma \le \sigma_{Yield}$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$



where: E is the Young's Modulus
n is the Poisson Ratio

The generalized Hooke's Law also reveals that strain can exist without stress. For example, if the member is experiencing a load in the y-direction (which in turn causes a stress in the y-direction), the Hooke's Law shows that strain in the x-direction does not equal to zero. This is because as material is being pulled outward by the y-plane, the material in the x-plane moves inward to fill in the space once occupied, just like an elastic band becomes thinner as you try to pull it apart. In this situation, the x-plane does not have any external force acting on them but they experience a change in length. Therefore, it is valid to say that strain exist without stress in the x-plane.

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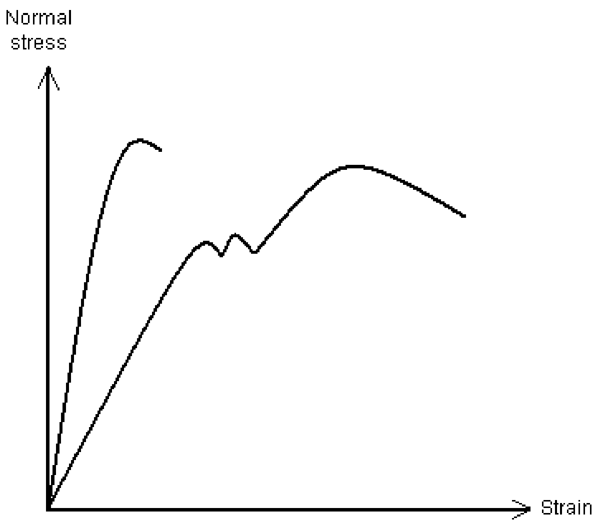
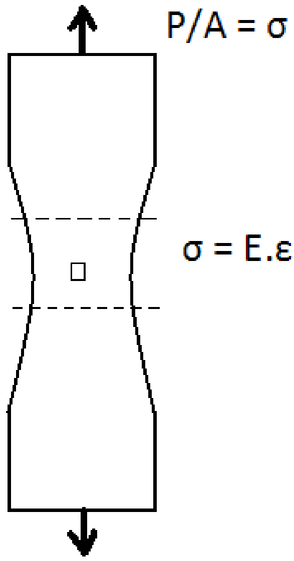
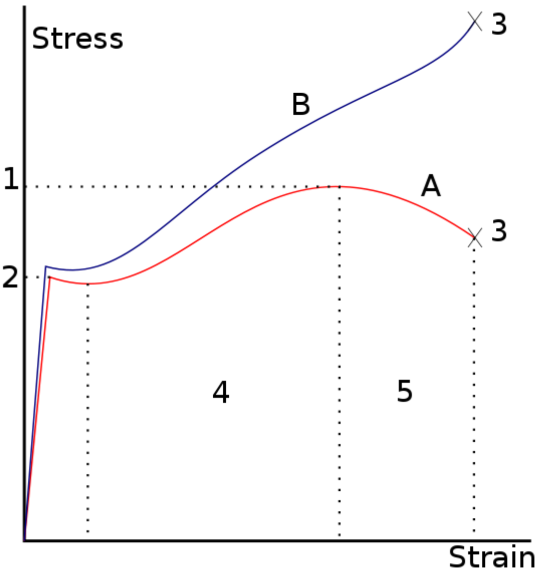
http://www.engineering.com/Library/ArticlesPage/tabid/85/articleType/A rticleView/articleId/208/Generalized-Hookes-Law.aspx



Stress-strain diagram



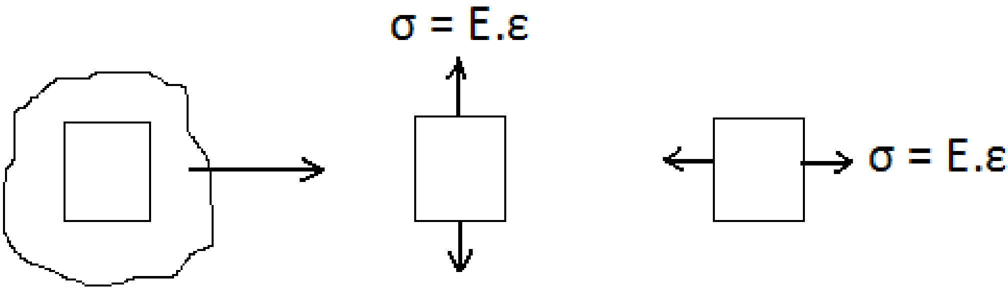


Stress–strain curve for low-carbon steel. Hooke's law is only valid for the portion of the curve between the origin and the yield point(2).

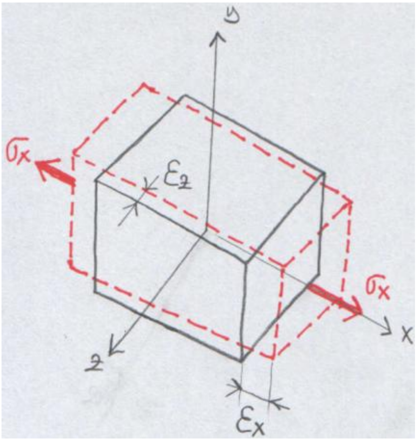
1. Ultimate strength
2. Yield strength-corresponds to yield point.
3. Rupture
4. Strain hardening region
5. Necking region.

A: Apparent stress (F/A0)
B: True stress (F/A)

- We need to connect all six components of stres to six components of strain.
- Restrict to linearly elastic-small strains.
- An isotropic materials whose properties are independent of orientation.



Consider an elment on which there is only one component of normal stres acting.



$$\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = 0$$

$$\varepsilon_x = \frac{1}{E}\sigma_x$$

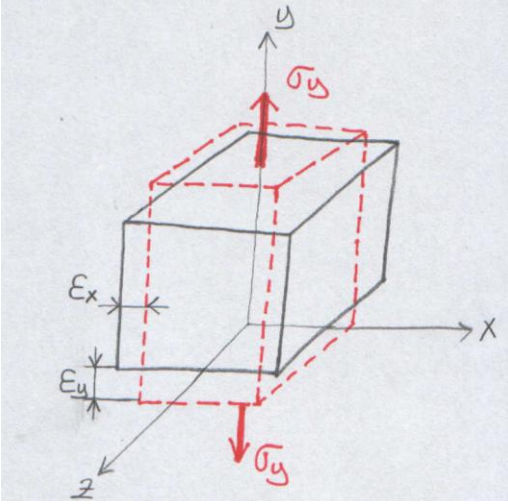
In addition to normal strain there is a lateral contraction

$\varepsilon_y = \varepsilon_z = - \nu\ \varepsilon_y = -\nu\ .\ \sigma_x /E$

There is no shear strain due to normal stres in isotropic materials.

$\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$ *(γ= gamma)*

- Now σ_y is applied



$\varepsilon_y = 1/E \,.\, \sigma_y$ because of isotropy

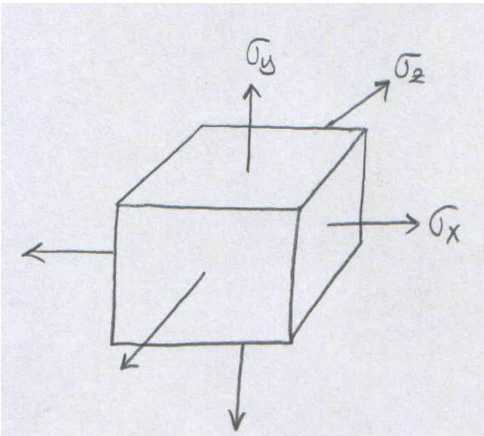
$\varepsilon_x = \varepsilon_z = -\nu\, \varepsilon_y = -\nu.\sigma_y/E$

Similar result for loading in the z direction

$$\varepsilon_z = \frac{\sigma_z}{E}$$

$\varepsilon_x = \varepsilon_y = -\nu\, \varepsilon_z = -\nu.\sigma_z/E$

$\varepsilon_x = \sigma_x/E - \nu/E \,.\, \gamma_y - \nu/E \,.\, \sigma_x$



* normal strains

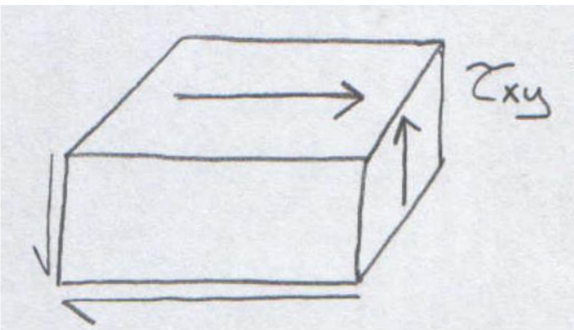
$\varepsilon_x = 1/E\,(\sigma_x - \nu(\sigma_y + \sigma_z))$

$\varepsilon_y = 1/E\,(\sigma_y - \nu(\sigma_x + \sigma_z))$

$\varepsilon_z = 1/E\,(\sigma_z - \nu(\sigma_x + \sigma_y))$

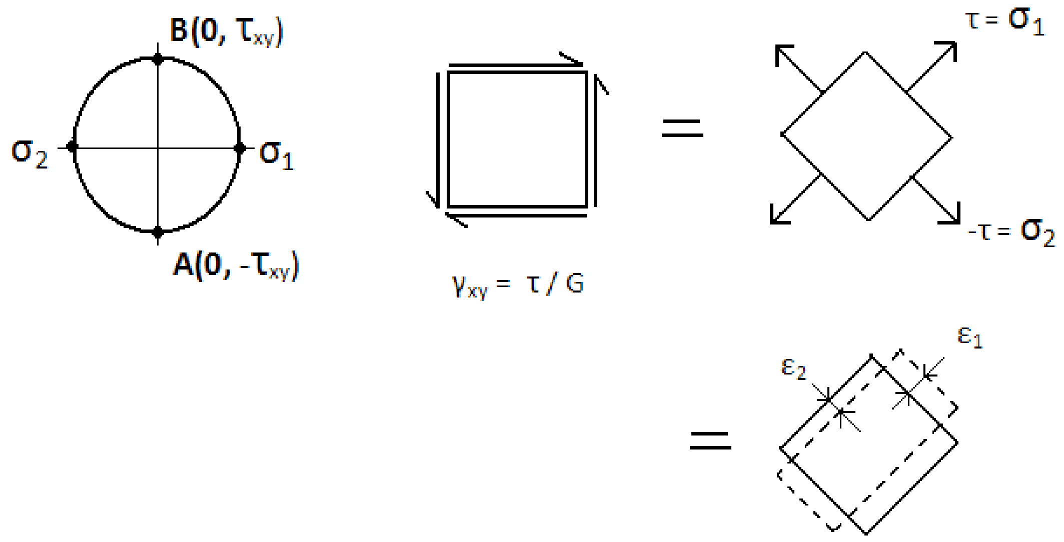
Shear Stres

Each shear stres component produces only its corresponding shear strain component.



$\gamma_{xy} = \tau_{xy}/G$ *(G: shear modulus)*

Relationship Between G, E and ν



$$\varepsilon_1 = \frac{\sigma_1}{E} - \frac{\upsilon}{E}\sigma_2$$

$$\varepsilon_2 = \frac{\sigma_2}{E} - \frac{\upsilon}{E}\sigma_1 = \tau\frac{(1+\upsilon)}{E}$$

$$\frac{\gamma_{xy}}{2} = \frac{\varepsilon_1 - \varepsilon_2}{2} = \frac{2(1+\upsilon)\tau}{E}$$

$$G = \frac{E}{2(1+\upsilon)}$$

Just 2 independent elastic constant

ε_{xx}	ε_{yy}	ε_{zz}	$\varepsilon_{xy} = \gamma_{xy}/2$	$\varepsilon_{xz} = \gamma_{xz}/2$	$\varepsilon_{yz} = \gamma_{yz}/2$
ε_{11}	ε_{22}	ε_{33}	ε_{12}	ε_{13}	ε_{23}

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \hline \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ \hline & & & 1/2G & & \\ & & & & 1/2G & \\ & & & & & 1/2G \end{array}\right] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \hline \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}
$$

Hooke's Law in Compliance Form

By convention, the 9 elastic constants in orthotropic constitutive equations are comprised of 3 Young's modulii E_x, E_y, E_z, the 3 Poisson's ratios ν_{yz}, ν_{zx}, ν_{xy}, and the 3 shear modulii G_{yz}, G_{zx}, G_{xy}.

The **compliance matrix** takes the form,

$$
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}
$$

where $\frac{\nu_{yz}}{E_y} = \frac{\nu_{zy}}{E_z}, \ \frac{\nu_{zx}}{E_z} = \frac{\nu_{xz}}{E_x}, \ \frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}$.

Note that, in orthotropic materials, there is no interaction between the normal stresses σ_x, σ_y, σ_z and the shear strains ε_{yz}, ε_{zx}, ε_{xy}

The factor 1/2 multiplying the shear modulii in the compliance matrix results from the difference between shear strain and engineering shear strain, where
$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = 2\varepsilon_{xy}$, etc.

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{bmatrix} = \left[\begin{array}{ccc|ccc} 2G+\lambda & \lambda & \lambda & & & \\ \lambda & 2G+\lambda & \lambda & & & \\ \lambda & \lambda & 2G+\lambda & & & \\ \hline & & & 2G & 0 & 0 \\ & & & 0 & 2G & 0 \\ & & & 0 & 0 & 2G \end{array}\right] = \begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{bmatrix}$$

$$\sigma_{11} = (2G+\lambda).\varepsilon_{11} + \lambda.(\varepsilon_{22} + \varepsilon_{33})$$

$$\sigma_{11} = 2G.\varepsilon_{11} + \lambda.(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

$$\sigma_{22} = (2G+\lambda).\varepsilon_{22} + \lambda.(\varepsilon_{11} + \varepsilon_{33})$$

$$\sigma_{22} = 2G.\varepsilon_{22} + \lambda.(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

$$\sigma_{33} = (2G+\lambda).\varepsilon_{33} + \lambda.(\varepsilon_{11} + \varepsilon_{22})$$

$$\sigma_{33} = 2G.\varepsilon_{33} + \lambda.(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

where

$$\lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} \qquad G = \frac{E}{2(1+\upsilon)}$$

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$ i,j= 1,2,3......

i=1 j=2

$$\sigma_{12} = 2G\varepsilon_{12} + \lambda\delta_{12}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$ *here (δ=0)*

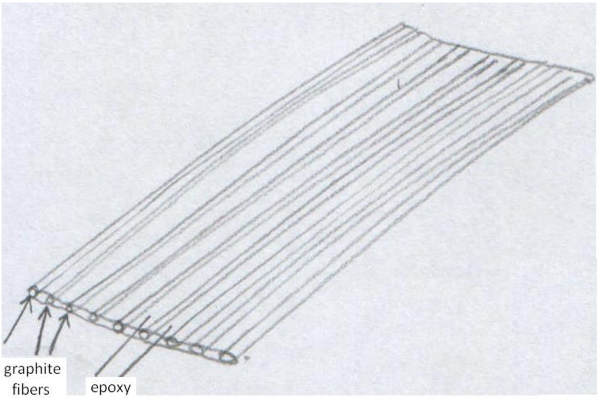
i=1 j=1

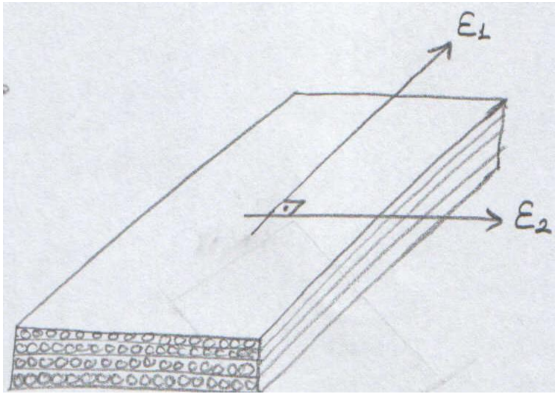
$$\sigma_{11} = 2G\varepsilon_{11} + \lambda\delta_{11}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$ *here (δ=1)*

$$\sigma_{11} = 2G\varepsilon_{11} + \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

Materials with different properties in different directions are called **anisotropic**.

<u>Exp :</u>
Composites materials





$$\left[\begin{array}{c}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \hline \varepsilon_{12}\\ \varepsilon_{13}\\ \varepsilon_{23}\end{array}\right] = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & \cdots & \\ C_{31} & C_{32} & C_{33} & C_{34} & \cdots & \\ \hline \vdots & \vdots & & & & \\ & & & & & \end{array}\right] = \left[\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \hline \sigma_{12}\\ \sigma_{13}\\ \sigma_{23}\end{array}\right]$$

If there are axes of symmetry in 3 perpendicular directions, material is called **ORTHOTROPIC** materials.

An **orthotropic material** has two or three mutually orthogonal two-fold axes of rotational symmetry so that its mechanical properties are, in general, different along the directions of each of the axes. Orthotropic materials are thus **anisotropic**; their properties depend on the direction in which they are measured. An **isotropic material**, in contrast, has the same properties in every direction.

One common example of an orthotropic material with two axes of symmetry would be a polymer reinforced by parallel glass or graphite fibers. The strength and stiffness of such a composite material will usually be greater in a direction parallel to the fibers than in the transverse direction. Another example would be a biological membrane, in which the properties in the plane of the membrane will be different from those in the perpendicular direction. Such materials are sometimes called transverse isotropic.

A familiar example of an orthotropic material with three mutually perpendicular axes is wood, in which the properties (such as strength and stiffness) along its grain and in each of the two perpendicular directions are different. Hankinson's equation provides a means to quantify the difference in strength in different directions. Another example is a metal which has been rolled to form a sheet; the properties in the rolling direction and each of the two transverse directions will be different due to the anisotropic structure that develops during rolling.

It is important to keep in mind that a material which is anisotropic on one length scale may be isotropic on another (usually larger) length scale. For instance, most metals are polycrystalline with very small grains. Each of the individual grains may be anisotropic, but if the material as a whole comprises many randomly oriented grains, then its measured mechanical properties will be an average of the properties over all possible orientations of the individual grains.

Generalized Hooke's Law (Anisotropic Form)

Cauchy generalized Hooke's law to three dimensional elastic bodies and stated that the 6 components of stress are linearly related to the 6 components of strain.

The stress-strain relationship written in matrix form, where the 6 components of stress and strain are organized into column vectors, is,

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}=\begin{bmatrix}S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}, \quad \varepsilon = \mathbf{S}\cdot\sigma$$

or,

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{bmatrix}, \quad \sigma = \mathbf{C}\cdot\varepsilon$$

where **C** is the **stiffness matrix**, **S** is the **compliance matrix**, and **S** = $\mathbf{C}^{-1}$.

In general, stress-strain relationships such as these are known as **constitutive relations**.

In general, there are 36 stiffness matrix components. However, it can be shown that conservative materials possess a strain energy density function and as a result, the stiffness and compliance matrices are symmetric. Therefore, only 21 stiffness components are actually independent in Hooke's law. The vast majority of engineering materials are conservative.

Please note that the **stiffness** matrix is traditionally represented by the symbol **C**, while **S** is reserved for the **compliance** matrix. This convention may seem backwards, but perception is not always reality. For instance, Americans hardly ever use their feet to play (American) football.

http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/hooke.cfm

15 Governing Equation

1-) Equations of equilibrium (3)

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + B_1 = 0 \qquad i=1$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + B_2 = 0 \qquad i=2$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + B_3 = 0 \qquad i=3$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} + B_i$$

2-) Strain Displacement Equations (6)

$$\varepsilon_{11} = \frac{\partial U_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial U_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial U_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial U_1}{\partial x_2} + \frac{\partial U_2}{\partial x_1}\right) \qquad \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial U_1}{\partial x_3} + \frac{\partial U_3}{\partial x_1}\right) \qquad \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial U_2}{\partial x_3} + \frac{\partial U_3}{\partial x_2}\right)$$

2-D Strain Compatibility

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_2^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

3-) Generalized Hook's Law-Stress-Strain (6)

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - \upsilon(\sigma_{22} - \sigma_{33})\right) \qquad \varepsilon_{12} = \frac{1}{2G}\sigma_{12}$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - \upsilon(\sigma_{11} - \sigma_{33})\right) \qquad \varepsilon_{13} = \frac{1}{2G}\sigma_{13}$$

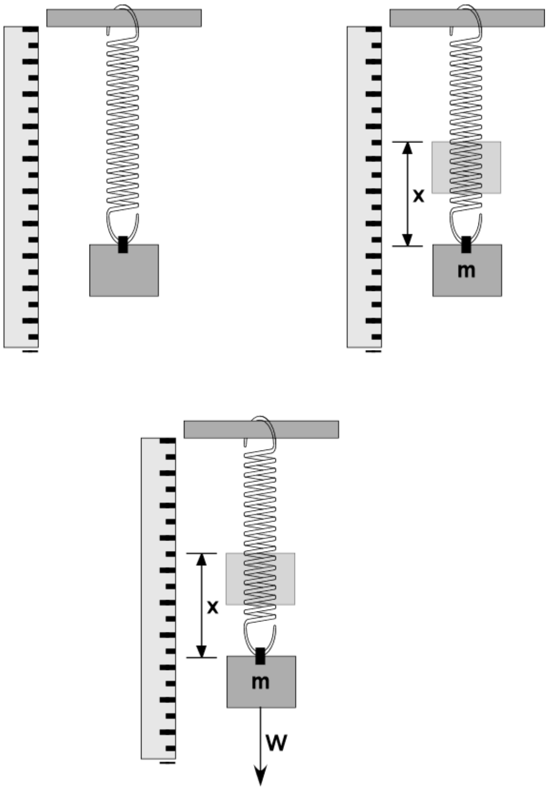
$$\varepsilon_{33} = \frac{1}{E}\left(\sigma_{33} - \upsilon(\sigma_{11} - \sigma_{22})\right) \qquad \varepsilon_{23} = \frac{1}{2G}\sigma_{23}$$

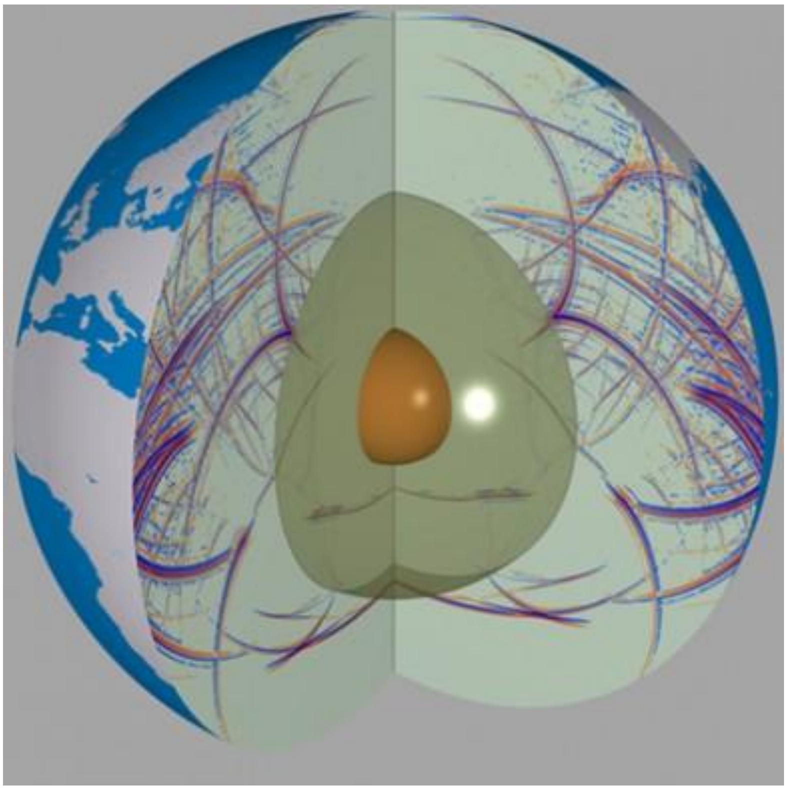
Question :
I have a spring, ruler,3 known masses, and 1 unknown mass. How would I find the unknown mass using these materials? Is it possible to solve using Hooke's Law? It would be very helpful if you guys can provide some equations or include any diagrams. Also how would I derive the needed equations from a graph?
Answer:
Hook's Law says "the restoring force of the spring is proportional to the extension or compression of the spring from its equilibrium." In formula form its F=-kx (the negative indicates that the force is in the opposite direction from the extension, x).

So, for every spring, there is a constant, k. Use your known masses and find how much of an "x" they will get on your spring. Now you have three sets of F and x. How are they related? Through "k".

Find k. Now you have k and you can measure the x of the unknown mass to get its weight (F).

Graphically: think "slope."





Hooke's law for isotropic continua, elastic wave equation, reflection and refraction methods for imaging the Earth's internal structure, plane waves in an infinite medium and interaction with boundaries, body wave seismology, inversion of travel-time curves, generalized ray theory, crustal seismology, surface waves and earthquake source studies