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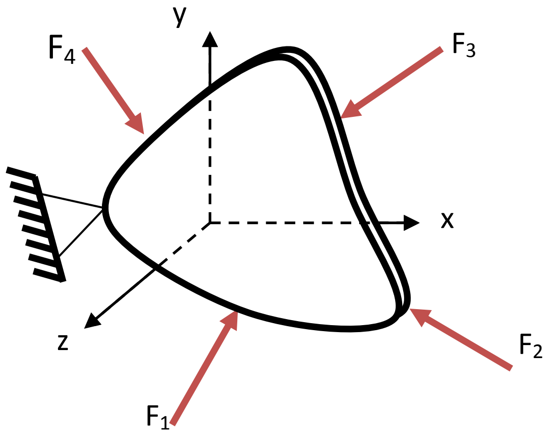
2D PLANE PROBLEMS $\left(\frac{\partial}{\partial z}=0,\ \tau_{XZ}=\tau_{YZ}=0\right)$



8 unknowns: σ_x, σ_y, τ_{xy}, ε_x, ε_y, ε_{xy}, u, v

Stress- Strain Eqns.		Equilibrium Eqns.	Strain-Disp. Eqns.
Pl- σ (σ_z=0)	Pl-ε ($\varepsilon_z=0$)		
$\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)$ $\varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)$ $\varepsilon_{xy}=\frac{\tau_{xy}}{2G}$	$\varepsilon_x=\frac{1-\nu^2}{E}(\sigma_x-\frac{\nu}{1-\nu}\sigma_y)$ $\varepsilon_y=\frac{1-\nu^2}{E}(\sigma_y-\frac{\nu}{1-\nu}\sigma_x)$	$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+B_X=0$ $\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+B_Y=0$	$\varepsilon_x=\frac{\partial u}{\partial x}$ $\varepsilon_y=\frac{\partial v}{\partial y}$ $\varepsilon_{xy}=\frac{1}{2}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x})$ Compatibility Eqs $\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}$

Note: To convert from plain σ to plain ε, replace E by $\frac{E}{1-\nu^2}$ and ν by $\frac{\nu}{1-\nu}$

If we replace the strains in the compatibility eqns. by the stress from Hooke's Law. It leads below Eqn.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0 \qquad (*)$$

Together with the Equilibrium Eqns. & the BC's we can solve for σ_x, σ_y, τ_{xy}.

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Airy Stress Function

(No body forces, i.e. $B_x = B_y = 0$)

1) Write compatibility equations, in terms of stresses, using Hooke's Law. Then you'll get 3 equations (2 equilibrium equations and 1 compatibility equation) and 3 unknowns (σ_x , σ_y, τ_{xy})
2) Define Airy stress function by:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

Equilibrium equations are automatically satisfied by the Airy stress function.

Compatibility equation reduces to:

$$\nabla^4 \Phi = 0$$

$$(\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4})$$

Step by step Solution

$\Phi_2 = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$ is given

1) Check that $\nabla^4 \Phi = 0$
2) Calculate stresses $\sigma_x = c_2$, $\sigma_y = a_2$, $\tau_{xy} = -b_2$
3) Check BC's

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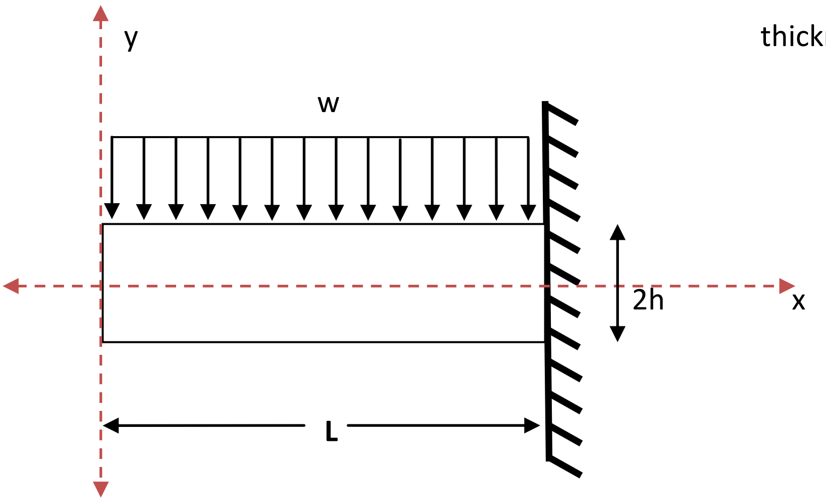
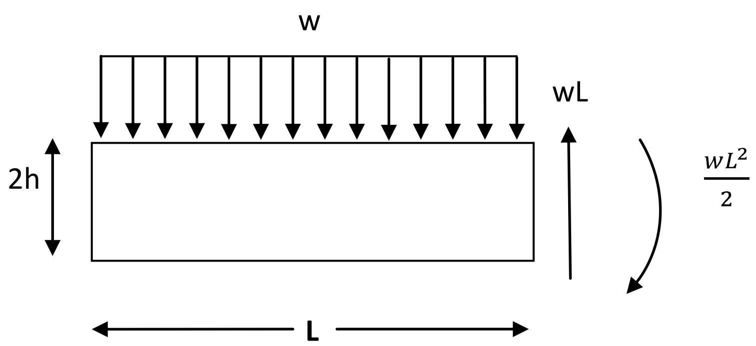
Example:



$$\Phi = Axy + Bx^2 + Cx^2y + Dy^3 + Exy^3 + Fx^2y^3 + Gy^5$$

1

FBD:



Analysis:

Step 1: Check biharmonic Eqn.

$$\nabla^4\Phi = 24Fy + 120Gy = 0$$

$$F = -5G$$

Step 2: Find stresses.

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = 6Dy + 6Exy - 30Gx^2y + 20Gy^3$$

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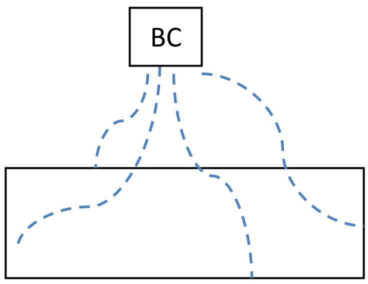
Continue to step 2

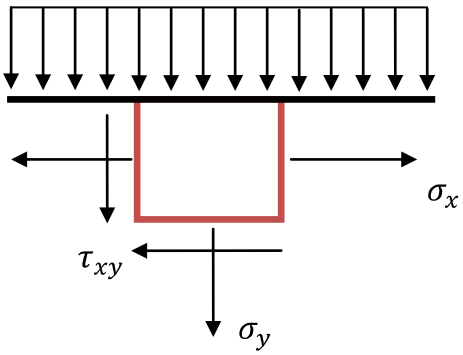
$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = 2B + 2Cy - 10Gy^3$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -A - 2Cx - 3Ey^2 + 30Gxy^2$$

Step 3: Check BC's



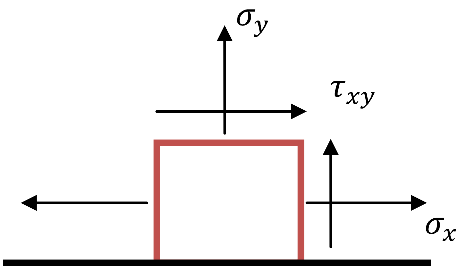
At $y = h$



$\sigma_y = \frac{-w}{t}$ (1)

$\tau_{xy} = 0$ (2)

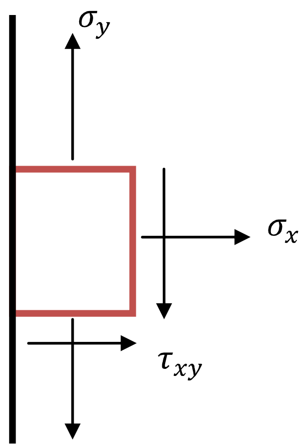
At $y = -h$



$\sigma_y = 0$ (3)

$\tau_{xy} = 0$ (4)

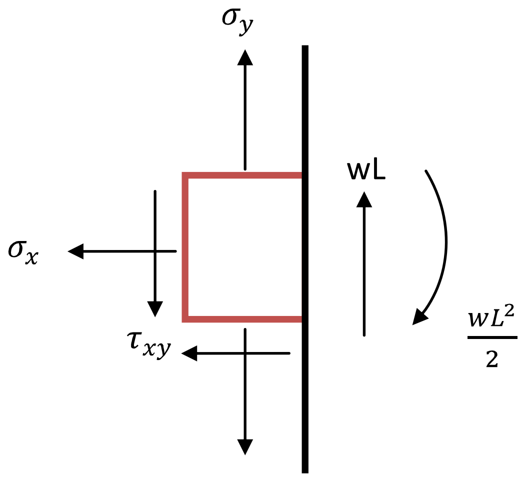
At $x = 0$



$\sigma_x = 0$ (5)

$\tau_{xy} = 0$ (6)

At $x = L$



$\int_{-h}^{h} \sigma_x t dy = 0$ (7)

$\int_{-h}^{h} \tau_{xy} t dy = wL$ (8)

$\int_{-h}^{h} (\sigma_x t) y dy = \frac{wL^2}{2}$ (9)

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From Eqn. (6) at $x = 0\,, \quad \tau_{xy} = 0$

$$-A - 3Ey^2 = 0 \quad \Rightarrow \quad A, E = 0$$

Egn. (5) at $\boxed{x = 0,\ \sigma_x = 0}$ → ignore

From Eqn. (2) or (4)

$$\tau_{xy} = 0 = -2Cx + 30Gxh^2$$

$$C = 15Gh^2$$

Use (3)

$$\sigma_y = 0 = 2B - 2Ch - 10Gh^3 = 2B - 30Gh^3 + 10Gh^3$$

$$B = 10Gh^3$$

Use (1)

$$\sigma_y = \frac{-w}{t} = 2B + 2Ch - 10Gh^3 = 2B + 20Gh^3$$

$$\frac{-w}{t} = 40Gh^3$$

$$G = \frac{-w}{40th^3}$$

Now take the integrals.

$$B = \frac{-wh^3}{6I}$$

$$\boxed{I = \frac{2}{3}th^3}$$

$$G = \frac{-w}{60I}$$

$$C = \frac{-wh^2}{4I}$$

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The Airy solution in rectangular coordinates

The Airy function procedure can then be summarized as follows:

1. Begin by finding a scalar function $\phi(x_1, x_2)$ (known as the Airy potential) which satisfies:

$$\nabla^4 \phi \equiv \frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_1^4} = C(\nu)\left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2}\right)$$

Where

$$C(\nu) = \begin{cases} \dfrac{1-\nu}{1-2\nu} & \text{(Plane Strain)} \\ \dfrac{1}{1-\nu} & \text{(Plane Stress)} \end{cases}$$

In addition ϕ must satisfy the following traction boundary conditions on the surface of the solid

$$\frac{\partial^2 \phi}{\partial x_2^2} n_1 - \frac{\partial^2 \phi}{\partial x_1 \partial x_2} n_2 = t_1 \qquad \frac{\partial^2 \phi}{\partial x_1^2} n_2 - \frac{\partial^2 \phi}{\partial x_1 \partial x_2} n_1 = t_2$$

Where (n_1, n_2) are the components of a unit vector normal to the boundary?

2. Given ϕ, the stress field within the region of interest can be calculated from the

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} - \Omega \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} - \Omega \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = 0 \qquad \text{(Plane Stress)}$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \qquad \text{(Plane Strain)}$$

formula $\sigma_{23} = \sigma_{13} = 0$

3. If the strains are needed, they may be computed from the stresses using the elastic stress strain relations.

4. If the displacement field is needed, it may be computed by integrating the strains, following the procedure described in Section 2.1.20. An example (in polar coordinates) is given in Section 5.2.4 below.

Although it is easier to solve for ϕ than it is to solve for stress directly, this is still not a trivial exercise. Usually, one guesses a suitable form for ϕ, as illustrated below. This may seem highly unsatisfactory, but remember that we are essentially integrating a system of PDEs. The general procedure to evaluate any integral is to guess a solution, differentiate it, and see if the guess was correct.

REFERENCE

1. http://solidmechanics.org/text/Chapter5_2/Chapter5_2.htm