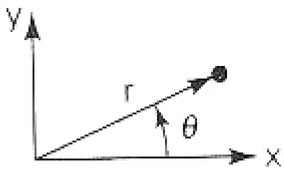
Lecture notes 13 Date: 18-Nov-2009

2-D plane formulation in polar coordinates

Reference: Ugural and Fenster, Sections 3.8 and 3.11

Change from Cartesian coordinates to Polar coordinates:



$$x = r\cos\theta, \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta, \qquad \theta = \tan^{-1}\frac{y}{x} \tag{a}$$

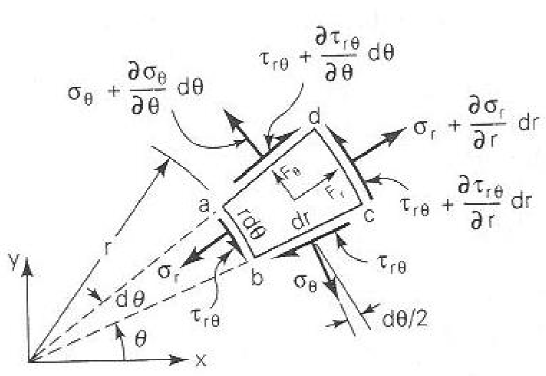
These equations yield

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r} \tag{b}$$

Any derivatives with respect to x and y in the Cartesian system may be transformed into derivatives with respect to r and θ by applying the *chain rule*:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} \tag{c}$$

Equations of Equilibrium in Polar Coordinates:



$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$
$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0$$

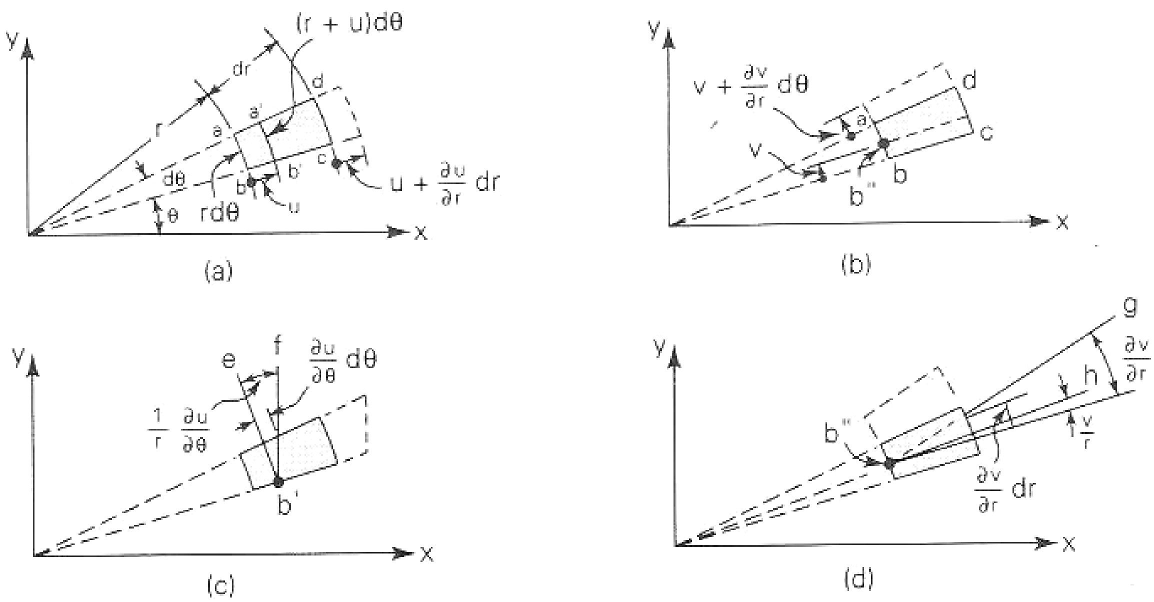
Airy Stress function in Polar Coordinates

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r\,\partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \theta}\right)$$

Strain-Displacement Relations can be obtained from the following figures



$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} \qquad \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}$$

Hooke's Law does not change:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta}$$

Compatibility equation gives the Biharmonic Equation for the Airy Stress Function in polar coordinates:

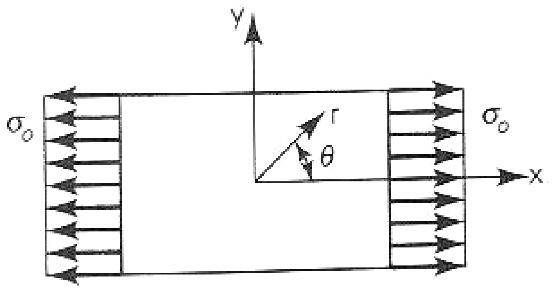
$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r\, \partial \theta} + \frac{1}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta}$$

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}$$

$$\nabla^4 \Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\nabla^2 \Phi) = 0$$

Example 1:

We will write the Airy Stress function and the stresses in polar coordinates for a plate pulled in the x-direction by a stress σ_o.



$$\sigma_x = \sigma_o, \qquad \sigma_y = \tau_{xy} = 0$$

The Airy stress function that would give this stress state is $\Phi = \frac{\sigma_o}{2} y^2$

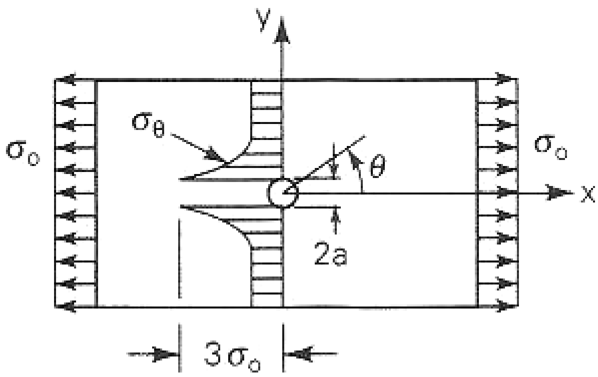
Then using $\quad y = r \sin\theta$

$$\Phi = \tfrac{1}{4}\sigma_o r^2 (1 - \cos 2\theta)$$

The stresses can be obtained from the Airy Stress Function (they can also be obtained by using the stress transformation equations):

$$\sigma_r = \tfrac{1}{2}\sigma_o(1 + \cos 2\theta)$$
$$\sigma_\theta = \tfrac{1}{2}\sigma_o(1 - \cos 2\theta)$$
$$\tau_{r\theta} = -\tfrac{1}{2}\sigma_o \sin 2\theta$$

Example 2: Use the Airy stress function to guess at a solution for a hole in a plate problem.



Using the Airy stress function for a plate with no hole, we guess the solution for the plate with a hole to be of the form:

$$\Phi = f_1(r) + f_2(r)\cos 2\theta$$

Inserting the Airy stress function into the biharmonic equation in polar coordinates we obtain:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 f_1}{dr^2} + \frac{1}{r}\frac{df_1}{dr}\right) = 0$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f_2}{dr^2} + \frac{1}{r}\frac{df_2}{dr} - \frac{4f_2}{r^2}\right) = 0$$

which can be rewritten as

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df_1}{dr}\right)\right]\right\} = 0$$

$$r\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}\left\{r^3\frac{d}{dr}\left[\frac{1}{r^3}\frac{d}{dr}(r^2 f_2)\right]\right\}\right) = 0$$

The solutions are:

$$f_1 = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$$

$$f_2 = c_5 r^2 + c_6 r^4 + \frac{c_7}{r^2} + c_8$$

The stresses are

$$\sigma_r = c_1(1 + 2\ln r) + 2c_2 + \frac{c_3}{r^2} - \left(2c_5 + \frac{6c_7}{r^4} + \frac{4c_8}{r^2}\right)\cos 2\theta$$

$$\sigma_\theta = c_1(3 + 2\ln r) + 2c_2 - \frac{c_3}{r^2} + \left(2c_5 + 12c_6 r^2 + \frac{6c_7}{r^4}\right)\cos 2\theta$$

$$\tau_{r\theta} = \left(2c_5 + 6c_6 r^2 - \frac{6c_7}{r^4} - \frac{2c_8}{r^2}\right)\sin 2\theta$$

To find the constants we use the boundary conditions:

c_1 and c_6 become zero because the stresses must be finite as $r \to \infty$. In addition, they must assume the forms for the plate with no hole giving us the expressions for c_2 and c_5

$$\sigma_o = -4c_5, \qquad \sigma_o = 4c_2$$

At r=a, $\sigma_r = \tau_{r\theta} = 0$ giving us the relations:

$$2c_2 + \frac{c_3}{a^2} = 0, \qquad 2c_5 + \frac{6c_7}{a^4} + \frac{4c_8}{a^2} = 0, \qquad 2c_5 - \frac{6c_7}{a^4} - \frac{2c_8}{a^2} = 0$$

Finally, the stresses are:

$$\sigma_r = \tfrac{1}{2}\sigma_o\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta\right]$$

$$\sigma_\theta = \tfrac{1}{2}\sigma_o\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau_{r\theta} = -\tfrac{1}{2}\sigma_o\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

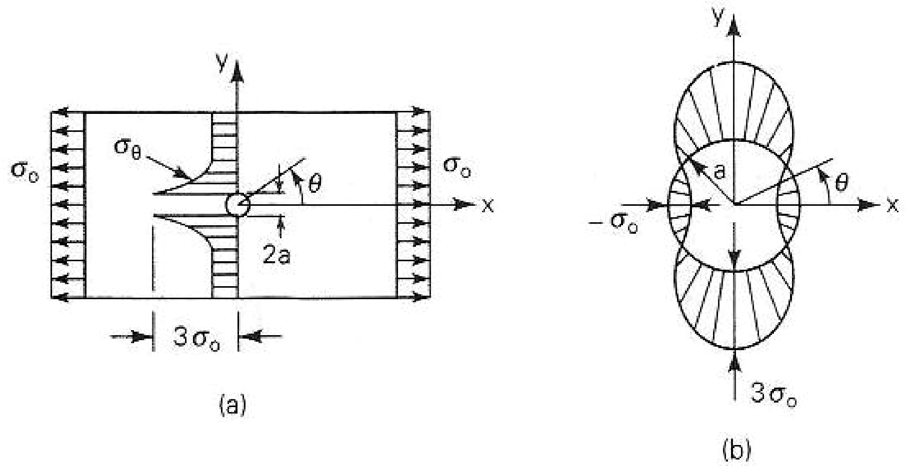


FIGURE 3.13. *Example 3.4. Circular hole in a plate subjected to uniaxial tension: (a) tangential stress distribution for* $\theta = \pm\pi/2$*; (b) tangential stress distribution along periphery of the hole.*

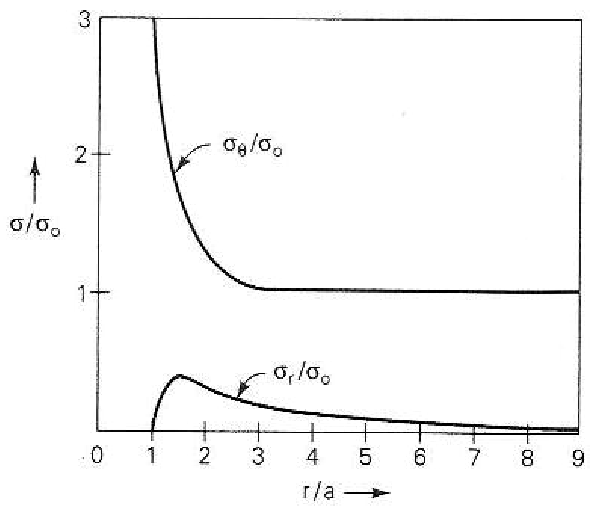


FIGURE 3.14. *Example 3.4. Graph of tangential and radial stresses for* $\theta = \pi/2$ *versus the distance from the center of the plate shown in Fig. 3.13a.*