1- Mechanics of Materials Approach
2- Prandtl Stress Function Approach

-more sufficient for axial cylinder problem

1-) Mechanics of Material Approach to Torsion of Circular Bars
Assumptions:

1) All plane sections perpendicular to torsion axis z remain perpendicular.

2) Cross sections are undistorted in their individual planes
   - Shearing strain $\gamma$ varies linearly from the center to a maximum at the outer surface

3) Material is homogenous

![Diagram of torsion with assumptions and equations]

\[ T = \int \tau \cdot \rho \, dA \]

\[ T = \int_0^r \frac{\rho}{r} \cdot \tau_{\text{max}} \cdot \rho \, dA = \frac{\tau_{\text{max}}}{r} \int \rho^2 \, dA \]

Where \( \int \rho^2 \, dA \) is polar moment of inertia

\[ J = \int \rho^2 \, dA \quad \text{and} \quad \tau_{\text{max}} = \frac{T \cdot r}{J} \]

\[ \tau(\rho) = \frac{T \cdot \rho}{J} \]

Note: Polar moment of inertia for circle of radius \( r \):

\[ J = \frac{\pi r^4}{2} \]

**Angle of Twist \( \Phi \):**

\[ \Phi \cdot r = \gamma_{\text{max}} \cdot L \]

\[ \gamma_{\text{max}} = \frac{\tau_{\text{max}}}{G} \quad \text{(Hooke’s law)} \]

\[ \gamma_{\text{max}} = \frac{T \cdot r}{J \cdot G} \quad \text{so,} \quad \gamma_{\text{max}} = \frac{T \cdot r}{J \cdot G} \]

\[ \Phi \cdot r = \frac{T \cdot L}{J \cdot G} \quad \text{J.G : Torsional rigidity} \]
2) theory of elasticity approach to torsion of bar

\[ u = -(r\theta z)\sin\alpha = -y\theta z \]
\[ v = (r\theta z)\cos\alpha = x\theta z \]

where the angular displacement of AP at a distance \( Z \) from the left end is \( \theta z \)

\[ \varepsilon_x = \gamma_{xy} = \varepsilon_y = \varepsilon_z = 0 \]
\[ \gamma_{yz} = \frac{\partial w}{\partial x} = y\theta, \quad \gamma_{zy} = \frac{\partial w}{\partial y} = x\theta \]
\[ \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \]
\[ \tau_{zx} = G\left(\frac{\partial w}{\partial x} - y\theta\right) \quad \ldots \ldots 1 \]
\[ \tau_{zy} = G\left(\frac{\partial w}{\partial y} + x\theta\right) \quad \ldots \ldots 2 \]

assuming negligible body forces

\[ \frac{\partial \tau_{zx}}{\partial z} = 0, \quad \frac{\partial \tau_{zy}}{\partial z} = 0, \quad \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \]

Differentiating the equation 1 and 2 w.r.t. \( y \) and \( x \) respectively, we have

\[ \frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x} = H \]

Where

\[ H = -2G\theta \]
**Stress function**

As in the case of a beam the torsion problem formulated in the preceding is commonly solved by introducing a single stress function. If a function \( \phi(x,y) \), the Prandtl stress function is assumed to exist, such that

\[
\begin{align*}
\tau_{zx} &= \frac{\partial \phi}{\partial y}, \\
\tau_{zy} &= -\frac{\partial \phi}{\partial x}
\end{align*}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = H
\]

**Boundary conditions**

The cosine of the angle between \( z \) and a unit normal \( n \) to the surface is zero that is \( \cos(n,z) = 0 \) we have

\[
\tau_{zx} l + \tau_{zy} m = 0
\]

\( \tau \) must be tangent to the surface

\[
l = \cos(n,x) = \frac{dy}{ds}, \quad m = \cos(n,y) = -\frac{dx}{ds}
\]

so on the boundary we have

\[
\frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} = \frac{\partial \phi}{\partial s} = 0 \quad \text{(on the boundary)}
\]
References

Lecture notes

Ugural - Frenster