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TORSION

Prandt's Stress Function

$\tau_{zx} = \frac{\partial \emptyset}{\partial y}$ $\quad$ $\tau_{zy} = \frac{\partial \emptyset}{\partial x}$ $\quad$ and $\quad \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = -2G\theta$

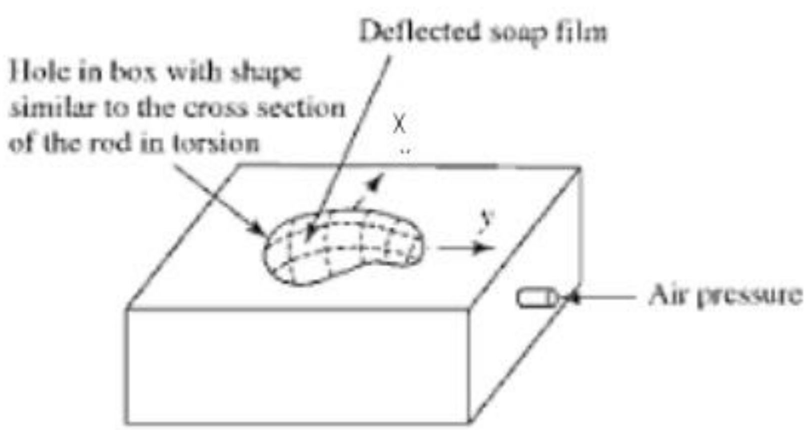
Bcs $\quad \emptyset = 0 \quad$ on the boundaries

$$T = 2 \iint \emptyset dxdy = 2(Area\ under\ \emptyset\ curve)$$

$\tau_{max} = \frac{Tr}{J}$ $\quad$ twist ➔ $\emptyset = \theta L = \frac{TL}{JG}$

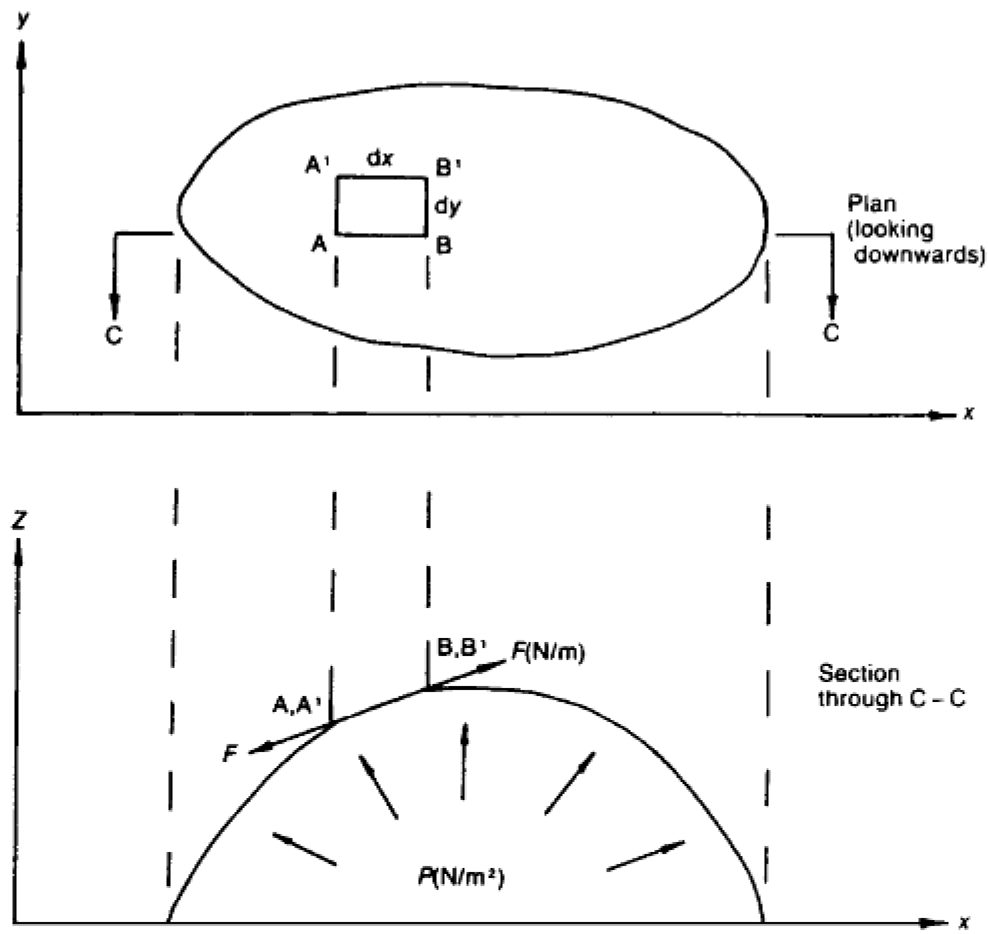
PRANDTL'S MEMBRANE ANALOGY

Consider an edge-supported homogeneous membrane, given its boudnary contour by a hole cut in a plate.The shape of the hole is the same as that of the twisted bar to be studied; the sizes need not be identical.



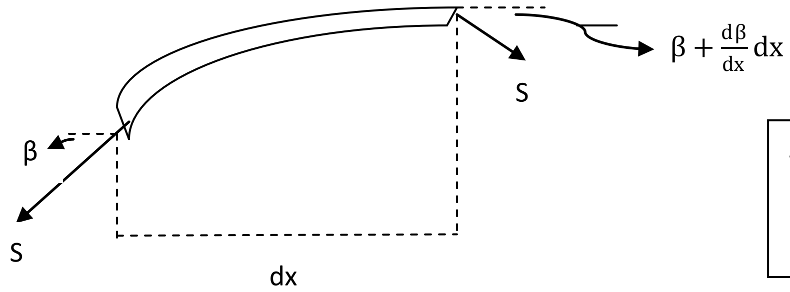
$\emptyset$ satisfies the same equation from describes the deflection of a membrane (or soap film) subjected to a pressure

$(\emptyset ==> z\text{: the height of the soap film})$



Equation for the deflection z; consider equilibrium of an infinitesmall element ABCD;

Let the tensile forces per unit membrane length to be denoted by S=const



Small deflection change

B=slope=dz/dx

$$\sum Fz = 0$$

$$(-Sdx)\frac{dz}{dx} + (Sdy)(\frac{dz}{dx} + \frac{d^2z}{dx^2}dx) - (Sdx)\frac{dz}{dy} + Sdx(\frac{dz}{dy} + \frac{d^2z}{dy^2}dy) + pdxdy = 0$$

Leading to

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = \frac{-p}{S} \; [1]$$

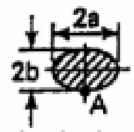
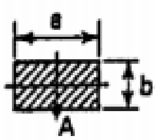
This is again Poisson's equation. Upon comparison of poisson's equation (Ugural 6.9), and (Ugural 6.8) The quantities shown in following table are aboserved to be analogous.The Membrane subject to the conditions outlined, thus represent the Ø surface.In view of the derivation , the restirction with regard to smallnes of slope must be borne in mind.

Analogous qualities between torsion and membrane problems.

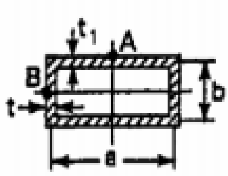
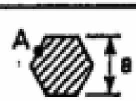
MEMBRANE	**TORSION**
z	Ø
$\frac{1}{S}$	G
p	2θ
$\frac{-dz}{dx}, \frac{dz}{dy}$	τ_{zy} ,τ_{zx}
2.(Volume beneath membrane)	T

The membrane analogy providesmore than a useful experimeantal technique.

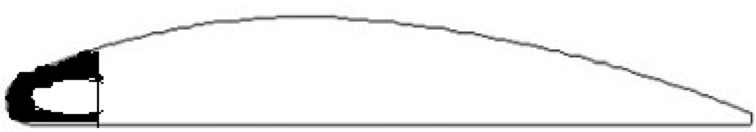
For reference purposes following table presents the shear and angle of twist for a number of commonly encountered shapes.Note that the values of coefficients α and β depend on the ratio of the length of the long side or depth a to the width b of the short side of a rectangular section.For thin sections, where a is much greater than b, their values approach 1/3.

Cross section	*Maximum shearing stress*	*Angle of twist per unit length*
 For circular bar: $a = b$	$\tau_A = \frac{2T}{\pi a b^2}$	$\theta = \frac{(a^2 + b^2)T}{\pi a^3 b^3 G}$
 Equilateral triangle	$\tau_A = \frac{20T}{a^3}$	$\theta = \frac{46.2T}{a^4 G}$
	$\tau_A = \frac{T}{\alpha a b^2}$	$\theta = \frac{T}{\beta a b^3 G}$

a/b	β	α
1.0	0.141	0.208
1.5	0.196	0.231
2.0	0.229	0.246
2.5	0.249	0.256
3.0	0.263	0.267
4.0	0.281	0.282
5.0	0.291	0.292
10.0	0.312	0.312
∞	0.333	0.333

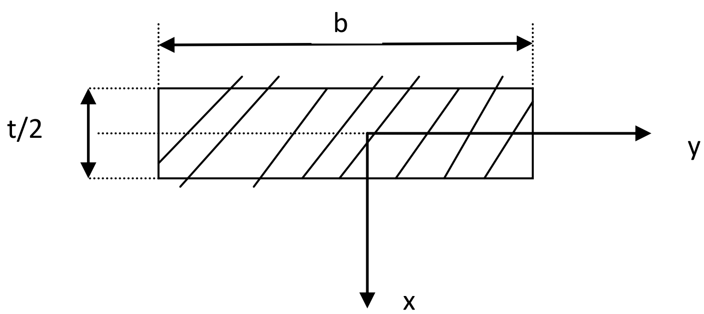
Cross section	*Maximum shearing stress*	*Angle of twist per unit length*
	$\tau_A = \frac{T}{2abt_1}$ $\tau_B = \frac{T}{2abt}$	$\theta = \frac{(at + bt_1)T}{2tt_1 a^2 b^2 G}$
 For circular tube: $a = b$	$\tau_A = \frac{T}{2\pi abt}$	$\theta = \frac{\sqrt{2(a^2 + b^2)}T}{4\pi a^2 b^2 tG}$
 Hexagon	$\tau_A = \frac{5.7T}{a^3}$	$\theta = \frac{8.8T}{a^4 G}$

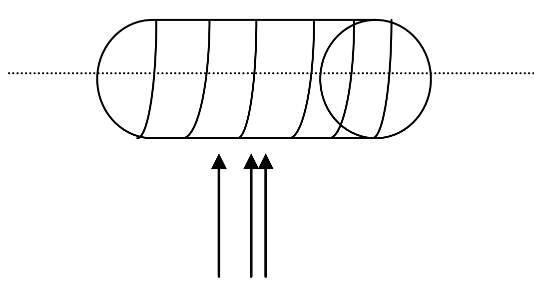
Augusta helicopter refer blade


(Eurocopter)

TORSION OF THIN WALLED MEMBERS WITH OPEN CROSS SECTIONS

In applying the analogy to a bar of narrow rectangular cross section, it is usual to assume a constan cylindrical membrane shape over the entire dimension b (see following figure).





Subject to this compression; $\frac{dz}{dy} = 0$ and eqn [1] reduces to $\frac{d^2z}{dx^2} = \frac{-p}{S}$

Integrate first

$$\frac{dz}{dx} = \frac{-p}{S}x + C_1$$

Integrate again; $z = -\frac{px^2}{25} + C_1x + C_2$

Known that BC's; x=0; $\frac{dz}{dx} = 0$

x=t/2 z=0

Using these BC's , the result of integral is

$$z = \frac{1}{2}\frac{p}{S}\left[\left(\frac{t}{2}\right)^2 - x^2\right]$$

The volume under the curve z(x,y) is

$V = \iint z dx dy = \frac{pbt^3}{12S}$

P can replaced by 2θ and 1/S by G

Consequently; $T = 2V = \frac{1}{3}bt^3G\theta$

T is torsional rigidity for a thin rectangular section is therefore;

T=JGθ $, J_c = \frac{bt^3}{3}$ $C = \frac{T}{\theta} = \frac{1}{3}bt^3G = J_cG$

J_c =effective polar moment of inertia of the section.The analogy also requires that

$$\tau_{zy} = \frac{-dz}{dx} = 2G\theta x[2]$$

Twist angle per unit length $\theta = \frac{3T}{bt^3G}$

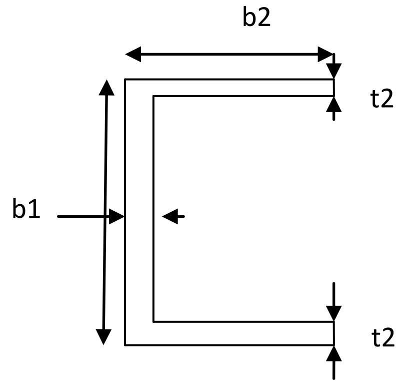
Maximum shear stress occurs at +/- t/2

$\tau_{max} = \tau \Big|_{x=+-t/2} = G\theta t = \frac{3T}{bt^2}$

Finally;

$\tau = \frac{1}{3}bt^2\tau_{max}$

According to eqn [2], the shearing stress is linear in x,



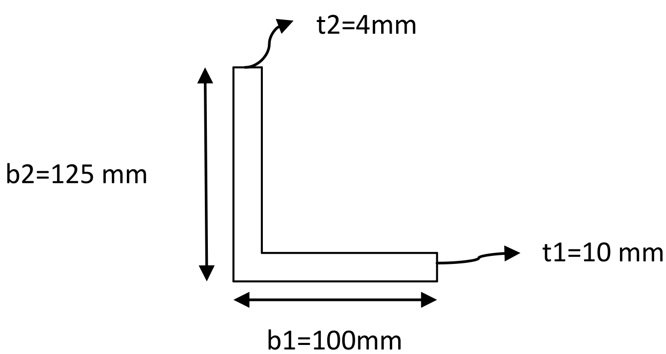
Calculate the polar moment of inertia

$$J_c = \sum \frac{bt^3}{3} = \frac{b_1 t_1{}^3}{3} + \frac{b_2 t_2{}^3}{3} * 2$$

$$\Theta = \frac{T}{J_c G} = \frac{3T}{G} = \frac{1}{b_1 t_1{}^3 + 2 b_2 t_2{}^3} \qquad \text{t1=is the larger (t1,t2)}$$

<u>Example problem 6.16 (ugural)</u>

A steel bar (G=200 GPa), of cross section as shown in the following figure is subjected to a torque 500 N.m . Determine the maximum shearing stress and the angle of twist per unit length. The dimensions are b_1=125 mm,t_1=10 mm and t_2=4 mm.



$$J_c = \sum \frac{bt^3}{3} = \frac{1}{3} 100 * 10^3 + \frac{1}{3} 115 * 4^3 = 3{,}5787 * 10^{-8} m^4$$

Maximum shear stress at the lower leg

$\tau_{max}=\frac{Tt1}{J_c}=\frac{500*0,01}{3,5787*10^{-8}}=139,7\ MPa$

Also known that

$\Theta=\frac{T}{J_c G}=69,86*10^{-3}\ \frac{rad}{m}=40 degree/meter$

TORSION OF CLOSED THIN WALLED TUBES

Consider the tube shown the following figure. And an element is isolated is shown the second one where the possibility of varying thickness and shear stress along the perimeter of the tube has been assumed

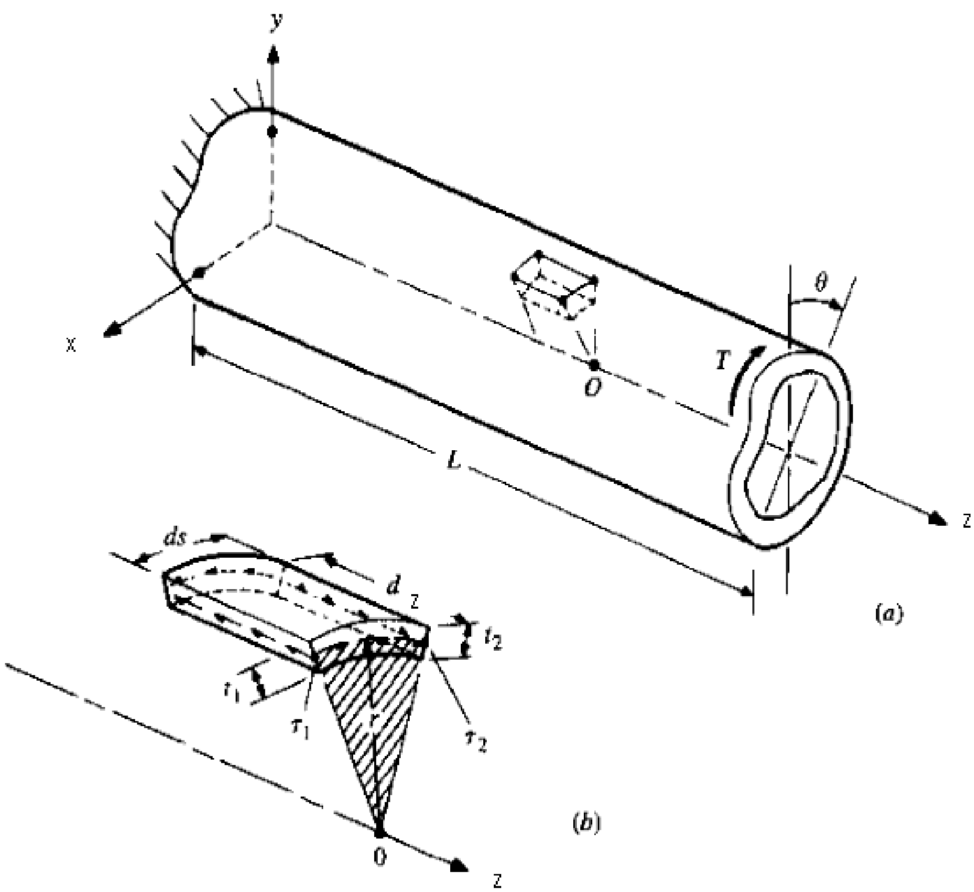
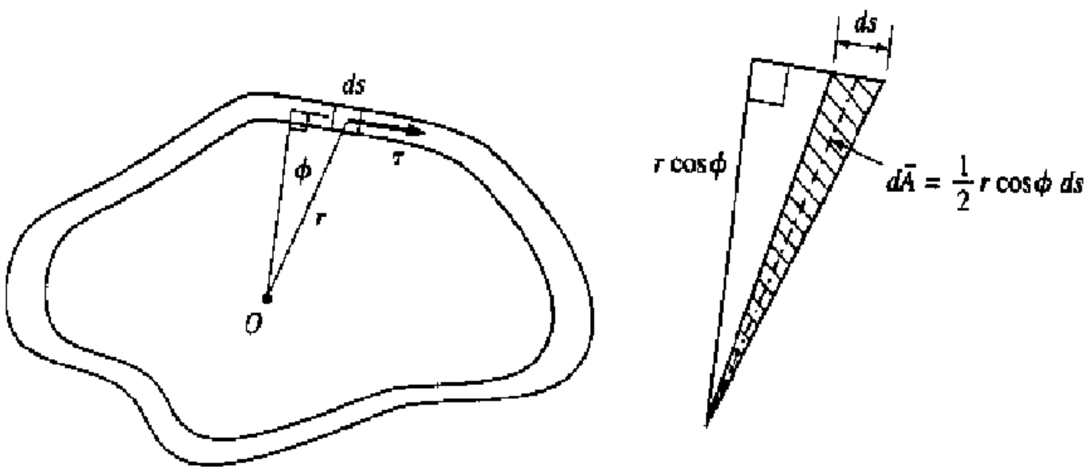


Figure 5.2-1 Closed, single cell tube in torsion.

$$\sum F_Z = 0$$

$$-\tau_1 t_1 dz + \tau_2 t_2 dz = 0 \quad ==> \ \tau_1 t_1 = \tau_2 t_2 = \tau t = q$$

Shear flow is a constant(force/length dz)



The net force on ds element

$$\tau(tds) = qds$$

q=constant

and the torque about O;

$(r\cos\emptyset) q ds$

The total torque is $T=\int_0^{2\pi}(r\cos\emptyset)q\,ds = q\int_0^{2\pi}\underbrace{r\cos\emptyset\, ds}_{2d\bar{A}}$

As a result of this $T=2Q\bar{A}$ ➔ $\tau = \frac{T}{2At}$

RECALL:

Angle of twist Θ is related to the shear strain

$\Theta=\frac{\gamma L}{r\cos\emptyset}$ ➔ $\gamma=\frac{\tau}{G}=\frac{q}{tG}$ $\longrightarrow$ $\Theta ds=\frac{q}{Gt}\frac{L}{r\cos\emptyset}$

Assume any angle of twist of each ds section is the angle of twist of the whole sections.

Averaging Θds wrt $\bar{A}$ gives

Total diameter

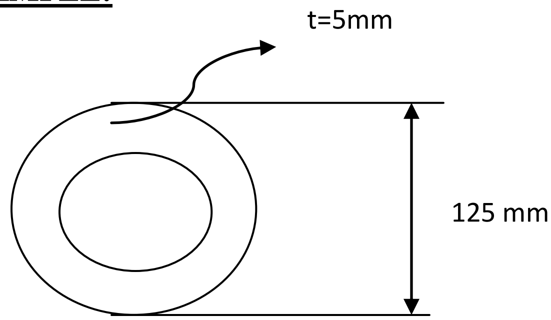
$$\Theta=\frac{1}{A}\int \Theta ds\, d\bar{A} = \frac{1}{2G}\frac{qL}{\bar{A}}\int_0^{\bar{s}}\frac{ds}{E}$$

As in terms of T

$$\Theta=\frac{TL}{G\bar{A}^2}\int_0^{\bar{s}}\frac{ds}{t}$$

$\Theta=\frac{TL}{GJ_e}$ $\quad J_e=\frac{4\bar{A}^2}{S\frac{ds}{t}}$

EXAMPLE:



GIVEN:

L=0,5m

T=1 kN.m

E=200 GPa

V=0,29

FIND: Shear stress of total angle of twist

Mean radius $r=\frac{125-5}{2}=60$ mm

$\bar{A}=\pi r^2 = 1,131*10^{-2}$

$q=t.\tau=\frac{T}{2A} \Rightarrow \tau = 8,84$ Mpa

Known that;

$$\Theta=\frac{TL}{G\bar{A}^2}\int_0^s \frac{ds}{t}=\frac{TL2\pi r}{G\bar{A}^2 t} = 9,5 * 10^{-4}$$

$$J=2\pi t r^3=\frac{\pi}{2}\,(r_0^4-r_i^4)=\frac{\pi}{2}\underbrace{(r_o-r_i)}_{\boxed{t}}\underbrace{(r_0+r_i)}_{\boxed{2r}}\underbrace{(r_o^2+r_i^2)}_{\boxed{2r^2}}$$

The result is 6,79*10^-3

REFERENCES

http://www.ecourses.ou.edu/cgi-bin/ebook.cgi?doc=&topic=me&chap_sec=02.3&page=theory

Budynas,R.G. (1999). *Advanced Strength and applied stress analysis* . Tsingua University press, Prentice hall.

Ugural and Fenster, Advanced Strength and Applied Elasticity, 4th Ed., Prentice Hall, PTR ,2003