04.01.10 Taner YILDIZ
Coşkun TUNCA

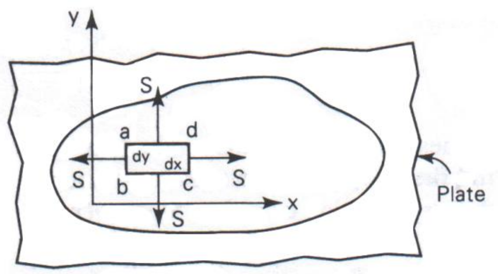
AE 361 APPLIED ELASTICITY
LECTURE NOTES

Prandtl Stress Function

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$ …………(*)

$T = 2 \iint \phi \, dx \, dy$

6.5 Prandtl's Membrane Analogy

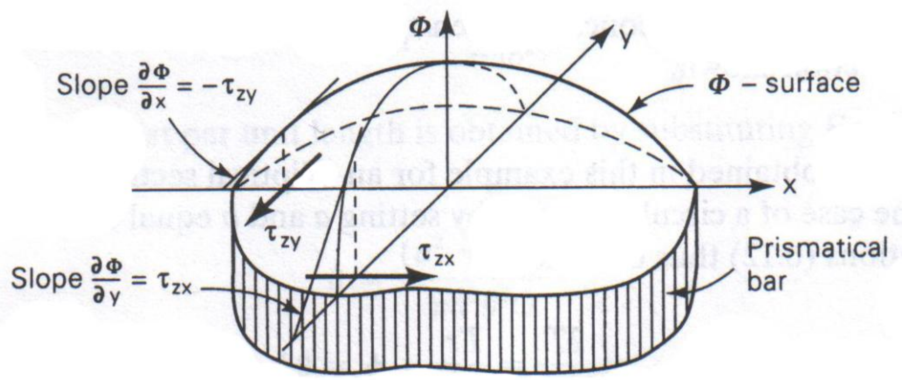


$\tau_{zx} = \frac{\partial \phi}{\partial y} \quad \tau_{zy} = -\frac{\partial \phi}{\partial x}$

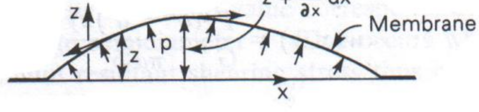
Equation (*) is the same form as the equation describing the deflection of a membrane (or soap film) subject to pressure.

$(\Phi = z = height\ of\ soap\ film)$

Derivation: Equation for the deflection z



Let the tensile force per unit membrane length be denoted by s = constant.



$\beta + \frac{\partial \beta}{\partial x} dx$

$z + \frac{\partial z}{\partial x} dx$

$\sum F_z = 0$

$$(-\int dy)\sin\beta + (\int dy)\left(\beta + \frac{\partial\beta}{\partial x}dx\right) - (\int dx)\alpha + (\int dx)\left(\alpha + \frac{\partial\alpha}{\partial y}dy\right) + P\,dx\,dy = 0 \quad ,\sin\beta \approx \beta$$

$$\int dy\ \frac{\partial^2 z}{\partial x^2}dx + \int dx\ \frac{\partial^2 z}{\partial y^2}dy + P\,dx\,dy = 0$$

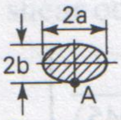
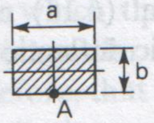
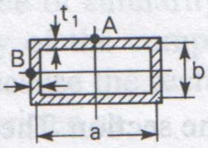
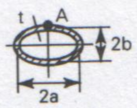
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{P}{s}$$

Comparison

TABLE 6.1

Membrane Problem	*Torsion Problem*
z	Φ
$\frac{1}{S}$	G
p	2θ
$-\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$	τ_{zy}, τ_{zx}
2 · (volume beneath membrane)	T

TABLE 6.2

Cross section	Maximum shearing stress	Angle of twist per unit length
 For circular bar: $a = b$	$\tau_A = \dfrac{2T}{\pi a b^2}$	$\theta = \dfrac{(a^2 + b^2)T}{\pi a^3 b^3 G}$
 Equilateral triangle	$\tau_A = \dfrac{20T}{a^3}$	$\theta = \dfrac{46.2T}{a^4 G}$
	$\tau_A = \dfrac{T}{\alpha a b^2}$	$\theta = \dfrac{T}{\beta a b^3 G}$
	$\tau_A = \dfrac{T}{2abt_1}$ $\tau_B = \dfrac{T}{2abt}$	$\theta = \dfrac{(at + bt_1)T}{2tt_1 a^2 b^2 G}$
 For circular tube: $a = b$	$\tau_A = \dfrac{T}{2\pi abt}$	$\theta = \dfrac{\sqrt{2(a^2 + b^2)}T}{4\pi a^2 b^2 t G}$
 Hexagon	$\tau_A = \dfrac{5.7T}{a^3}$	$\theta = \dfrac{8.8T}{a^4 G}$

a/b	β	α
1.0	0.141	0.208
1.5	0.196	0.231
2.0	0.229	0.246
2.5	0.249	0.256
3.0	0.263	0.267
4.0	0.281	0.282
5.0	0.291	0.292
10.0	0.312	0.312
∞	0.333	0.333

TORSION OF THIN WALLED MEMBERS with OPEN CROSS SECTION

$\frac{\partial z}{\partial y} = 0$

$\frac{\partial^2 z}{\partial x^2} = -\frac{P}{s}$

$z = -\frac{P}{2s}x^2 + C_1 x + C_2$

Boundary conditions:

at $x = 0$ $\frac{\partial z}{\partial y} = 0$ $z = \frac{1}{2}\frac{P}{s}\left[\left(\frac{t}{2}\right)^2 - x^2\right]$

at $x = \pm\frac{t}{2}$ $z = 0$

Volume of the parabolic cylindircal membrane is

V=$\iint_{-b/2}^{b/2} 2dxdy$=$\frac{Pbt^3}{12S}$

Define an effective polar moment of inertia

J_e=$\frac{1}{3}$bt^3 for rectangular cross section

T=2V=$\frac{1}{3}$bt^3Gθ= J_e λmax Gθ

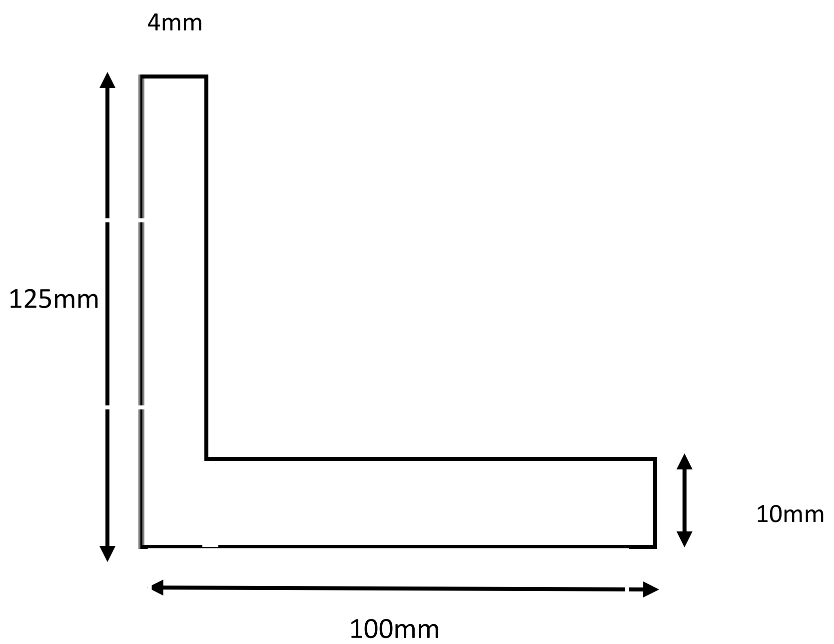
SHEAR STRESS

λzy=- $\frac{dz}{dx}$=2Gθx

Angle of Twist per unit length λmax=Gθt θ=$\frac{T}{J_e \mathrm{G}}$

Then T=$\frac{J_e\, \lambda\mathrm{max}}{t}$

Example



A stell bar G=200GPa of croos section shown is subject to a torque of 500 Nm

- Determine max shear stress?

Solution

$J_e = J_{e1} + J_{e2} = 1/3 b_1 t_1^3 + 1/3 b_2 t_2^3$

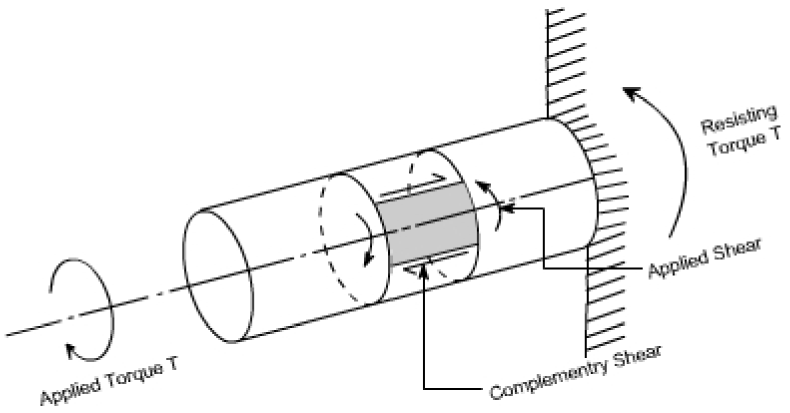
$= 1/3*(100)*(10)^3 + 1/3(125)*(4)^3$

$= 3,5787*10^4 \text{ mm}^4$

$\lambda max = \frac{Tt}{Je}$ where $t = t_1$

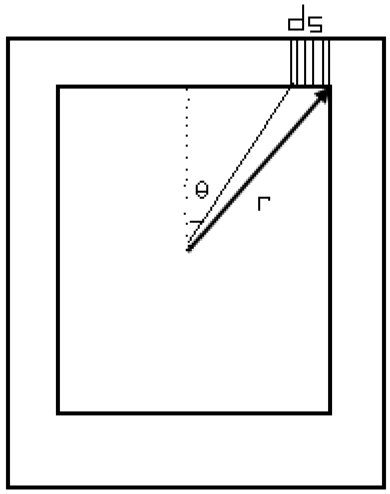
$= 500*(0,01)/3,578*10^{4=}$ 139,7 MPa

$\theta = \frac{T}{J_e G} = 69,86*10^{-3} rad$



z For this rectangular section force equation equals ,

$F_z=0$ $\lambda_1 dz t_1 + \lambda_2 dz t_2 = 0$ $\lambda_1 t_1 = \lambda_2 t_2 = q$=shear flow and it is constant



Area=$\frac{dSr\cos\theta}{2}$

the net force on the element is

$\lambda(ds.t)=qds$

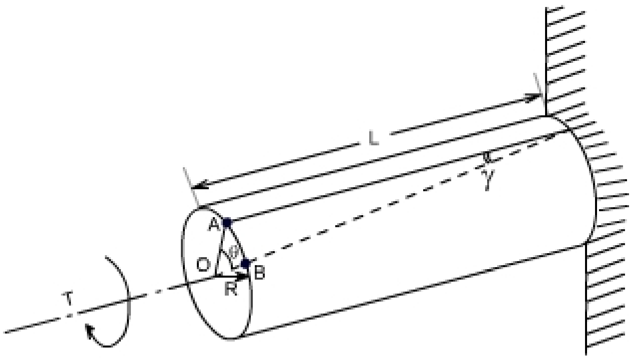
the total torque T_t 0 to $(r\cos\theta)qds$

$T=q\int_0^s (r\cos\ \theta)ds$

2dA

$T=2qA$ & $\lambda=\frac{T}{2At}$

Angle of Twist



γL= θr for ds element

γL = θdsrcosØ

$\frac{\lambda}{G}=\frac{qlt}{G}$

Θds=G$\frac{q}{t}\frac{L}{rcosØ}$

Angle of Twist=θ=$\frac{1}{A}\int \theta dsdA$

Θ=$\frac{1}{2G}\frac{ql}{A}\int\frac{ds}{t}$

Θ=$\frac{TL}{G4A^2}\int_0^s\frac{ds}{t}$

Θ=$\frac{TL}{GJ}$ $J_e=\frac{4A^2}{\int\frac{ds}{t}}$