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THICK – WALLED CYLINDERS

Axisymmetrically Loaded Members:

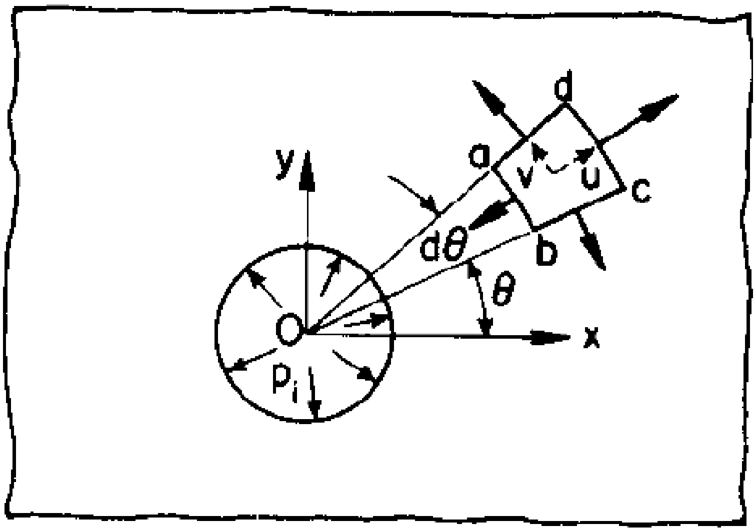

Figure 1

- P_i = initial pressure
- Θ= angular position coordinate
- r = radial position coordinate
- u = displacement in r- direction
- v = displacement in Θ - direction

Figure 1 shows that a large thin plate having a small circular hole subjected to uniform pressure. Due to no axial loading, σ_z is equal to zero. The stresses are clearly symmetrical about the z axis, and the deformations likewise display Θ independence.

Axisymmetric ≡ Nothing varies in the Θ - direction. Thus $\frac{\partial}{\partial\theta} = 0$.

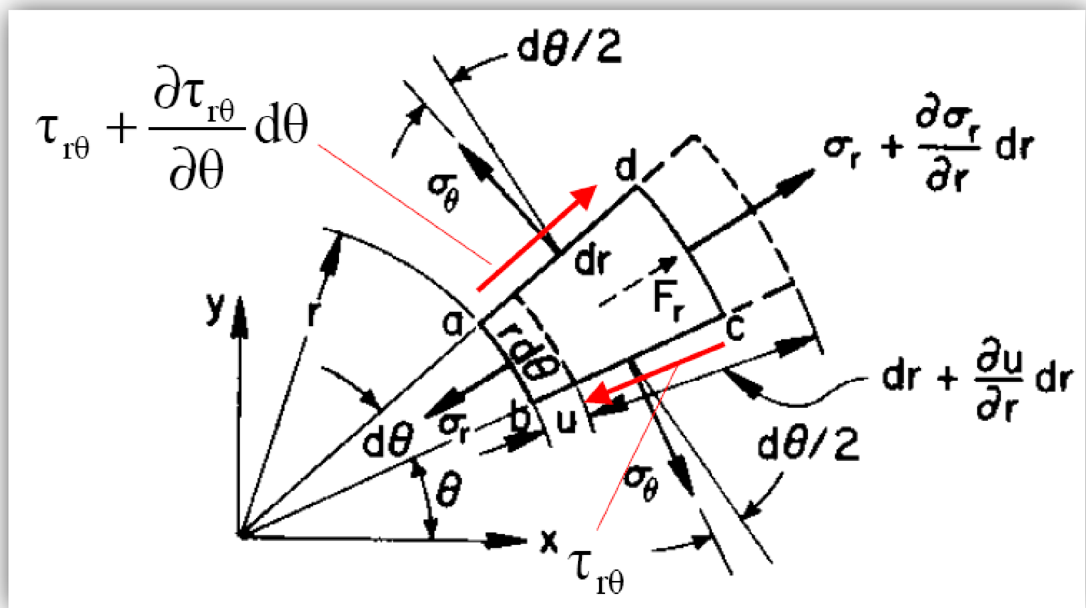

Figure 2

In the symmetrical field, there is no tangential displacement (v = 0). As you can see in the figure2, the element abcd moves radially as a consequence of loading, but not tangentially.

$$\frac{\partial \tau_{r\theta}}{\partial \theta} = 0 \; due\ to\ axisymmetric\ constraint$$

$$\tau_{r\theta} = 0 \; due\ to\ stress\ compatibility$$

Axisymmetric equation of equilibrium:

The polar equations of equilibrium:

$$\frac{d\sigma_r}{dr} + \frac{1}{r}\frac{d\tau_{r\theta}}{d\theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

$$\frac{1}{r}\frac{d\sigma_\theta}{d\theta} + \frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0$$

The polar equations of equilibrium reduce to:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \qquad (1)$$

Strain displacement equation:

The polar equations :

$$\varepsilon_r = \frac{du}{dr}$$

$$\varepsilon_\theta = \frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{dv}{dr} + \frac{1}{r}\frac{du}{d\theta} - \frac{v}{r}\right)$$

The polar equations reduce to:

$$\varepsilon_r = \frac{du}{dr} \qquad (2)$$

$$\varepsilon_\theta = \frac{u}{r} \qquad (3)$$

$$\gamma_{r\theta} = 0 \qquad (4)$$

Hooke's law:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \vartheta\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \vartheta\sigma_r)$$

$$\varepsilon_{r\theta} = \frac{\tau_{r\theta}}{2G} = 0$$

$\rightarrow$

$$\sigma_r = \frac{E}{1-\vartheta^2}(\varepsilon_r + \vartheta\varepsilon_\theta) \quad (5)$$

$$\sigma_\theta = \frac{E}{1-\vartheta^2}(\varepsilon_\theta + \vartheta\varepsilon_r) \quad (6)$$

Thick walled cylinders:

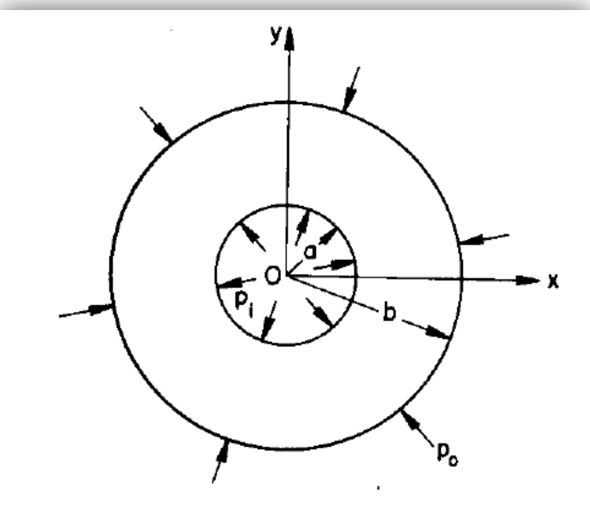

Figure 3

a = inside radius

b = outside radius

p_i = internal pressure

p_o = external pressure

Thick walled cylinder subject to uniform internal and external pressure, and the deformation is symmetrical about the z axis. Thus , the equilibrium and strain displacement equations which are seen above (eqn 1-2-3-4), apply to any point on a ring of unit length cut from the cylinder, as seen figure3.

According to Hooke's law;

Substituting eqn2-3 into Hooke's law

$$\sigma_r = \frac{E}{1-\vartheta^2}(\varepsilon_r + \vartheta\varepsilon_\theta) = \frac{E}{1-\vartheta^2}\left(\frac{du}{dr} + \vartheta\frac{u}{r}\right) \quad (7)$$

$$\sigma_\theta = \frac{E}{1-\vartheta^2}(\varepsilon_\theta + \vartheta\varepsilon_r) = \frac{E}{1-\vartheta^2}\left(\frac{u}{r} + \vartheta\frac{du}{dr}\right) \quad (8)$$

The equilibrium equation: $\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$

Substituting the above equations into equilibrium equation results in the following equildimensional equation in radial displacement:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \quad (9)$$

The solution of this equation: $k = lnr$, $\frac{dk}{dr} = \frac{1}{r}$

$$\frac{du}{dr} = \frac{du}{dk}\frac{dk}{dr} = \frac{du}{dk}\frac{1}{r}$$

$$\frac{d^2u}{dr^2} = \frac{d^2u}{drdk}\frac{1}{r} - \frac{1}{r^2}\frac{du}{dk} = \frac{1}{r^2}\left(\frac{d^2u}{dk^2} - \frac{du}{dk}\right)$$

Substituting these into (9)

$$\frac{d^2u}{dk^2} - u = 0 \ , \quad characteristic\ eqn\colon\ c^2 - 1 = 0\ \rightarrow c_{1,2} = \mp 1$$

$$u = c_1 e^k + c_2 e^{-k}$$

General solution is $u = c_1 r + c_2 \frac{1}{r}$ (10)

The radial and tangential stresses are written in terms of c_1 and c_2 by combining eqn (7-8) and (10):

$$\sigma_r = \frac{E}{1-\vartheta^2}\left(\frac{du}{dr} + \vartheta\frac{u}{r}\right) = \frac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) - c_2\left(\frac{1-\vartheta}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\vartheta^2}\left(\frac{u}{r} + \vartheta\frac{du}{dr}\right) = \frac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) + c_2\left(\frac{1-\vartheta}{r^2}\right)\right]$$

BC's:

$\sigma_{r,at\ r=a} = -p_i$, $\sigma_{r,at\ r=b} = -p_0$ The negative sign determines compressive stress.

$$-p_i = \frac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) - c_2\left(\frac{1-\vartheta}{a^2}\right)\right]$$

$$-p_o = \frac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) - c_2\left(\frac{1-\vartheta}{b^2}\right)\right]$$

Thus,

$$c_1 = \frac{1-\vartheta}{E}\frac{a^2p_i - b^2p_0}{b^2-a^2} \qquad c_2 = \frac{1+\vartheta}{E}\frac{a^2b^2(p_i-p_0)}{b^2-a^2}$$

Finally,

Lame' equations:

$$\begin{cases} \sigma_r = \dfrac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) - c_2\left(\dfrac{1-\vartheta}{r^2}\right)\right] = \dfrac{a^2p_i - b^2p_0}{b^2-a^2} - \dfrac{a^2b^2(p_i-p_0)}{(b^2-a^2)r^2} \\ \sigma_\theta = \dfrac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) + c_2\left(\dfrac{1-\vartheta}{r^2}\right)\right] = \dfrac{a^2p_i - b^2p_0}{b^2-a^2} + \dfrac{a^2b^2(p_i-p_0)}{(b^2-a^2)r^2} \\ u = \dfrac{1-\vartheta}{E}\dfrac{(a^2p_i - b^2p_0)r}{b^2-a^2} + \dfrac{1+\vartheta}{E}\dfrac{a^2b^2(p_i-p_0)}{(b^2-a^2)r} \end{cases}$$

- If $p_i > p_o$, the maximum value of σ_r occurs at r=a and is equal to p_i. However if $p_o > p_i$, the maximum σ_r occurs at r=b and equals p_o.
- The maximum σ_Θ occurs at either the inner or outer edge according to pressure ratio.

- The maximum shearing stress at any point is $\tau_{max} = \frac{\sigma_r - \sigma_\theta}{2} = \frac{a^2b^2(p_i-p_0)}{(b^2-a^2)r^2}$

The largest τ_{max} corresponds to r=a and p_o=0.

$$\tau_{max} = \frac{b^2p_i}{(b^2-a^2)}$$

- *<u>Longitudinal strain:</u>*

Assume the ends of cylinder are open and unconstrained. (σ_z=0)

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \vartheta\sigma_r - \vartheta\sigma_\theta) = -\frac{\vartheta}{E}(\sigma_r + \sigma_\theta)$$

$$\left.\begin{aligned} \sigma_r &= \frac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) - c_2\left(\frac{1-\vartheta}{r^2}\right)\right] \\ \sigma_\theta &= \frac{E}{1-\vartheta^2}\left[c_1(1+\vartheta) + c_2\left(\frac{1-\vartheta}{r^2}\right)\right] \end{aligned}\right\} \underbrace{\sigma_r + \sigma_\theta}_{constant} = \frac{2E}{1-\vartheta^2}[c_1(1+\vartheta)]$$

$$\varepsilon_z = -\frac{\vartheta}{E}\frac{2E}{1-\vartheta^2}[c_1(1+\vartheta)] = -\frac{2\vartheta c_1}{1-\vartheta}$$

$$\varepsilon_z = -\frac{2\vartheta}{E}\frac{a^2p_i - b^2p_0}{b^2-a^2} = constant$$

➢ *<u>Longitudinal stress:</u>*

Assume the ends of cylinder are constrained. (ε_z=0)

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \vartheta\sigma_r - \vartheta\sigma_\theta) = 0$$

$$\sigma_z = \vartheta(\sigma_r + \sigma_\theta)$$

$$\underbrace{\sigma_r + \sigma_\theta}_{constant} = \frac{2E}{1-\vartheta^2}[c_1(1+\vartheta)]$$

$$\sigma_z = \vartheta\left(\frac{2Ec_1}{1-\vartheta}\right) = 2\vartheta\ \frac{a^2p_i - b^2p_0}{b^2 - a^2}$$

$$\sigma_z = constant$$

Assume the ends of cylinder are closed and unconstrained.

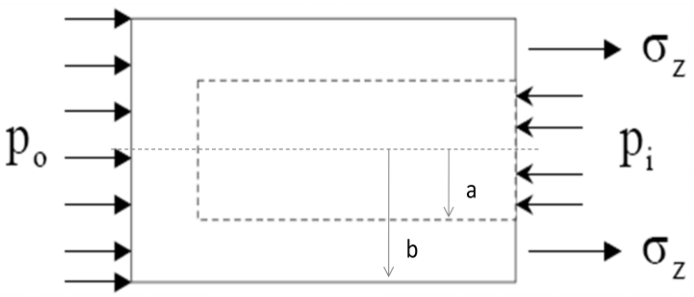

Figure 4

$$\rightarrow \Sigma F = 0:\ \ \sigma_z\pi(b^2 - a^2) + p_o\pi b^2 - p_i\pi a^2 = 0$$

$$\sigma_z = \frac{a^2p_i - b^2p_0}{b^2 - a^2}$$

<u>*Special cases:*</u>

Internal pressure only: If only p_i exists, p_o= 0.

$$\sigma_r = \frac{a^2p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{a^2p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$u = \frac{1-\vartheta}{E}\ \frac{(a^2p_i)r}{b^2 - a^2} + \frac{1+\vartheta}{E}\ \frac{a^2b^2(p_i)}{(b^2 - a^2)r}$$

If $\frac{b^2}{r^2} \geq 1$, σ_r is compressive for all r. At r = b, σ_r= 0. The maximum σ_r at r=a.

On the other hand σ_θ is tensile for all r, and also maximum σ_θ occurs at r=a.

External pressure only: If only p_o exists, p_i=0.

$$\sigma_r = -\frac{b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\frac{b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

$$u = \frac{1-\vartheta}{E}\frac{(-b^2 p_0)r}{b^2 - a^2} + \frac{1+\vartheta}{E}\frac{a^2 b^2(-p_0)}{(b^2 - a^2)r}$$

The maximum σ_r is found at r= b, and compressive for all r. On the other hand σ_θ is compressive for all r, and also maximum σ_θ occurs at r=a.

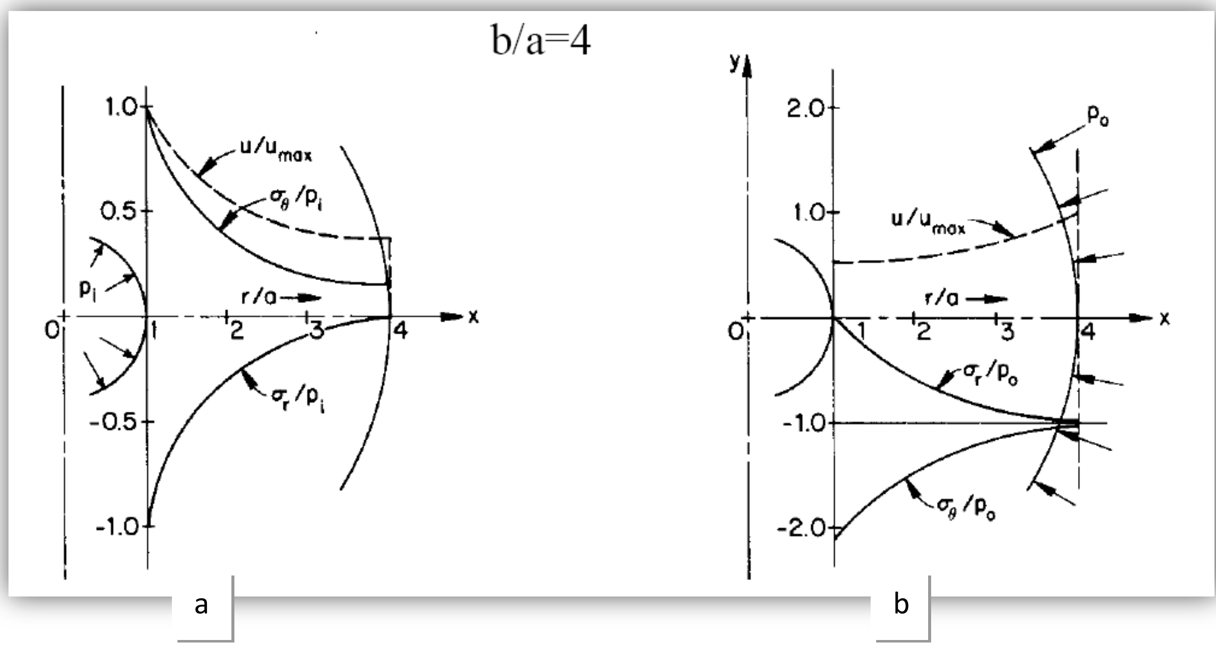

Figure 5: distribution of stress and displacement in a thick walled with b/a=4: (a) under internal pressure; (b) under external pressure.

Reference:

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