SOLUTION (3.15)

(a) For $\nabla^4\Phi = 0$, $e = -5d$ and a, b, c are arbitrary.

Thus

$$\Phi = ax^2 + bx^2y + cy^3 + d(y^5 - 5x^2y^3) \tag{1}$$

(b) The stresses:

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = 6cy + 10d(2y^3 - 3x^2y) \tag{2}$$

$$\sigma_y = \frac{\partial^2\Phi}{\partial x^2} = 2a + 2by - 10dy^3 \tag{3}$$

$$\tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -2bx - 30dxy^2 \tag{4}$$

Boundary conditions:

$$\sigma_y = -p \qquad \tau_{xy} = 0 \qquad \text{(at y=h)} \tag{5}$$

Equations (3), (4), and (5) give

$$b = -15dh^2 \qquad 2a - 40dh^3 = -p \tag{6}$$

$$\int_{-h}^{h}\sigma_x dy = 0 \qquad \int_{-h}^{h} y\sigma_x dy = 0 \qquad \int_{-h}^{h}\tau_{xy} dy = 0 \qquad \text{(at x=0)} \tag{7}$$

Equations (2), (4), and (7) yield

$$c = -2dh^2 \tag{8}$$

Similarly

$$\sigma_y = 0 \qquad \tau_{xy} = 0 \qquad \text{(at y=-h)}$$

give $$a = 20dh^3 \tag{9}$$

Solution of Eqs. (6), (8), and (9) results in

$$a = -\frac{p}{4} \qquad b = -\frac{3p}{16h} \qquad c = -\frac{p}{40h} \qquad d = \frac{p}{80h^3} \qquad e = -\frac{p}{16h^3} \tag{10}$$

The stresses are therefore

$$\sigma_x = -\frac{3py}{20h} + \frac{p}{8h^3}(2y^3 - 3x^2y)$$

$$\sigma_y = -\frac{p}{2} - \frac{3py}{8h} - \frac{py^3}{8h^3}$$

$$\tau_{xy} = \frac{3px}{8h}(1 - \frac{y^2}{h^2})$$