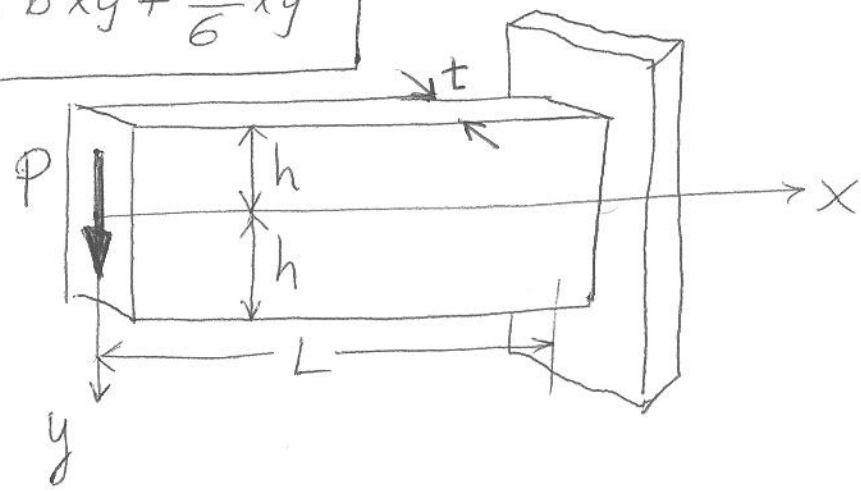
16-Nov-2009 (Monday)

$\boxed{\text{Quiz 2}}$ (Example 3.1 in Ugural & Fenster)

Show that the given Airy Stress Function solves the problem shown and find the constants.

$$\boxed{\Phi = b\,xy + \frac{d}{6}xy^3}$$



For the Solution:

1. Draw the FBD

2. Show that the Airy Stress function satisfies the Biharmonic Eq:

$$\nabla^4\Phi = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

3. Determine the Stresses:

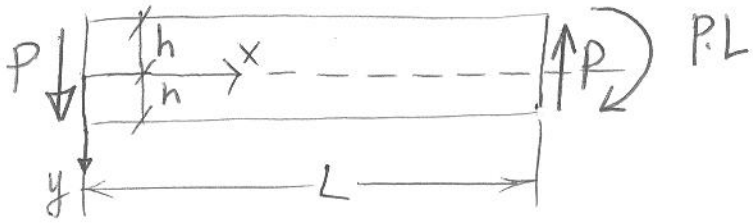
$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}$$

4. Write down all the boundary conditions (not all BCs will be used)

5. Determine the constants.

SOLUTION

(1) Free Body Diagram



(2) The given function satisfies the biharmonic equation:

$$\left.\begin{aligned} &\Phi = bxy + \frac{d}{6}xy^3 \\ &\frac{\partial^4\Phi}{\partial x^4} = 0, \quad \frac{\partial^4\Phi}{\partial y^4} = 0, \quad \frac{\partial^4\Phi}{\partial x^2\partial y^2} = 0 \end{aligned}\right\} \Rightarrow \nabla^4\Phi = 0 \checkmark$$

(3) The stresses are:

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = dxy$$

$$\sigma_y = \frac{\partial^2\Phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -b - \frac{d}{2}y^2$$

(4) The boundary conditions:

Top and bottom edges of the beam are not loaded:

1)
2) at $y = \pm h$ $\quad \sigma_y = 0$ (pointwise at each x).
3) $\quad \tau_{xy} = 0$
4)

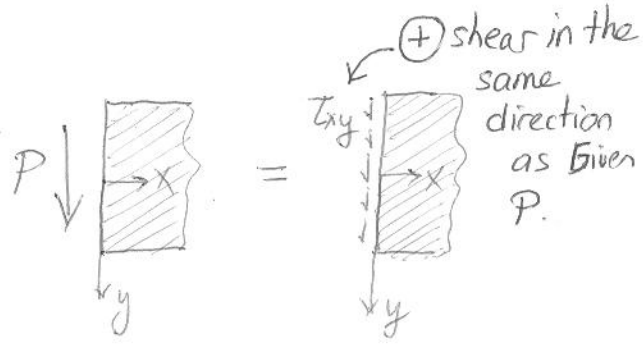
There is zero external loading in the x-dir at $x = 0$.

5) at $x = 0$ $\quad \sigma_x = 0$ (pointwise at each y)

The applied load ~~at~~ P must be equal to the resultant of the shearing forces distributed across the free end (at $x = 0$)

6) at $x = 0$

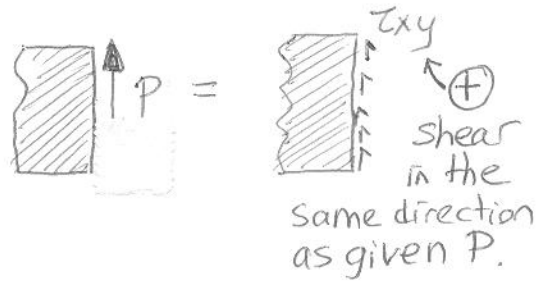
$$\oplus P = \oplus \int_{-h}^{h} \tau_{xy}\, t\, dy$$



Similarly, the resultant of the shearing tractions distributed at the other end, $x = L$, must be equal P:

7) at $x = L$

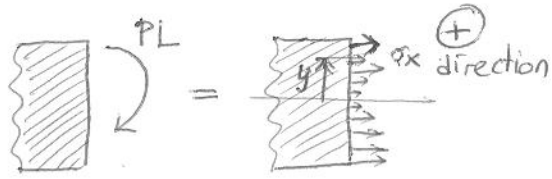
$$\oplus P = \oplus \int_{-h}^{h} \tau_{xy}(L, y)\, t\, dy$$



Resultant moment is given by the resultant of σ_x at $x = L$

8) at $x = L$

$$\oplus PL = \oplus \int_{-h}^{h} (\sigma_x(L, y)\, y)\, t\, dy.$$



(5) Determine the constants from the BCs:

1), 2) at $y=\pm h$ $\sigma_y = 0$ identically satisfied

3), 4) at $y=\pm h$ $\tau_{xy}(x,\pm h) = -b - \frac{d}{2}h^2 = 0$

$$\Rightarrow \boxed{b = -\frac{d}{2}h^2} \quad (*)$$

5) at $x=0 \Rightarrow \sigma_x(0,y) = d(0)y = 0$ satisfied.

6), 7) at $\left.\begin{matrix} x=0 \\ x=L \end{matrix}\right\}$ $\tau_{xy} = -b - \frac{d}{2}y^2$

$$P = \int_{-h}^{h}\left(-b - \frac{d}{2}y^2\right) t\, dy = -b\, 2ht - \frac{d}{6} 2h^3 t$$

(exactly the same expression since there is no x-dependence in the expression for τ_{xy})

Insert $(*)$ for b: $P = +\frac{d}{\cancel{2}}h^2(\cancel{2}ht) - \frac{d}{3}h^3 t = \frac{2}{3}h^3 t\, d.$

$$\Rightarrow \boxed{d = \frac{3P}{2h^3 t}} \Rightarrow \boxed{b = -\frac{3P}{4ht}}$$

8) Now check the last BC: at $x=L$

$$P = \int_{-h}^{h} \sigma_x(L,y)\cdot y\, t\, dy = \int_{-h}^{h} dL\, y \cdot y\, t\, dy = dL\frac{2h^3}{3}t$$

$$\Rightarrow \boxed{d = \frac{3}{2}\frac{P}{h^3 t}}$$ again the same result.

<u>Note</u>: Some of the BCs were identically satisfied because the author had chosen the Airy Stress function wisely.