Show that the given Airy Stress Function solves the problem shown and find the constants

$$\Phi = bxy + \frac{d}{6}xy^3$$

For the Solution:

1. Draw the FBD

2. Show that the Airy Stress Function satisfies the Biharmonic Eq:

$$\nabla^4 \Phi = \frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0$$

3. Determine the Stresses:

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}$$

4. Write down all the boundary conditions (not all BCs will be used)

5. Determine the constants.
1. **Free Body Diagram**

\[ P \downarrow \quad F_x \quad \overrightarrow{L} \]

\[ y \quad L \]

2. The given function satisfies the biharmonic equation:

\[ \Phi = bxy + \frac{d}{6}xy^3 \]

\[ \frac{\partial^4 \Phi}{\partial x \partial y^3} = 0, \quad \frac{\partial^4 \Phi}{\partial y^4} = 0, \quad \frac{\partial^4 \Phi}{\partial x^2 \partial y^2} = 0 \]

\[ \nabla^4 \Phi = 0 \]

3. The stresses are:

\[ \sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = dy \]

\[ \sigma_y = \frac{\partial^2 \Phi}{\partial x^2} = 0 \]

\[ \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = -b - \frac{d}{2}y^2 \]
The boundary conditions:

1) Top and bottom edges of the beam are not loaded:
   \[ \sigma_y = 0 \quad \text{(pointwise at each } x) \]
   \[ \tau_{xy} = 0 \]

2) There is zero external loading in the x-dir at \( x = 0 \):
   \[ \sigma_x = 0 \quad \text{(pointwise at each } y) \]

3) The applied load at \( P \) must be equal to the resultant of the shearing forces distributed across the free end (at \( x = 0 \))
   \[ \Theta P = \Theta \int_{-h}^{h} \tau_{xy} t \, dy \]

4) Similarly, the resultant of the shearing tractions distributed at the other end, \( x = L \), must be equal \( P \):
   \[ \Theta P = \Theta \int_{-h}^{h} \tau_{xy} (L, y) t \, dy \]

Resultant moment is given by the resultant of \( \sigma_x \) at \( x = L \):

5) \( \Theta P L = \Theta \int_{-h}^{h} (\sigma_x (L, y) y) t \, dy \).
(6) Determine the constants from the BCs:

1) at $y = \pm h$, $\sigma_y = 0$ identically satisfied
2) at $y = \pm h$, $\tau_{xy}(x, \pm h) = -b - \frac{d}{2} h^2 = 0$

   \[ \Rightarrow \quad b = -\frac{d}{2} h^2 \]  

3) at $x = 0 \Rightarrow \sigma_x(0, y) = d(0)y = 0$ satisfied.

4) at $x = 0$

   \[ \tau_{xy} = -b - \frac{d}{2} y^2 \]

5) \[ P = \int_{-h}^{h} (-b - \frac{d}{2} y^2) t \, dy = -b2h - \frac{d}{6}2h^3 + \]

   (exactly the same expression since there is no $x$-dependence in the expression for $\tau_{xy}$)

6) Insert (x) for $b$: \[ P = \int_{-h}^{h} \sigma_x(x, y) \cdot y \, t \, dy = \int_{-h}^{h} \tau_{xy} \cdot y \, t \, dy = \frac{d}{3} 2h^3 + d. \]

   \[ \Rightarrow \quad d = \frac{3}{2} \frac{P}{h^3} \quad \Rightarrow \quad b = -\frac{3}{4} \frac{P}{h^3} \]

7) Now check the last BC: at $x = L$

   \[ P_L = \int_{-h}^{h} \sigma_x(L, y) \cdot y \, t \, dy = \int_{-h}^{h} t \, \tau_{xy} \cdot y \, t \, dy = d \frac{2h^3}{3} \]

   \[ \Rightarrow \quad d = \frac{3}{2} \frac{P}{h^3} \] (again the same result).

\[ \text{Note: Some of the BCs were identically satisfied because the author had chosen the Airy Stress function wisely.} \]