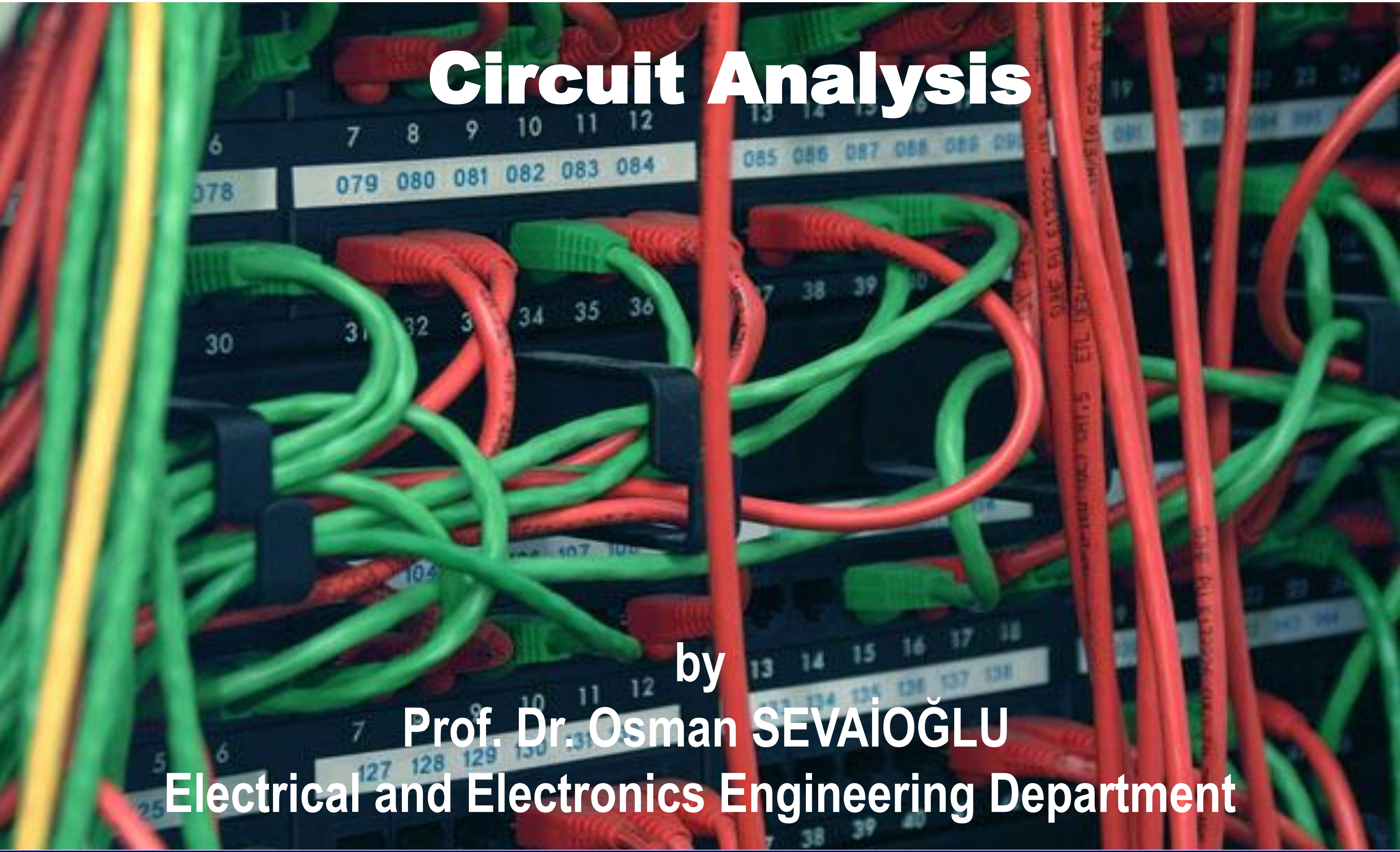


Circuit Analysis

by

Prof. Dr. Osman SEVAİOĞLU

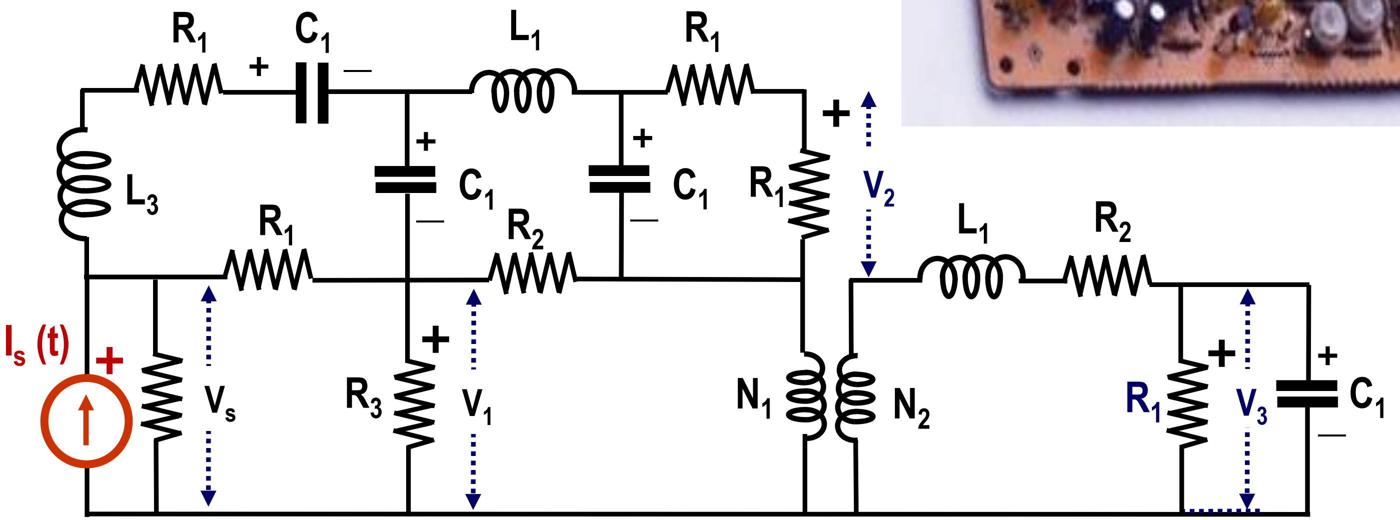
Electrical and Electronics Engineering Department

METU

What is an Electrical Circuit ?

Definition

An electrical circuit is a set of various system elements connected in a certain way, through which electrical current can pass

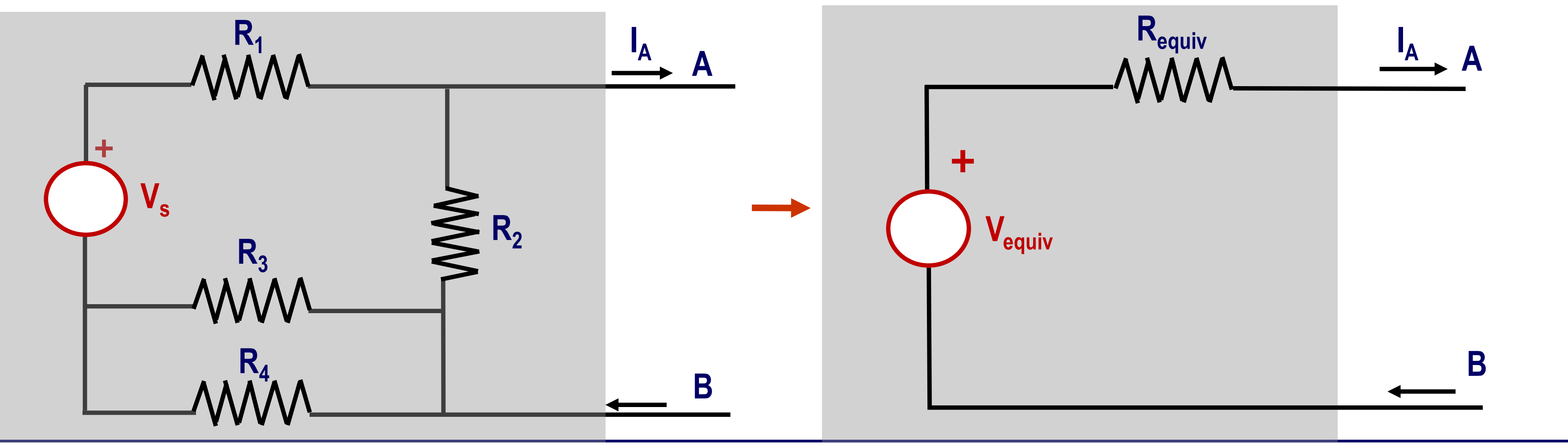


Thevenin Equivalent of an Electrical Circuit

Definition

"Thevenin Equivalent" of an electrical circuit is the simplified form of the circuit consisting of a voltage source in series with a resistance.

Given Circuit	Thevenin Equivalent Circuit

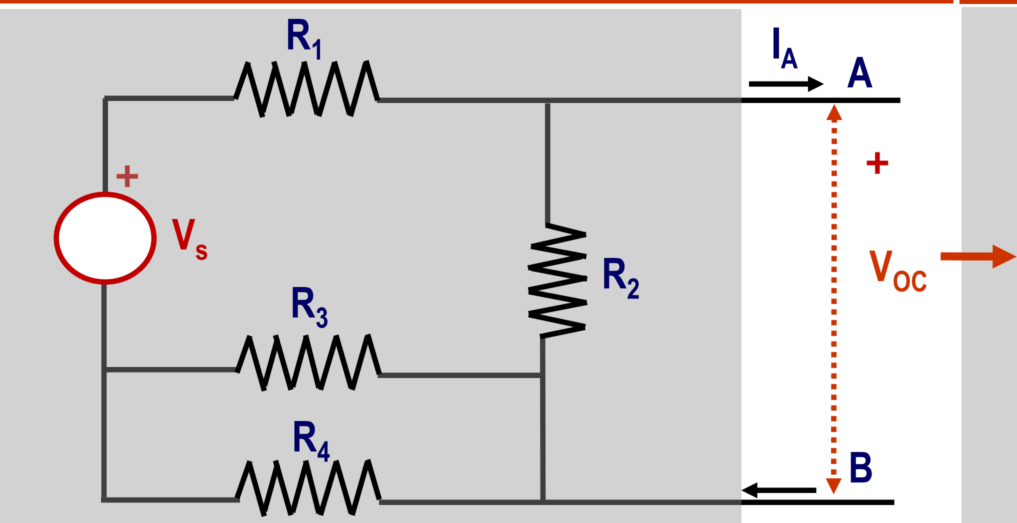


Calculation of Thevenin Equivalent of a Circuit

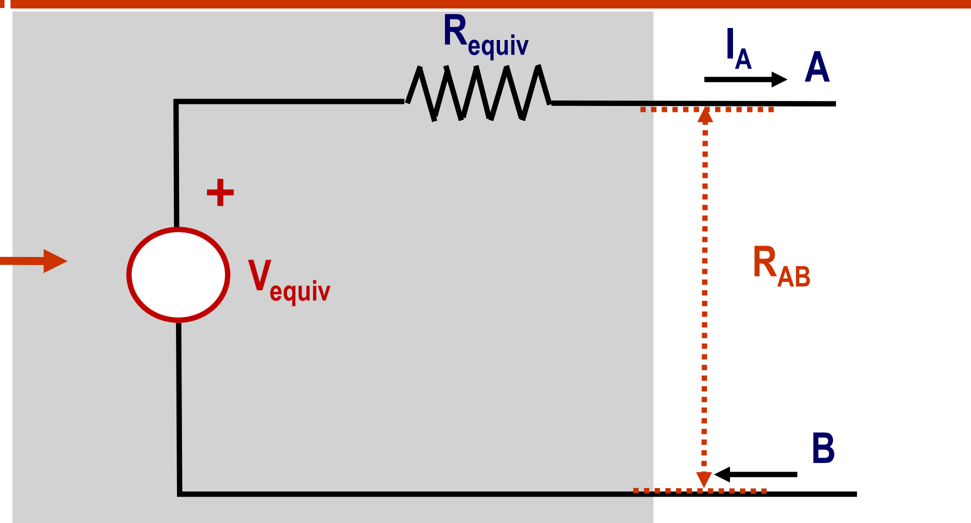
Method

1. Open circuit the terminals A–B of the given circuit,
2. Calculate the open circuit voltage V_{AB} seen at the terminals A and B,
3. Remove (kill) all the sources in the circuit,
4. Calculate the equivalent resistance $R_{AB} = R_{equiv}$ seen at the terminals A and B

Given Circuit



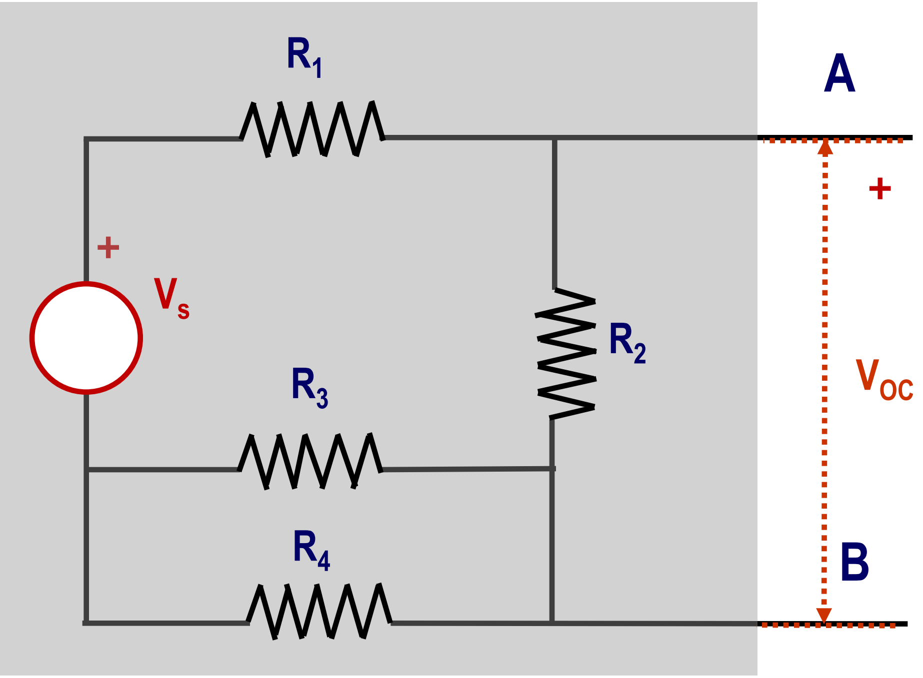
Thevenin Equivalent Circuit

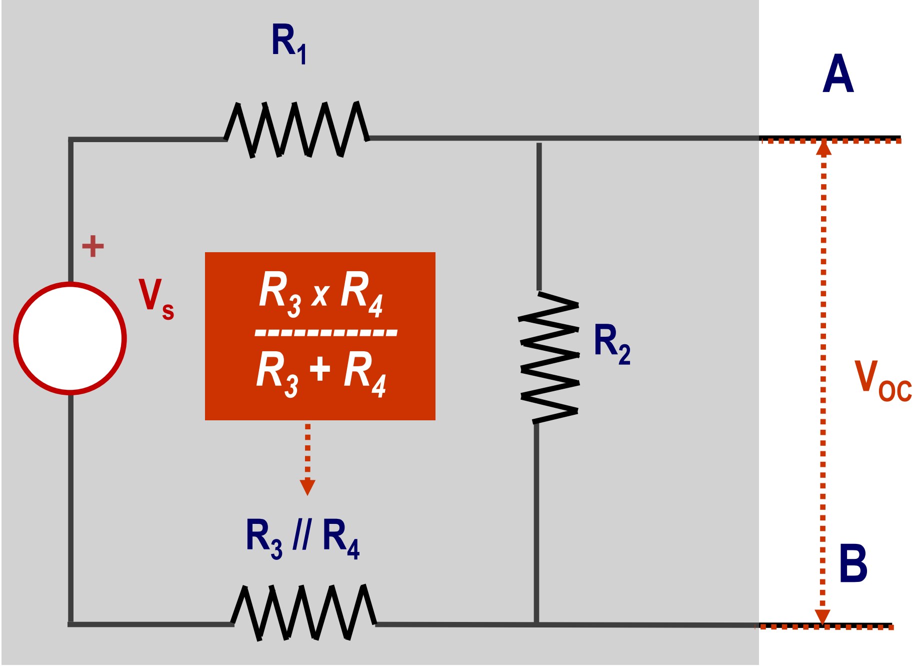


Calculation of the Thevenin Equivalent Voltage V_{equiv}

1. Open circuit the the terminals A-B,
2. Calculate the open circuit voltage V_{oc} seen at the terminals A-B

Given Circuit	*Simplified Circuit*





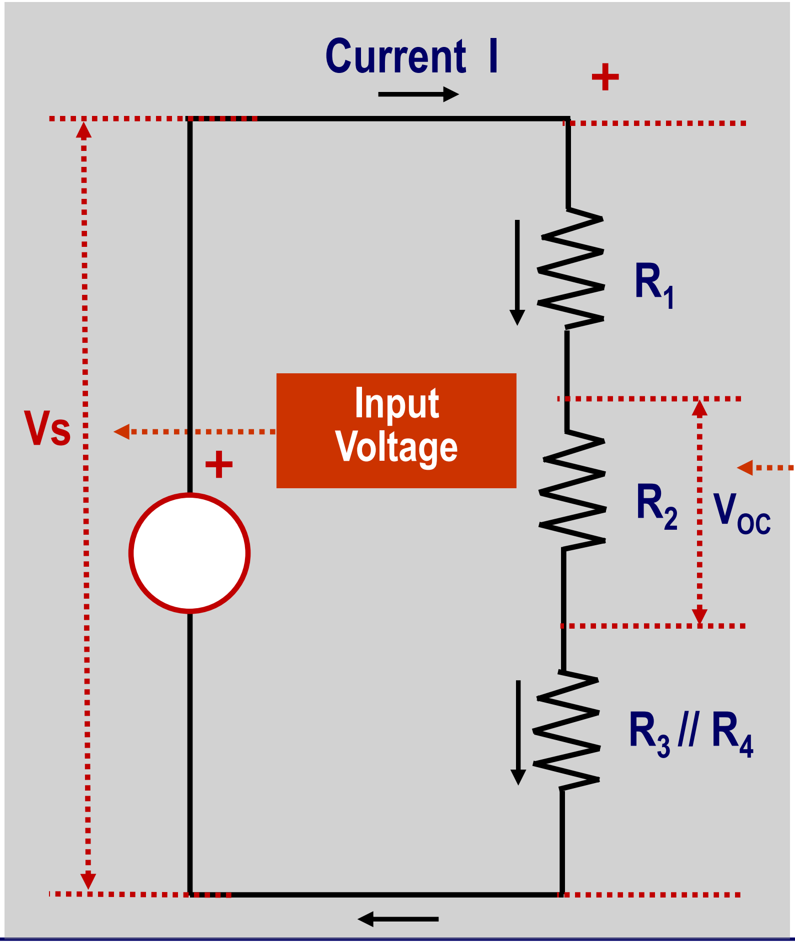
Calculation of the Thevenin Equivalent Voltage V_{equiv}

2. Calculate the voltage open circuit V_{oc} between the terminals A-B

$$\text{Voltage Division Ratio} = \frac{R_2}{R_1 + R_2 + (R_3 // R_4)}$$

$$V_{OC} = \frac{R_2}{R_1 + R_2 + (R_3 // R_4)} Vs$$

Alternative Representation of the Circuit



Circuit Analysis

Determination of Thevenin Equivalent Circuit by Calculation

3. Remove (kill) all the sources in the given circuit

Meaning of "Killing Voltage Source":

(a) Short Circuit all voltage sources

SC

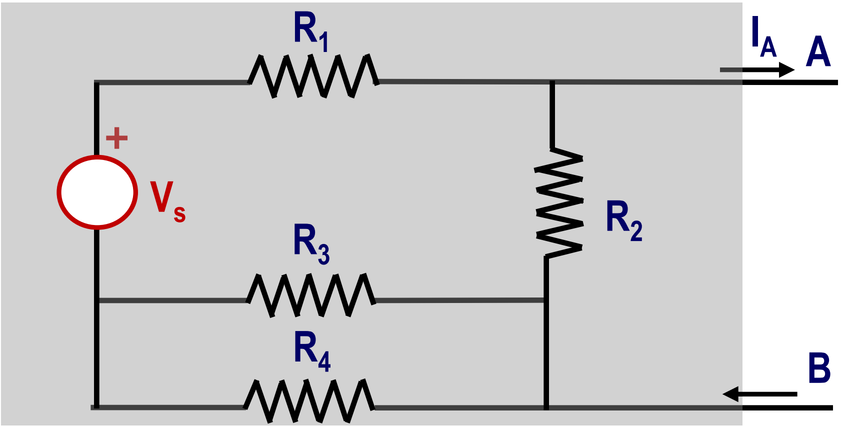
A very Important Rule:

Controlled (dependent) sources cannot be killed.

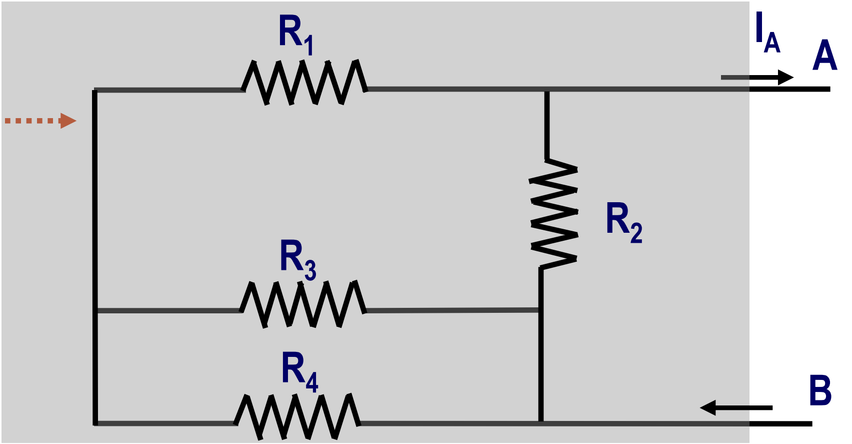
If you do, the result will be incorrect !

Hence, a circuit with these types of sources can NOT be simplified by using the Thevenin Equivalencing Method

Circuit with Voltage Source



Voltage Source Killed



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Circuit Analysis

Calculation of Thevenin Equivalent Resistance R_{equiv}

3. Remove (kill) all the sources in the given circuit

Meaning of "Killing Current Source":

(b) Open Circuit all current sources

OC

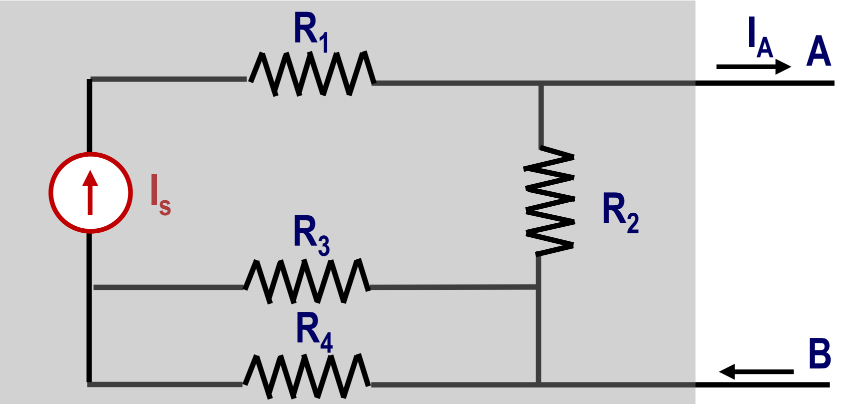
An Important Rule:

Controlled (dependent) sources cannot be killed.

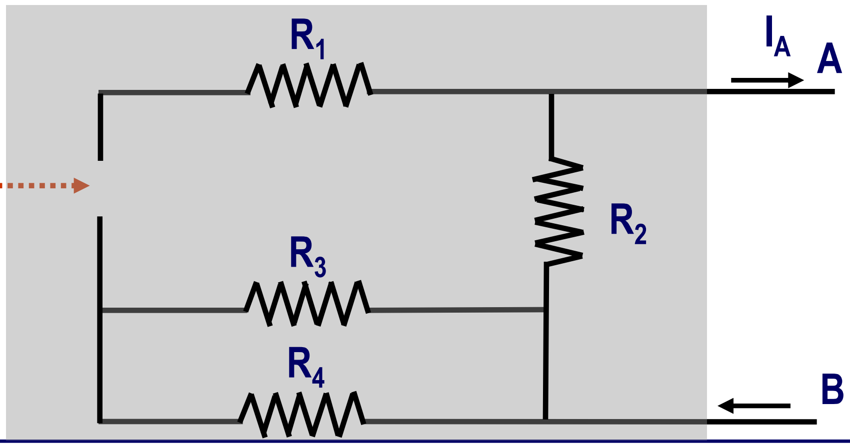
If you do, the result will be incorrect !

Hence, a circuit with these types of sources can NOT be simplified by using the Thevenin Equivalencing Method

A Circuit with Current Source



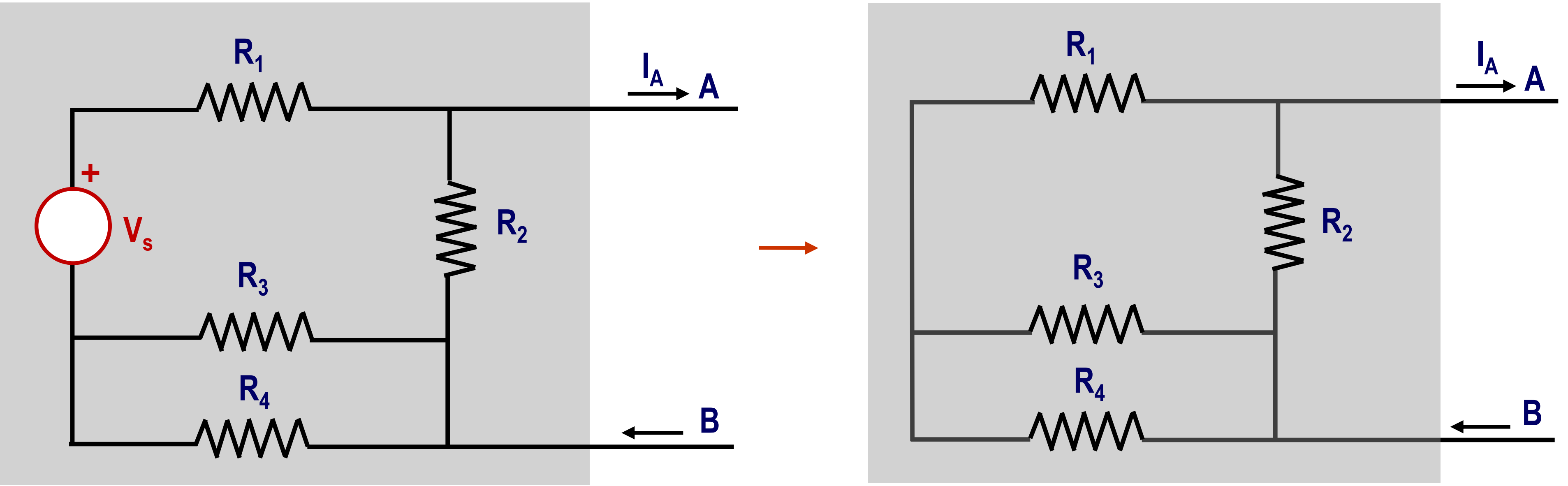
Current Source Killed



Calculation of Thevenin Equivalent Resistance R_{equiv}

3. Kill all the sources in the given circuit,
4. Calculate the equivalent resistance R_{AB} seen from the terminals A and B

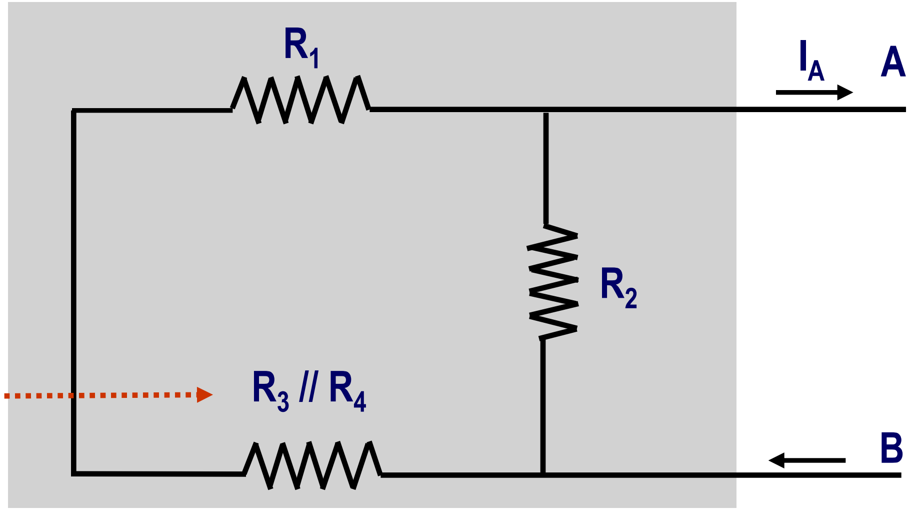
Given Circuit	*Calculate R_{equiv}*

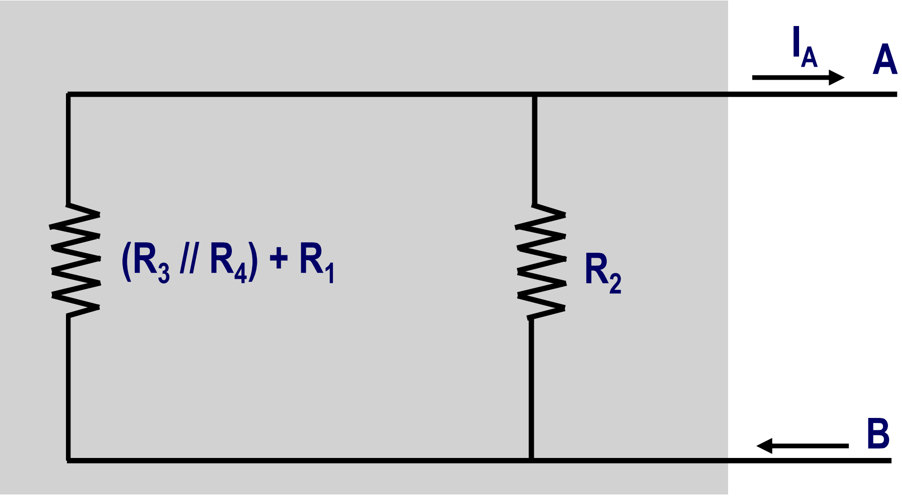


Calculation of Thevenin Equivalent Resistance R_{equiv}

4. Perform simplifications on the resulting circuit in order to find $R_{equiv.}$

$$\frac{R_3 \times R_4}{R_3 + R_4}$$





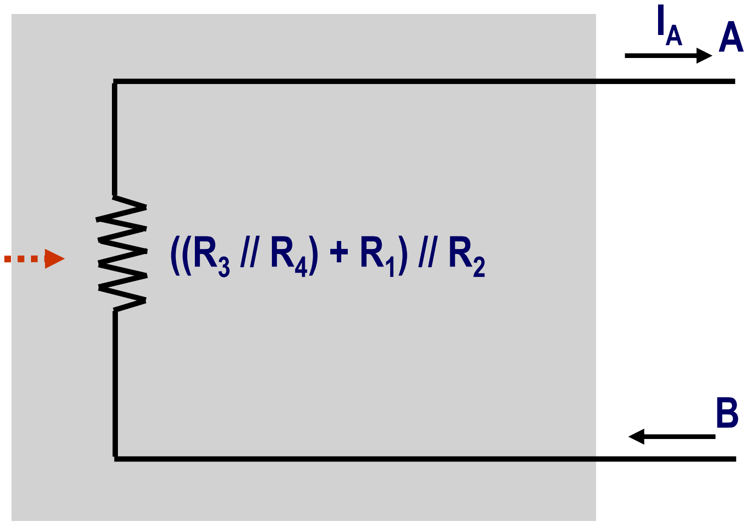
Calculation of Thevenin Equivalent Resistance R_{equiv}

4. Perform simplifications on the resulting circuit in order to find $R_{equiv.}$

$$R_{equiv} = ((R_3 // R_4) + R_1) // R_2$$

$$= ((R_3 \times R_4)/(R_3 + R_4) + R_1) // R_2$$

$$= \frac{((R_3 \times R_4)/(R_3 + R_4) + R_1) \times R_2}{((R_3 \times R_4)/(R_3 + R_4) + R_1) + R_2}$$



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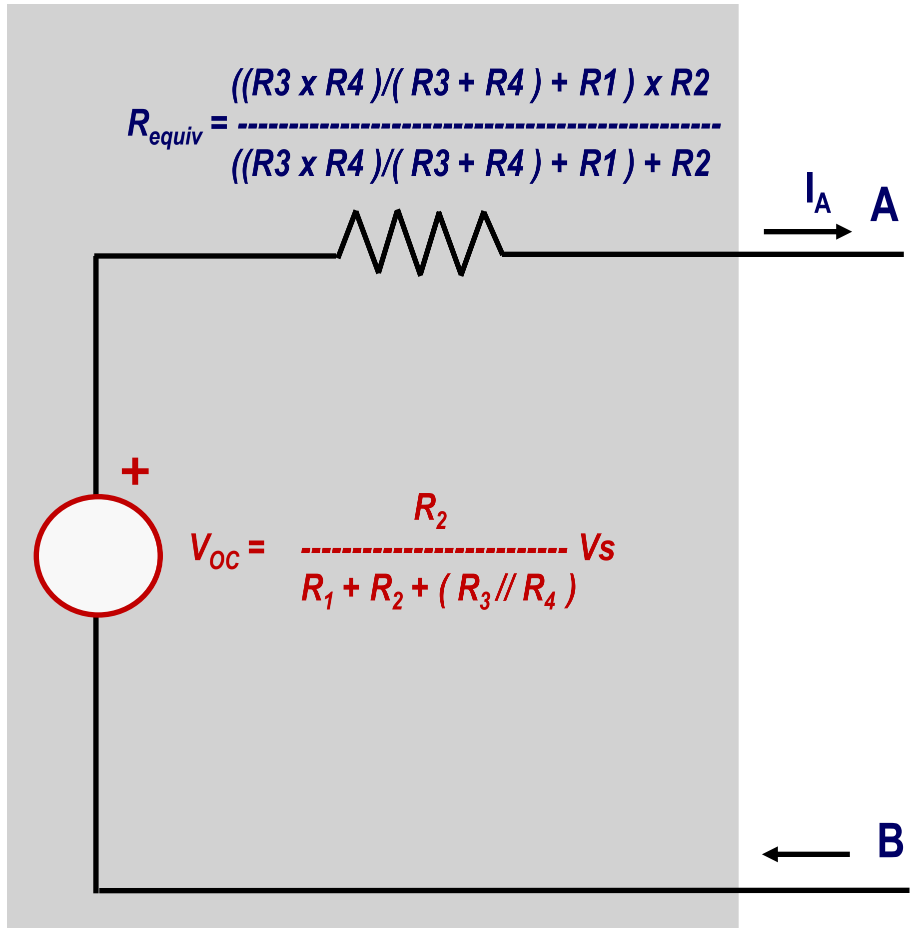
Circuit Analysis

Resulting Thevenin Equivalent Circuit

$$R_{equiv} = \frac{((R_3 \times R_4)/(R_3 + R_4) + R_1) \times R_2}{((R_3 \times R_4)/(R_3 + R_4) + R_1) + R_2}$$

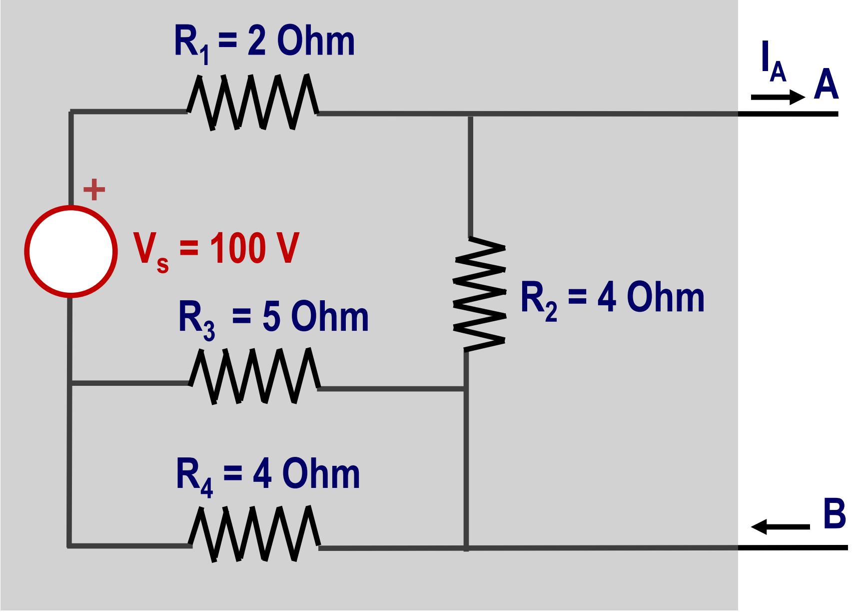
$$V_{OC} = \frac{R_2}{R_1 + R_2 + (R_3 // R_4)} Vs$$



Example

Example

Determine the "Thevenin Equivalent" of the circuit shown on the RHS



Calculation of R_{equiv}

$$R_{equiv} = (R_3 \,//\, R_4 + R_1) \,//\, R_2$$

$$= ((R_3 \times R_4)/(R_3 + R_4) + R_1) \,//\, R_2$$

$$= \frac{((R_3 \times R_4)/(R_3 + R_4) + R_1) \times R_2}{((R_3 \times R_4)/(R_3 + R_4) + R_1) + R_2}$$

$$= \frac{(2.2222 + 2) \times 4}{(2.2222 + 2) + 4} = \frac{16.8888}{8.22222} = 2.054 \text{ Ohm}$$

Circuit Analysis

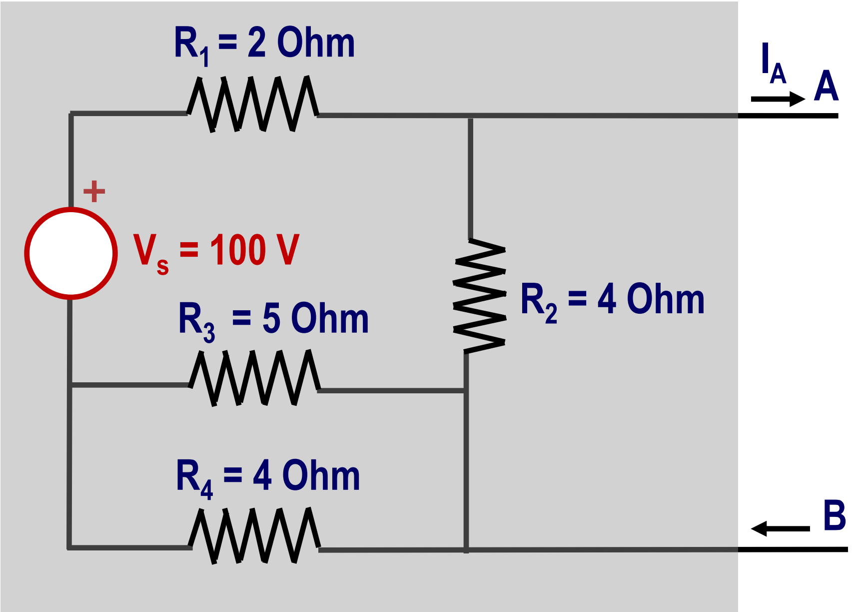
Example

Example

Calculation of V_{equiv}

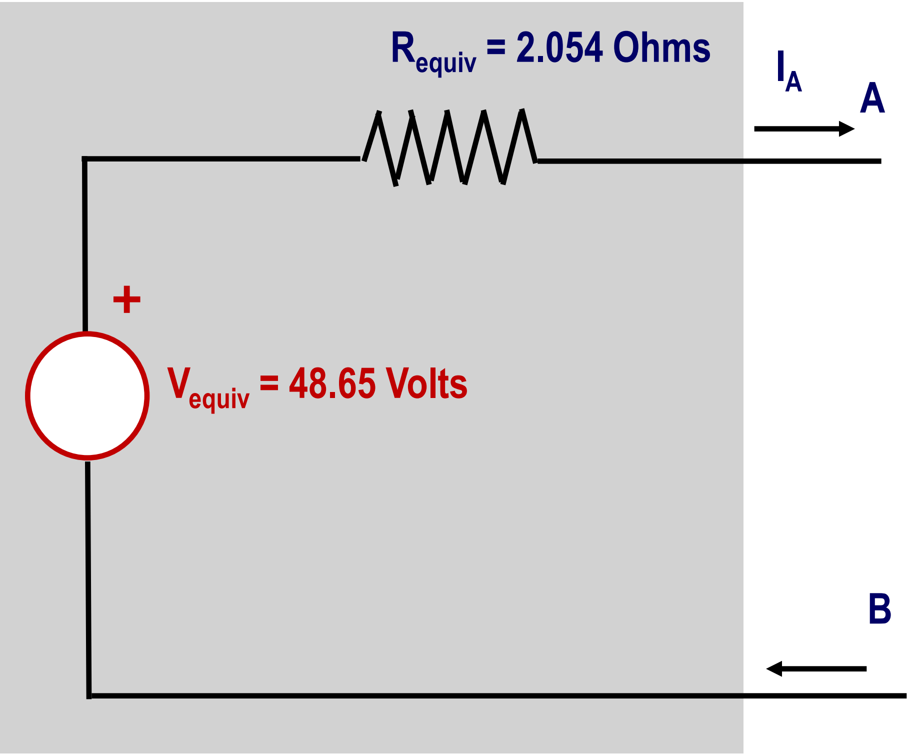
$$V_{OC} = \frac{R_2}{R_1 + R_2 + (R_3 // R_4)} Vs$$

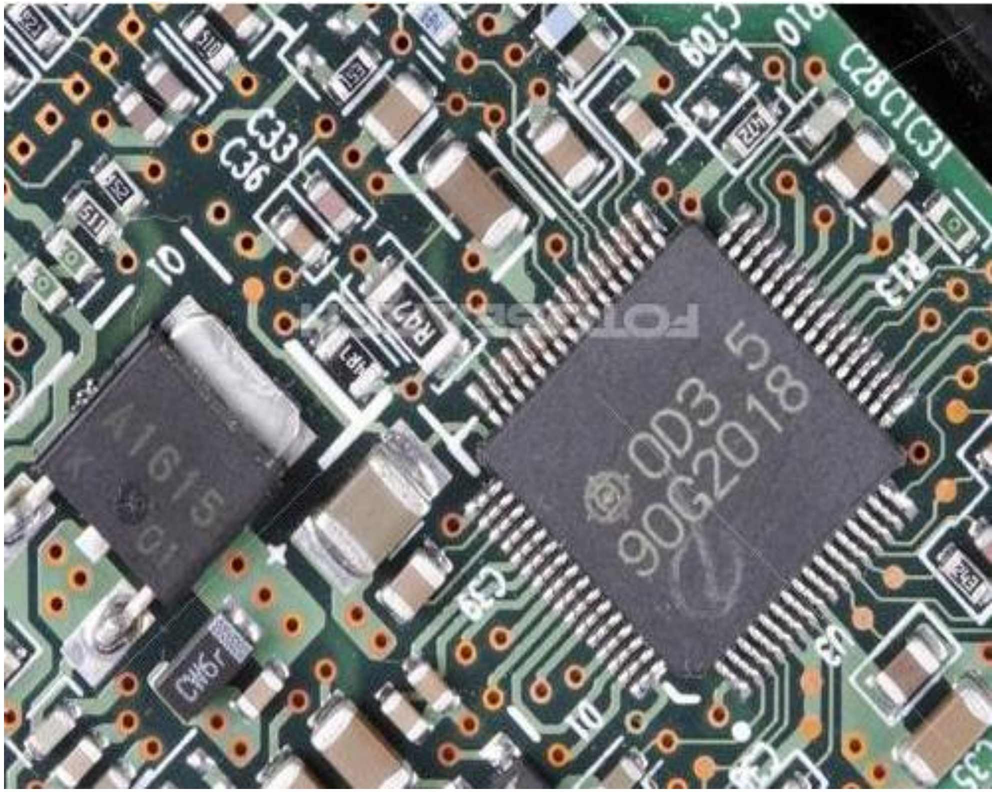
$$= 4 / (2 + 4 + 2.2222) * 100 = 48.65 \text{ Volts}$$



Example

Resulting Thevenin Equivalent Circuit





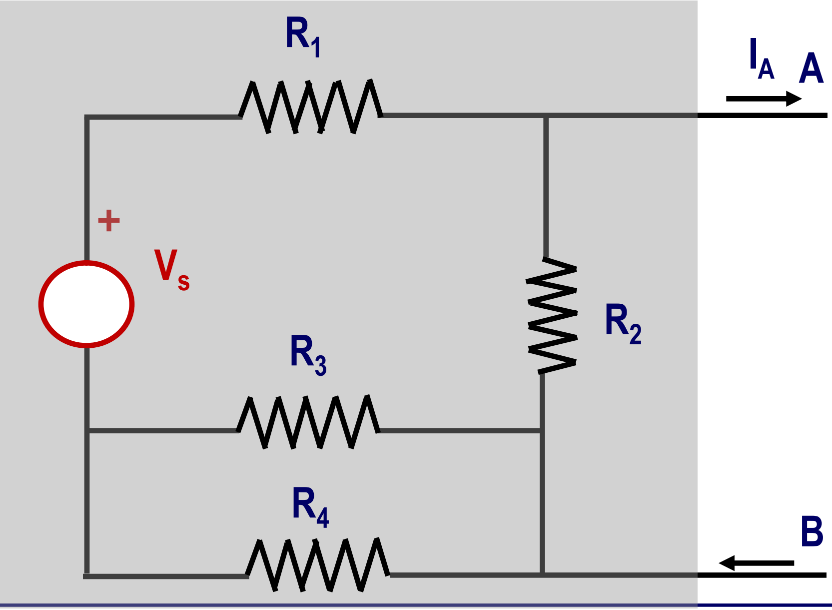
Circuit Analysis

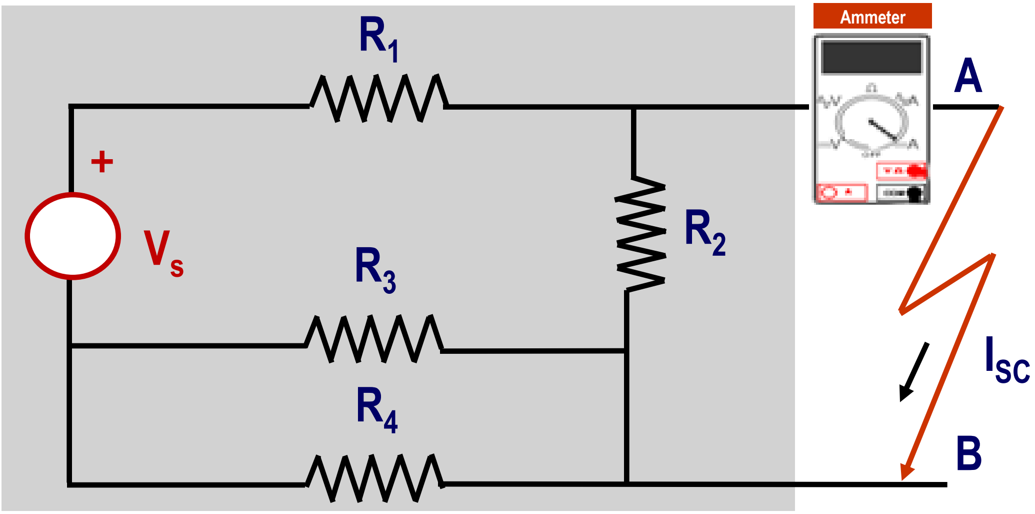
Determination of the Thevenin Equivalent Circuit by using Open and Short Circuit Tests

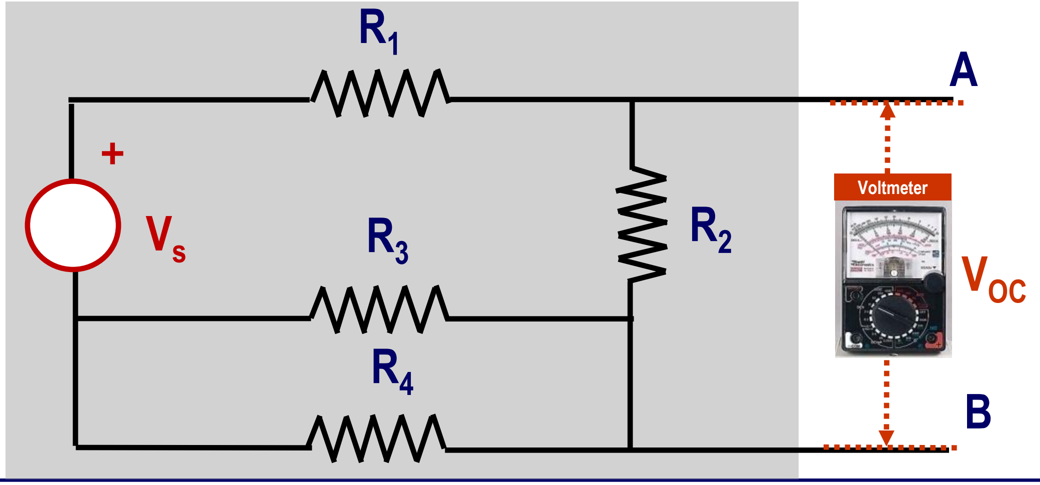
Procedure

a) Short circuit the terminals A and B and measure I_{sc}

b) Open circuit the terminals A and B and measure V_{OC}



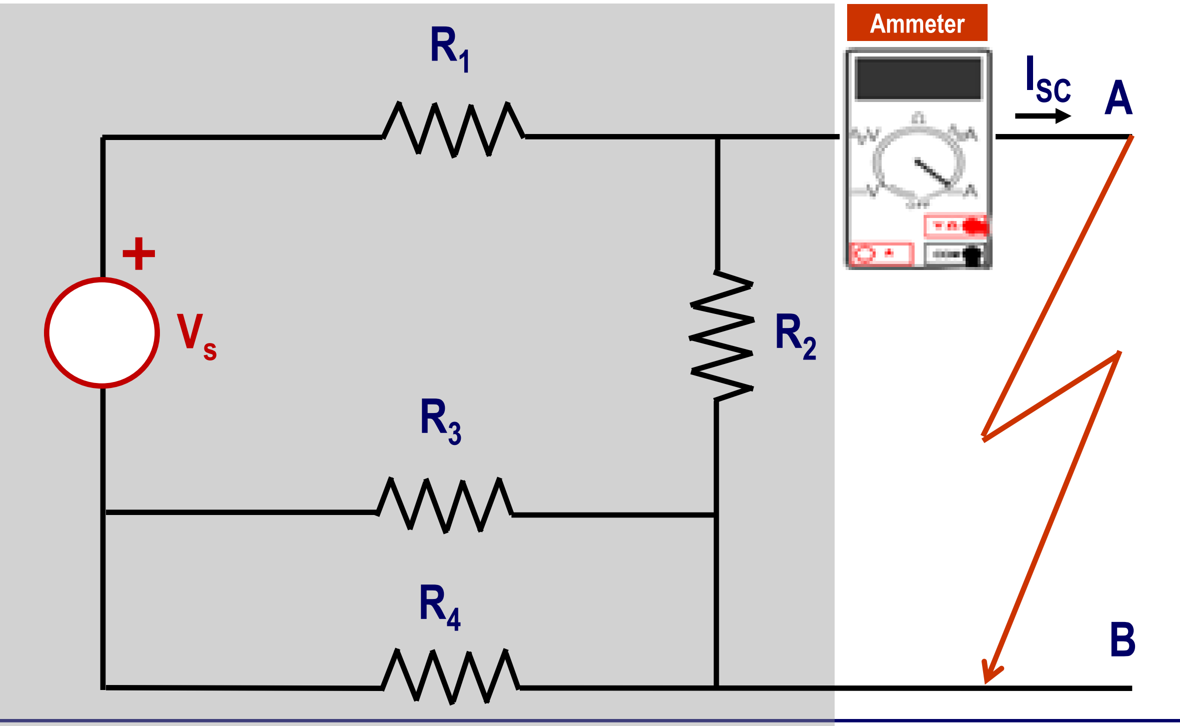




Short Circuit Test

Objective

The main objective of Short Circuit Test is to determine the current I_{sc} flowing when the terminals A and B are shorted



Procedure

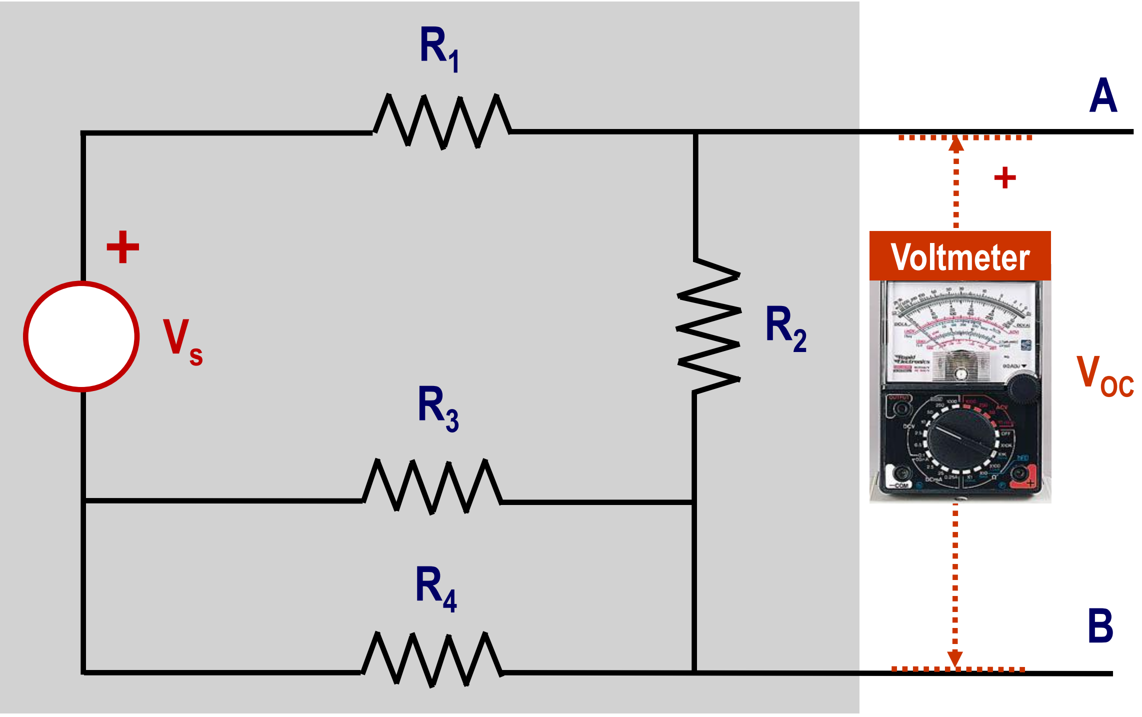
a) Short circuit the terminals A and B of the given circuit,
b) Measure the current I_{SC} flowing through the short circuit

Circuit Analysis

Open Circuit Test

Objective

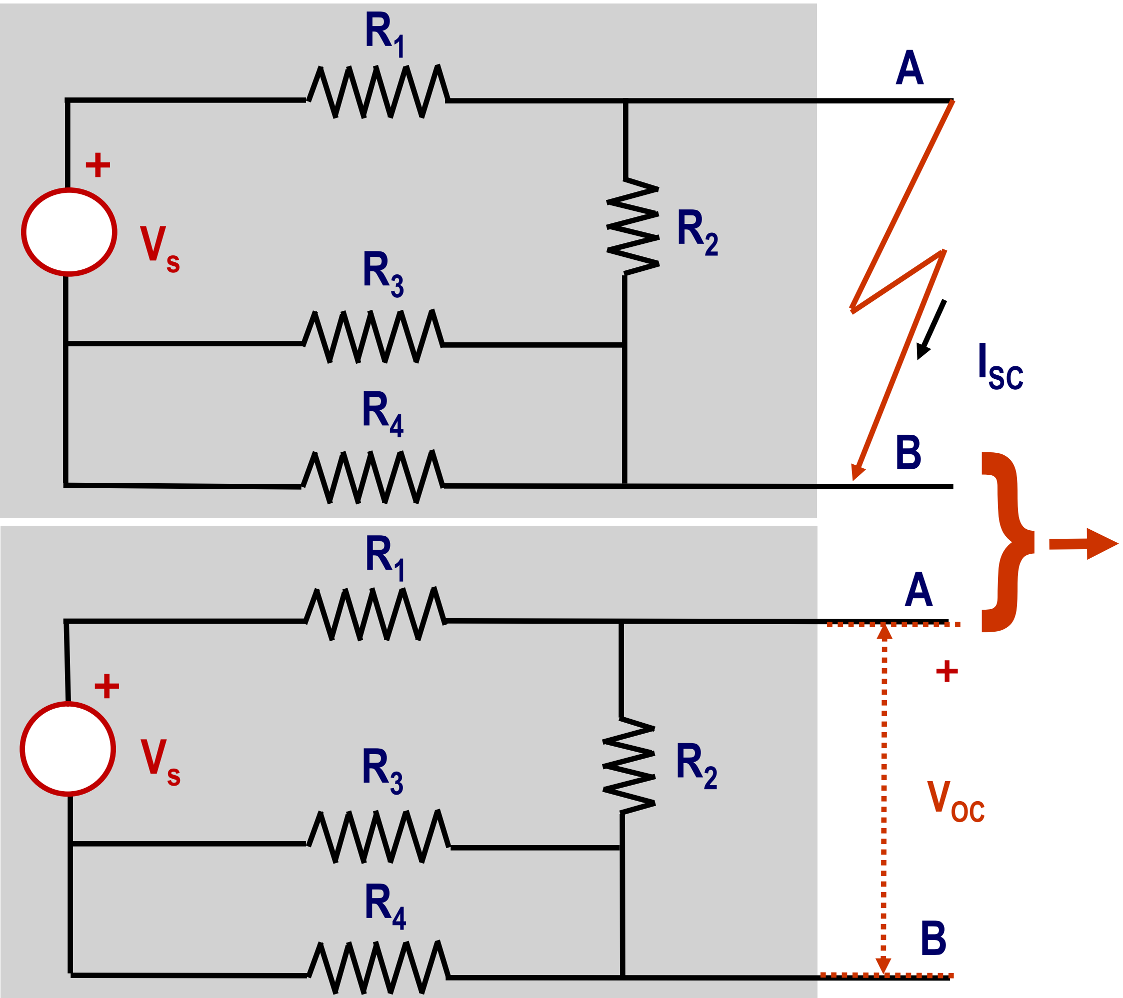
The main objective of Open Circuit Test is to determine the voltage at the terminals A and B when these terminals are open circuited



Procedure

a) Open circuit the terminals of the given circuit,
b) Measure the voltage V_{OC} between the terminals A and B of the given circuit

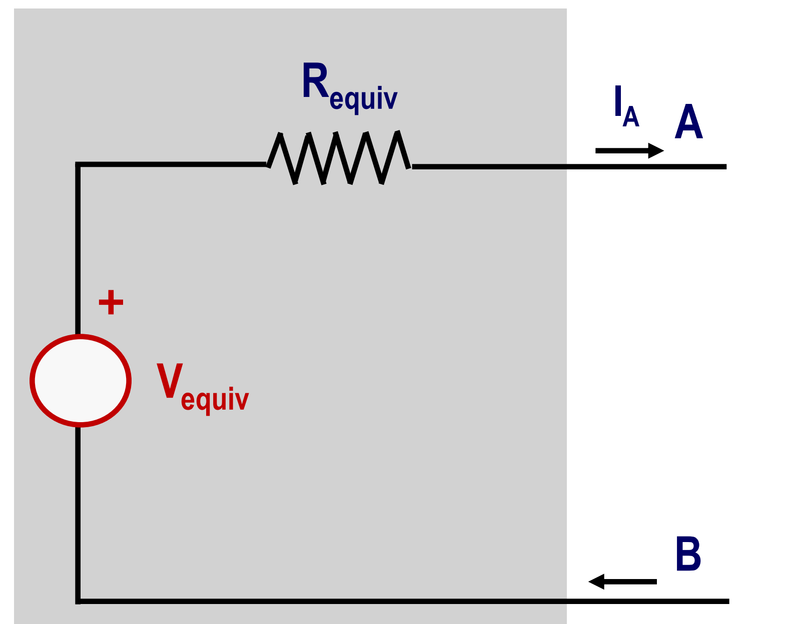
Determination of the Thevenin Equivalent Circuit by using Open and Short Circuit Tests



Procedure

a) Divide V_{OC} by I_{SC} and find $R_{equivalent}$,

b) $V_{equivalent} = V_{OC}$



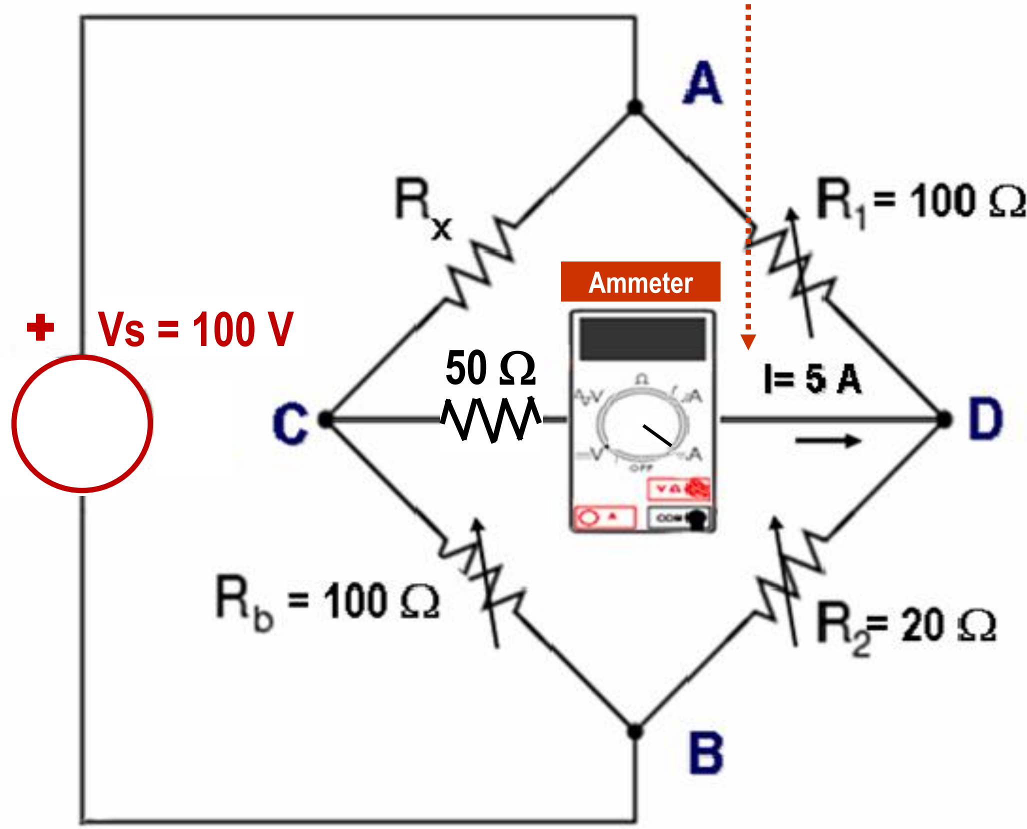
Example 1. Unbalanced Wheatstone Bridge

Example

Calculate the value of the unknown resistance R_x in the unbalanced Wheatstone Bridge shown on the RHS, if the current read by the ammeter is 5 Amp.

Since 5 Amp passes through the ammeter, the bridge is unbalanced, hence, cross multiplication of branches are not equal

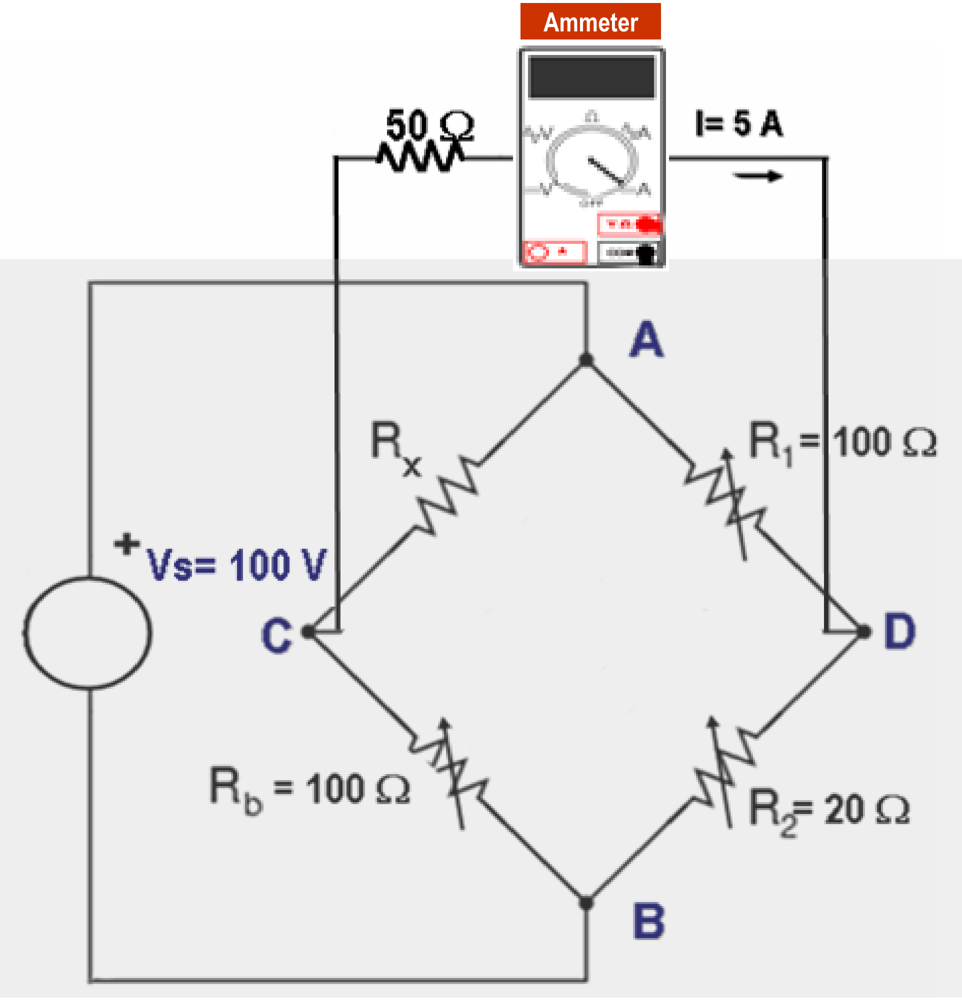
Please note that the bridge is unbalanced, i.e. current flows in the ammeter



Example 1. Unbalanced Wheatstone Bridge

Solution

First, take out the ammeter and 50 Ohm resistance connected to terminals C and D



Example 1. Unbalanced Wheatstone Bridge

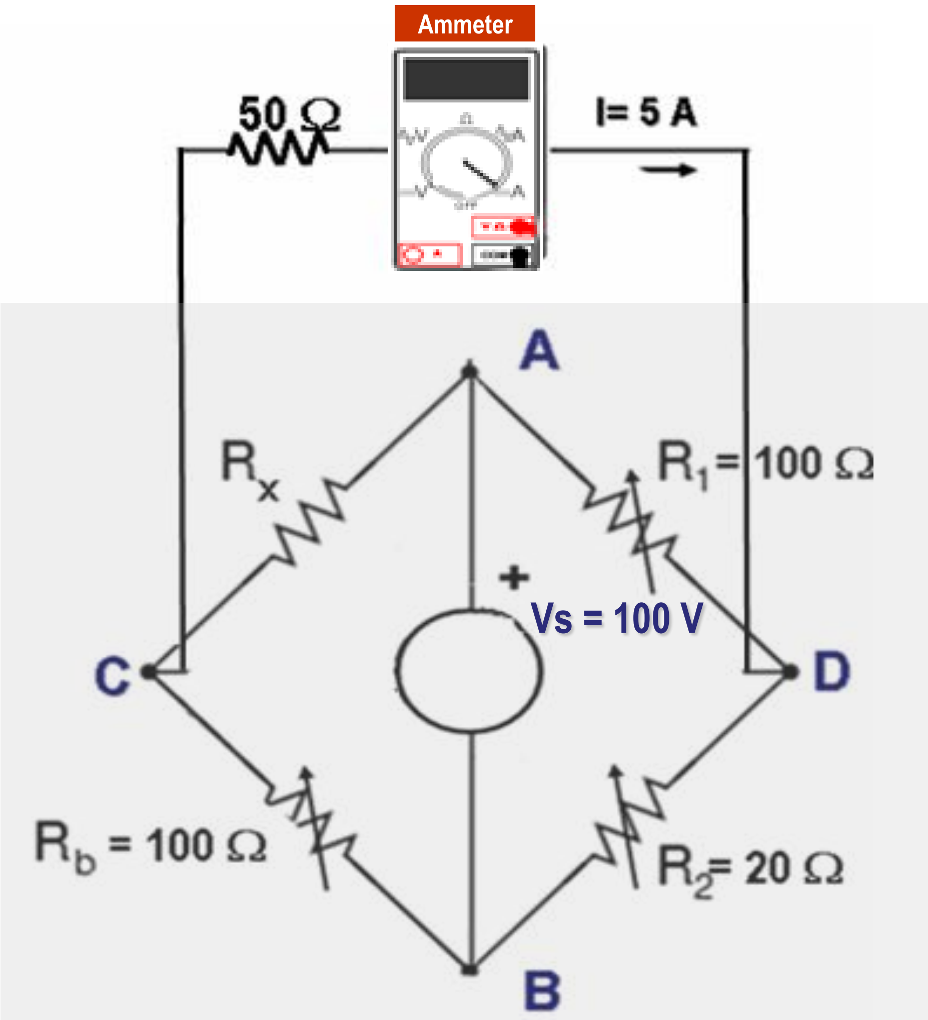
Solution

Determine the Thevenin Equivalent of the source side of the circuit, i.e. the rest of the circuit after the ammeter and 50 Ohm resistance are taken out

1. Kill all the sources in the given circuit

<u>Meaning of the The Term: "Killing Sources"</u>

Means Short Circuiting the voltage source in the circuit on the RHS

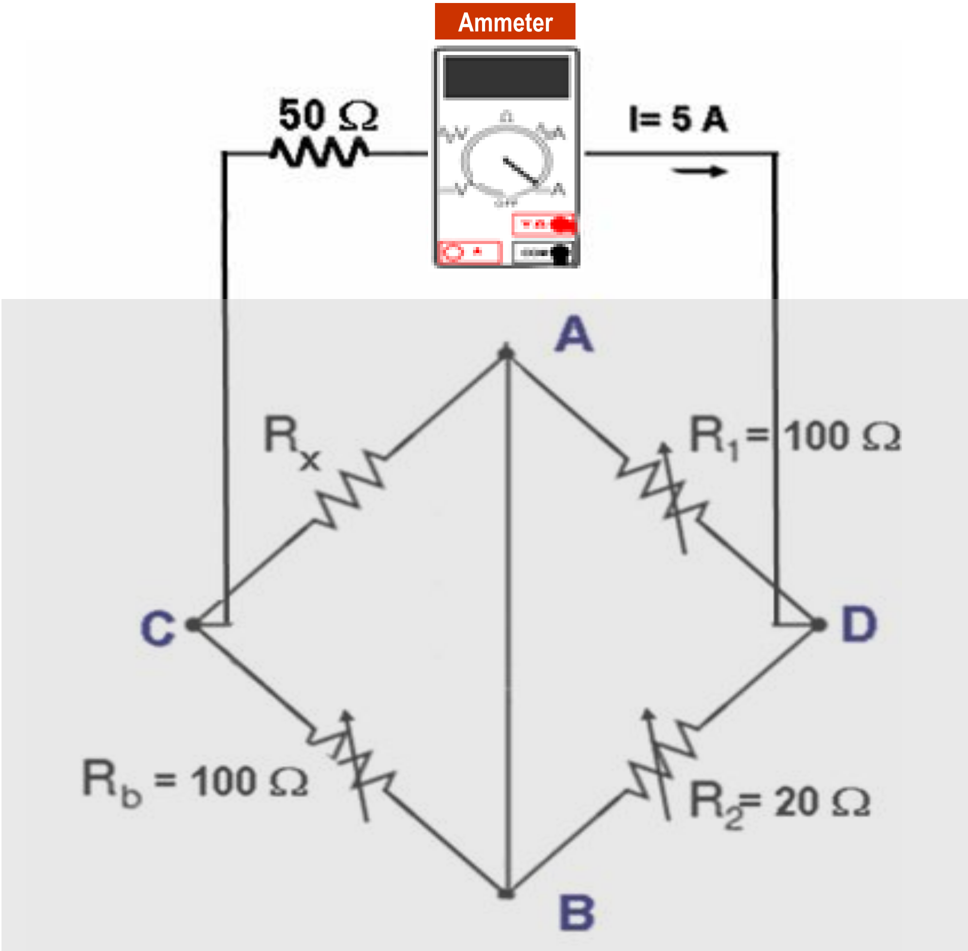


Example 1. Unbalanced Wheatstone Bridge

Solution

Determine the Thevenin Equivalent of the source side of the circuit, i.e. rest of the circuit after the ammeter and 50 Ohm resistance are taken out

2. Calculate the equivalent resistance of the rest of the circuit

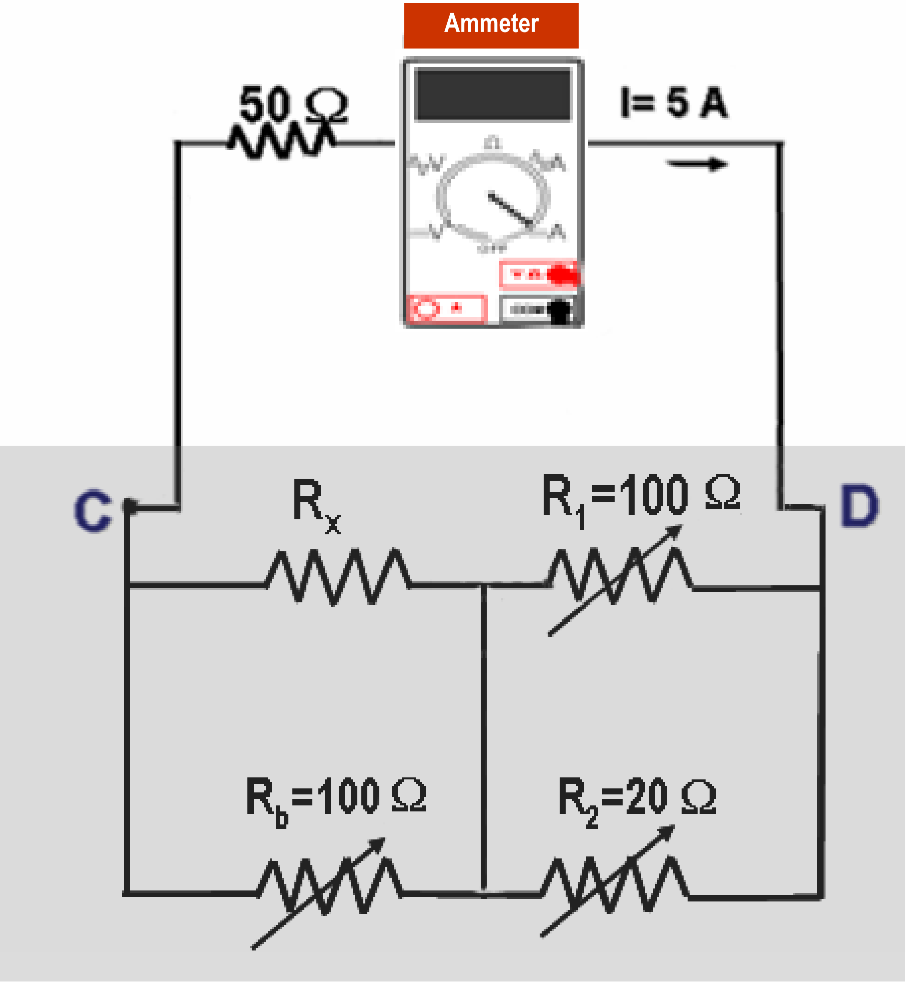


Example 1. Unbalanced Wheatstone Bridge

Solution

Determine the Thevenin Equivalent of the source side of the circuit, i.e. the rest of the circuit after the ammeter and 50 Ohm resistance are taken out

2. Calculate the equivalent resistance of the rest of the circuit

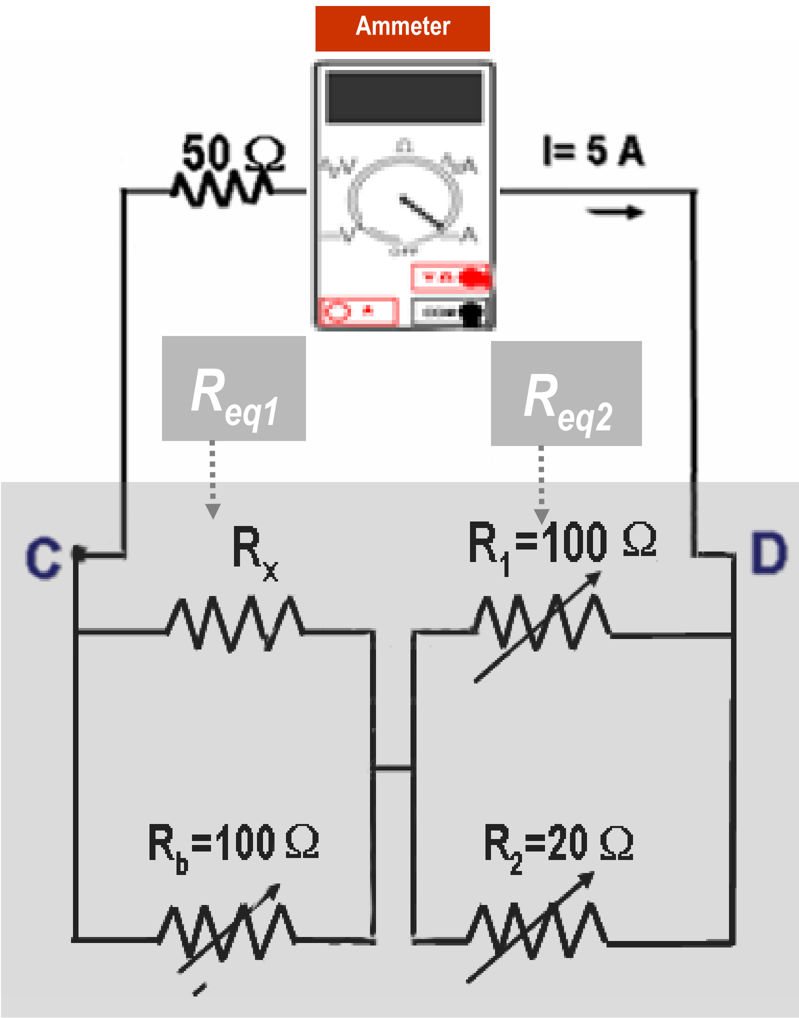


Example 1. Unbalanced Wheatstone Bridge

Solution

Determine the Thevenin Equivalent of the source side of the circuit, i.e. the rest of the circuit after the ammeter and 50 Ohm resistance are taken out

$$R_{eq} = R_{eq1} + R_{eq2}$$
$$= (R_X // R_b) + (R_1 // R_2)$$
$$= (R_x \, x100)/(R_x+100) + (100 \times 20)/(100+20)$$



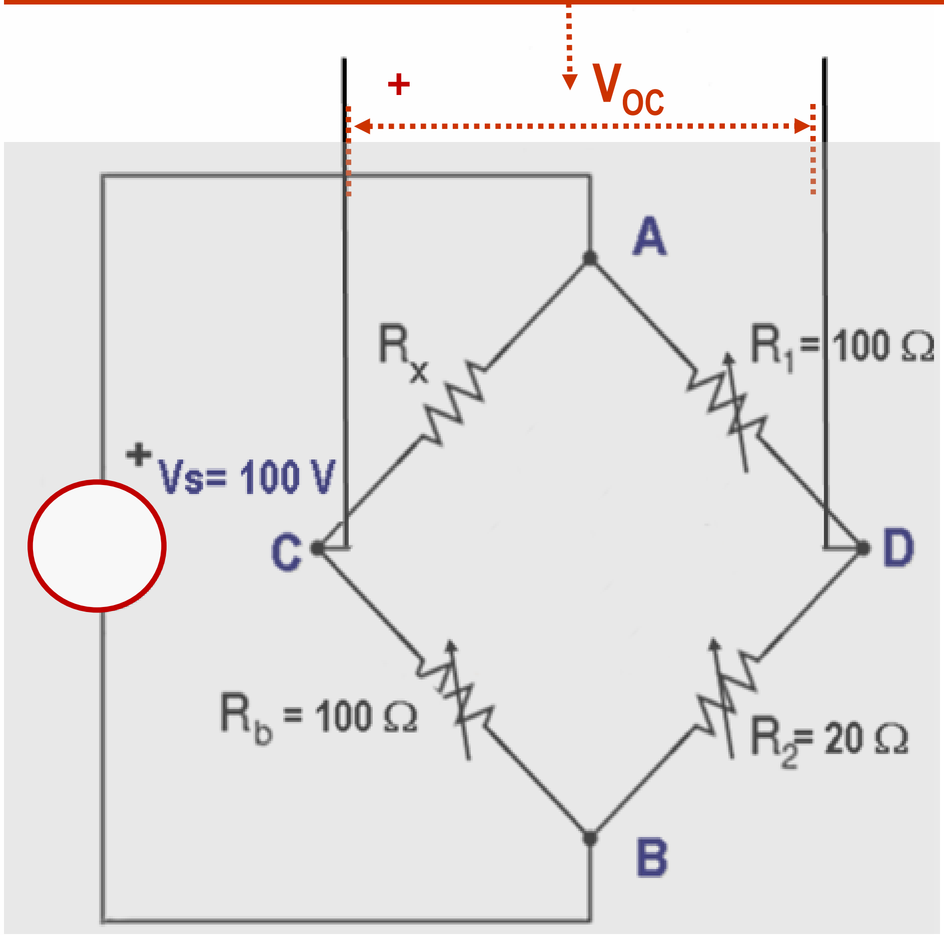
Example 1. Unbalanced Wheatstone Bridge

Solution

Determine the Thevenin Equivalent of the source side of the circuit, i.e. the rest of the circuit after the ammeter and 50 Ohm resistance are taken out

2. Restore back the source,
3. Open circuit the terminals C and D and calculate the Thevenin Equivalent Voltage

Thevenin Equivalent Voltage



Example 1. Unbalanced Wheatstone Bridge

Solution

Determine the Thevenin Equivalent of the source side of the circuit, i.e. the rest of the circuit after the ammeter and 50 Ohm resistance are taken out

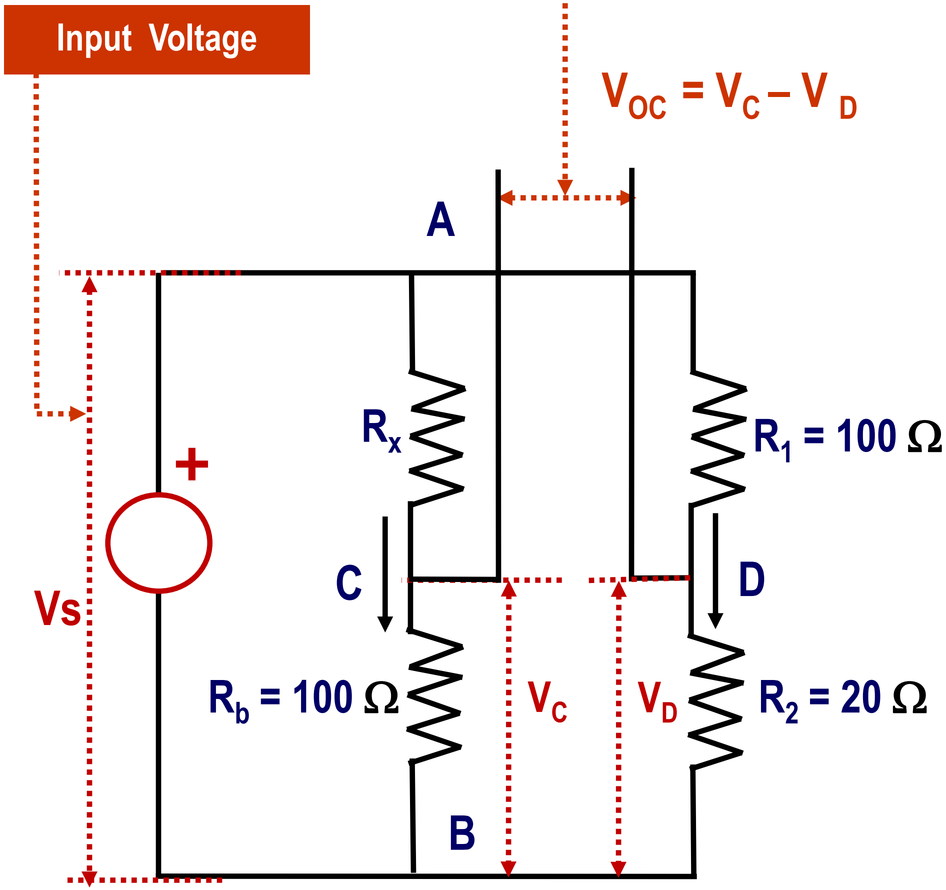
2. Restore back the source,
3. Open circuit the terminals C and D and calculate the Thevenin Equivalent Voltage

$V_C = 100\ V \times 100 /(100 + R_x)$

$V_D = 100\ V \times 20 / (100 + 20)$

$V_{OC} = V_C - V_D = 100\ (100/(100+R_x) - 100/6)$

Thevenin Equivalent Voltage



Example 1. Unbalanced Wheatstone Bridge

Solution

Draw the Thevenin Equivalent Circuit

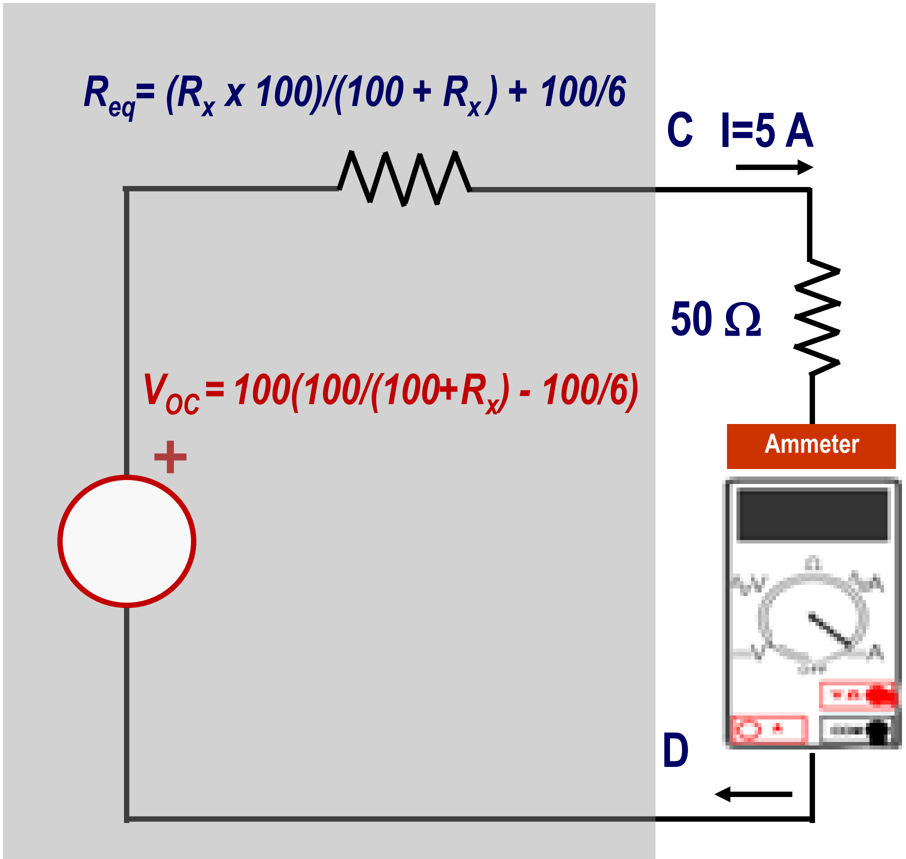
$V_{OC} = V_C - V_D = 100(100/(100+R_x)-100/6)$

$R_{eq} = (R_x x100)/(100 + R_x) + 100/6$

$I = V_{OC} / (R_{eq} + 50 \text{ Ohm})$
$= (100(100/(100 + R_x) - 100/6)) / (R_{eq} + 50)$
$= 5 \text{ Amp}$

Solve this equation for R_x

Homework



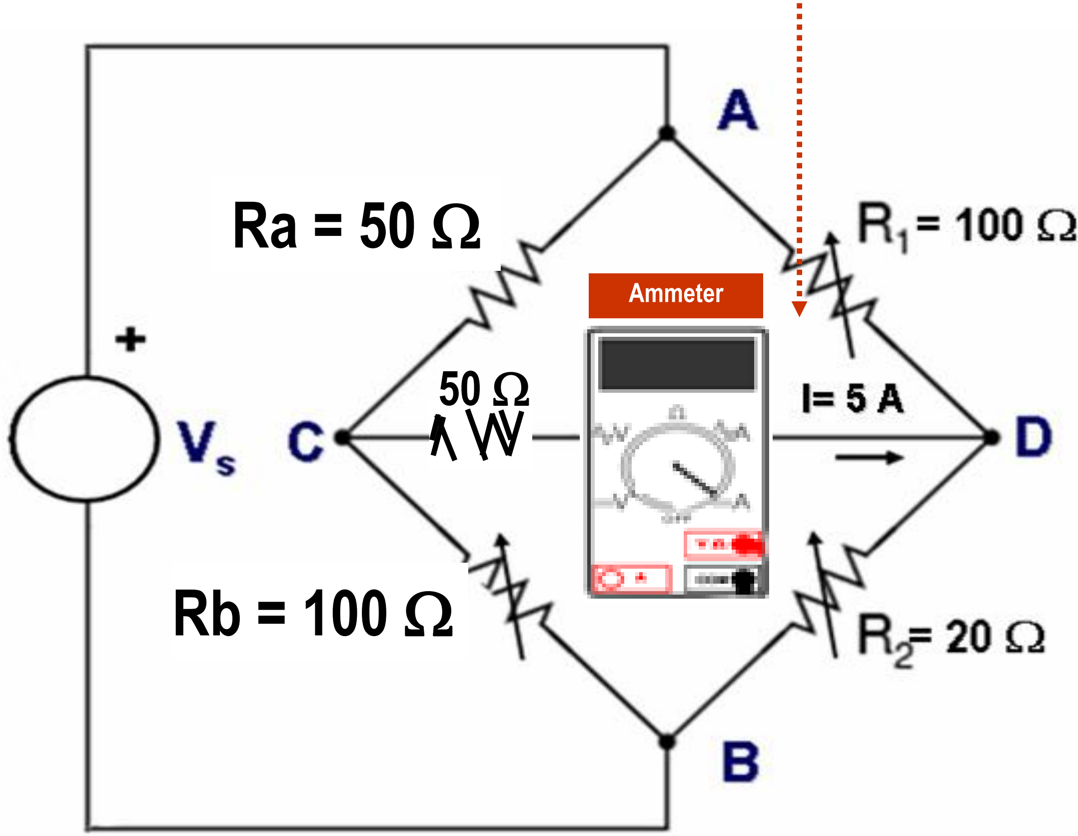
Example 2. Unbalanced Wheatstone Bridge

Example

Calculate the source voltage V_s by using the Thevenin Equivalent Circuit of the unbalanced Wheatstone Bridge shown on the RHS

Since the bridge is unbalanced, cross multiplication of branches are NOT equal, hence 5 Amp passes through the ammeter

Please note that the bridge is unbalanced, i.e. current flows in the ammmeter

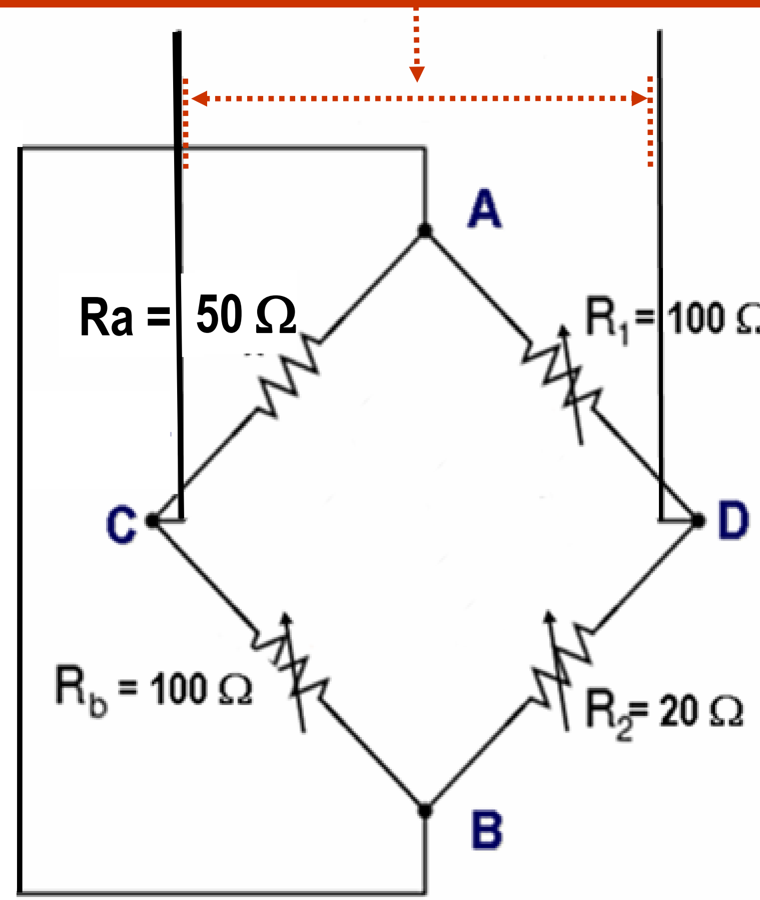


Example 2. Unbalanced Wheatstone Bridge

Solution

First, find the Thevenin Equivalent Resistance of the circuit other than the 50 Ω resistance in the middle of the Bridge

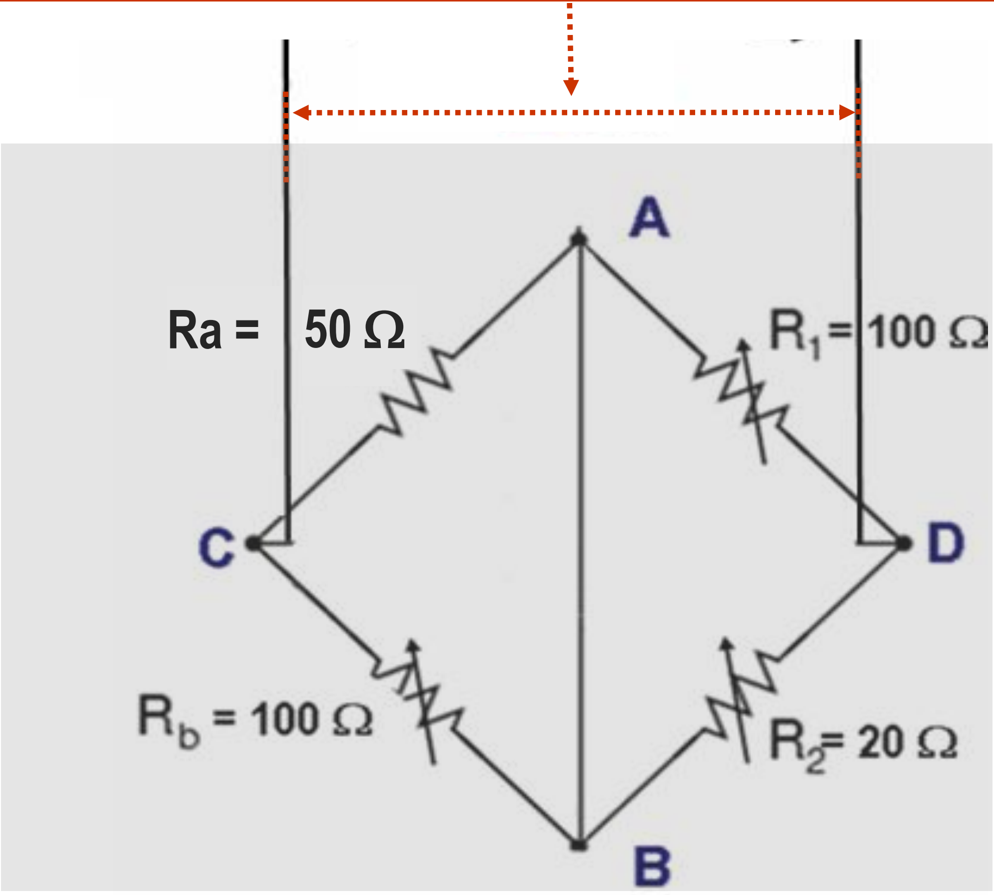
Thevenin Equivalent Resistance



Example 2. Unbalanced Wheatstone Bridge

Solution

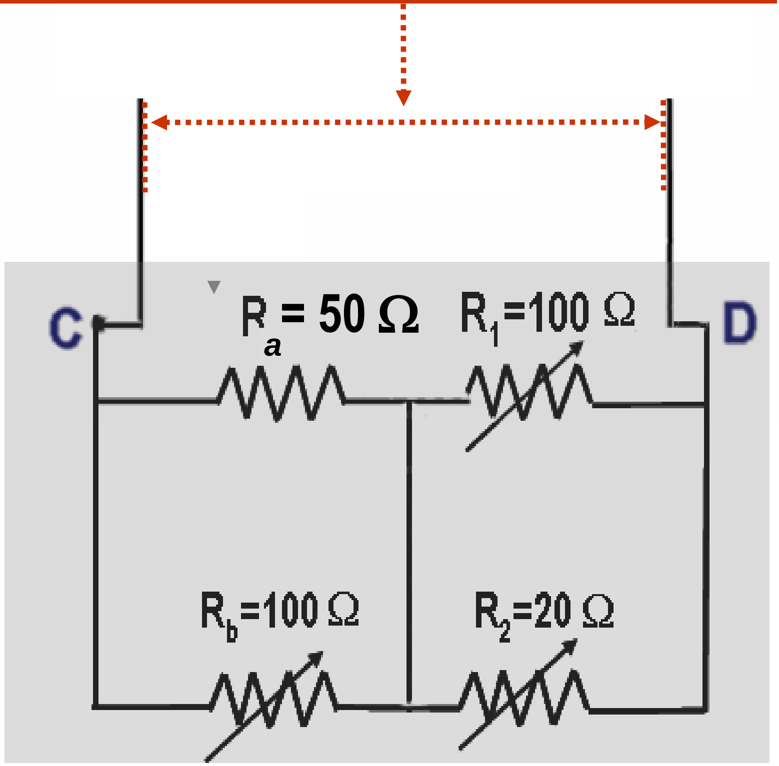
Thevenin Equivalent Resistance



Example 2. Unbalanced Wheatstone Bridge

Solution

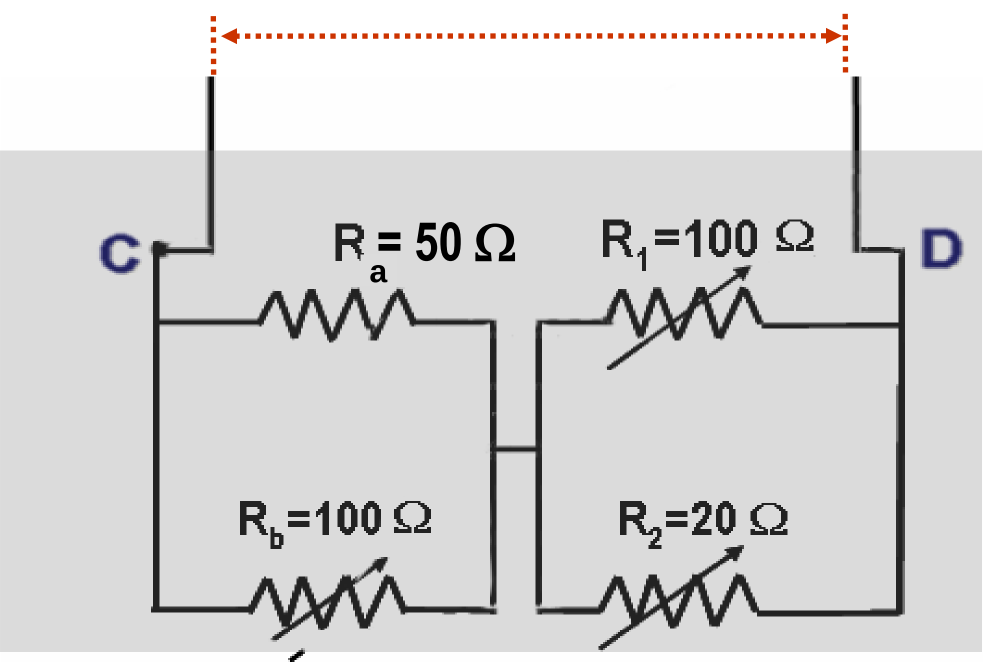
Thevenin Equivalent Resistance



Example 2. Unbalanced Wheatstone Bridge

Solution

Thevenin Equivalent Resistance



$$R_{Thev.} = (R_a \,//\, R_b) + (R_1 \,//\, R_2)$$
$$= (50//100) + (100//20)$$
$$= 33.333 + 16.667 = 50\ \Omega$$

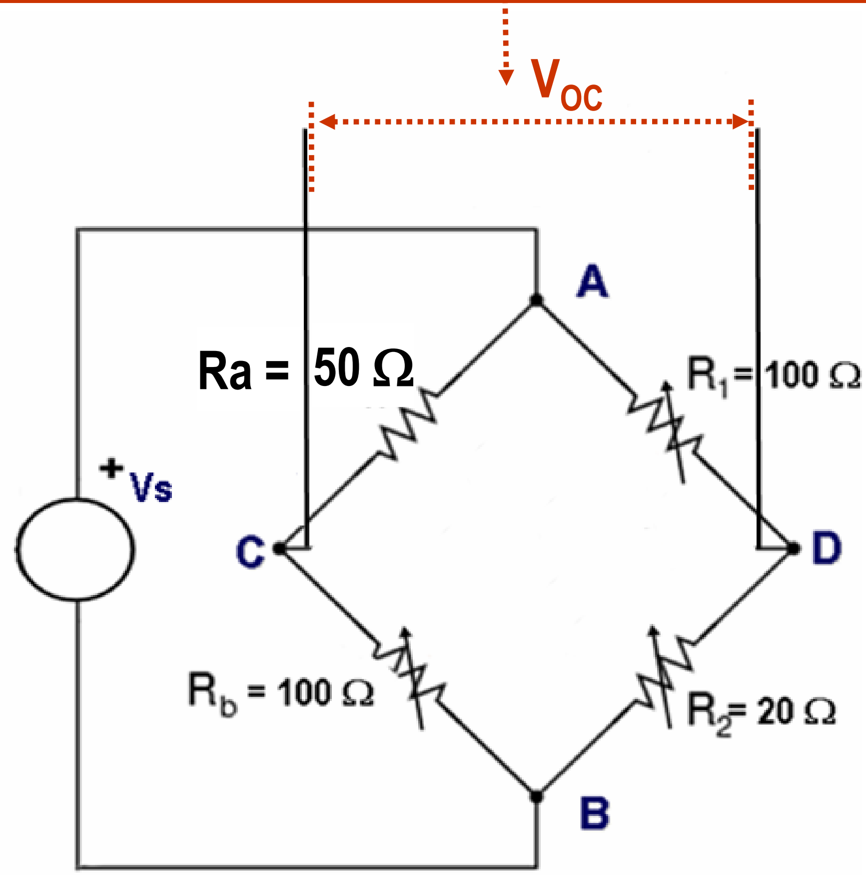
Example 2. Unbalanced Wheatstone Bridge

Solution

Thevenin Equivalent Voltage

Now, find the Thevenin Equivalent Voltage at the terminals C and D

1. Put back the source,
2. Open circuit the terminals C and D,
3. Calculate the Thevenin Equivalent Voltage at the terminals C and D



Example 2. Unbalanced Wheatstone Bridge

Solution

Now, find the Thevenin Equivalent Voltage at the terminals C and D

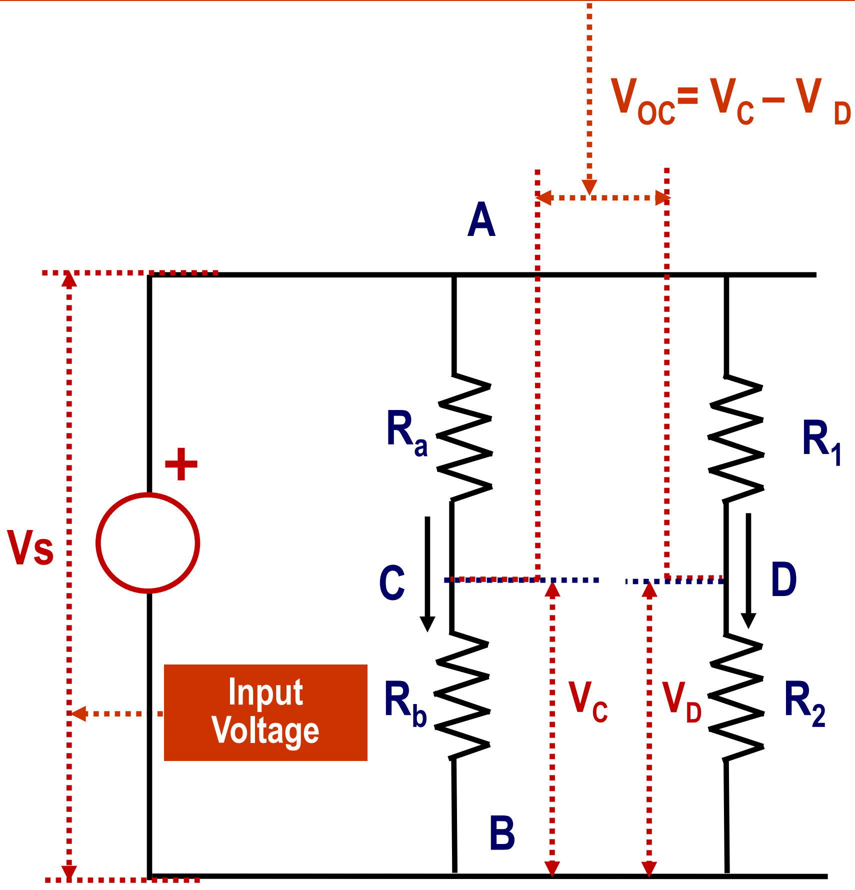
1. Put back the source,
2. Open circuit the terminals C and D,
3. Calculate the Thevenin Equivalent Voltage at the terminals C and D

$V_C = V_s\ 100 /(100 + 50) = (2/3)\ V_s$

$V_D = V_s \times 20 / (100 + 20) = (2 / 12)\ V_s = V_s / 6$

$V_{OC} = V_C - V_D = V_s\ (2/3 - 1/6) = V_s / 2$ Volts

Thevenin Equivalent Voltage



Example 2. Unbalanced Wheatstone Bridge

Solution

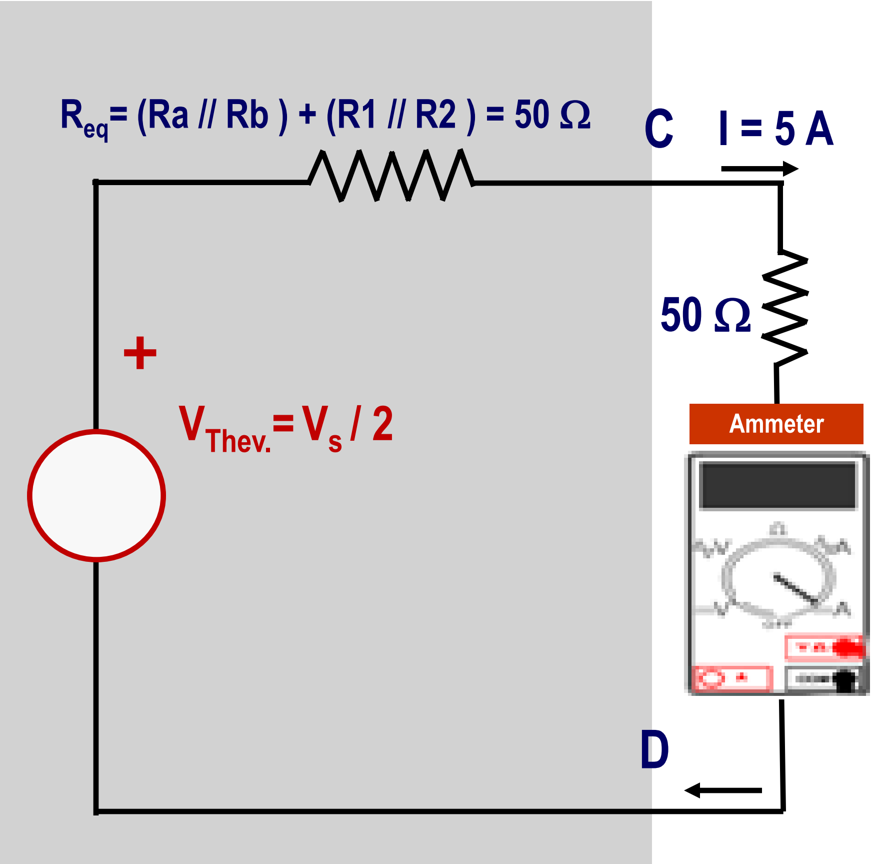
Now connect the resulting Thevenin Equivalent Resistance and Thevenin Equivalent Voltage Source

$I = (V_s / 2) / (50 + 50 \text{ Ohm})$
$= (V_s / 2) / 100 = V_s / 200$
$= 5 \text{ Amp}$

Solve this eq for V_s

Vs = 1000 Volts

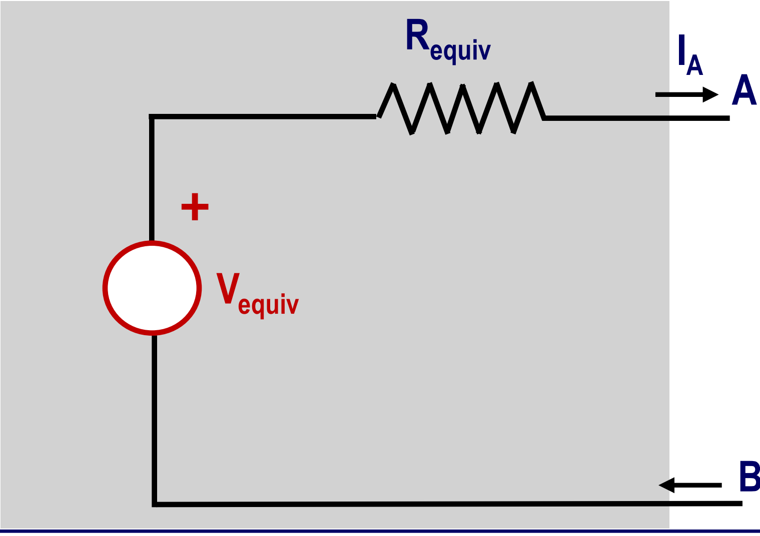
Thevenin Equivalent Circuit

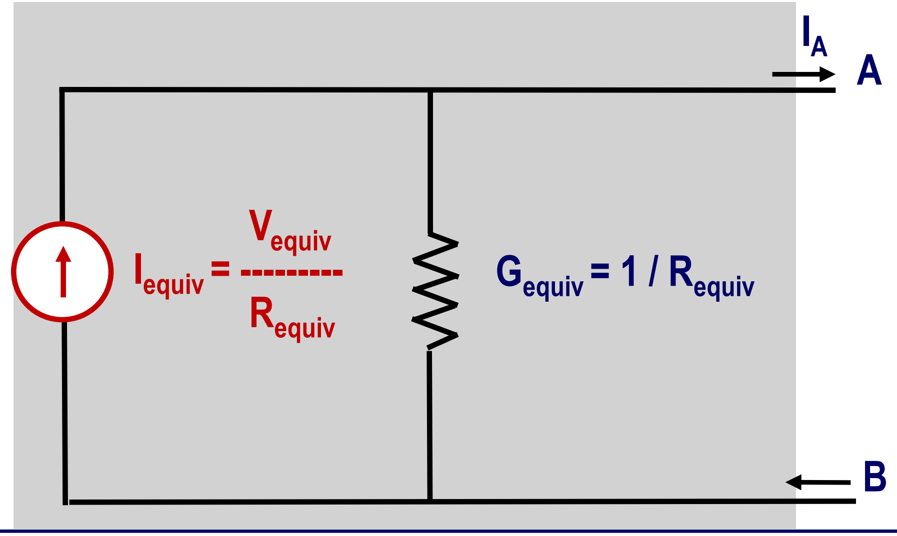


Norton Equivalent Circuit

Thevenin Equivalent Circuit can be converted to an alternative form with a current source I_{equiv} in parallel with an admittance G_{equiv}, called; **"Norton Equivalent Circuit"** or simply **"Norton Form"**

Thevenin Equivalent Circuit	*Norton Equivalent Circuit*



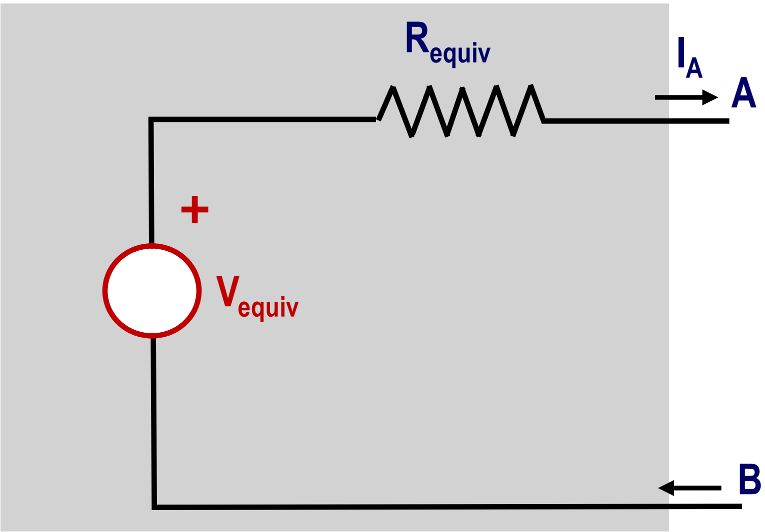


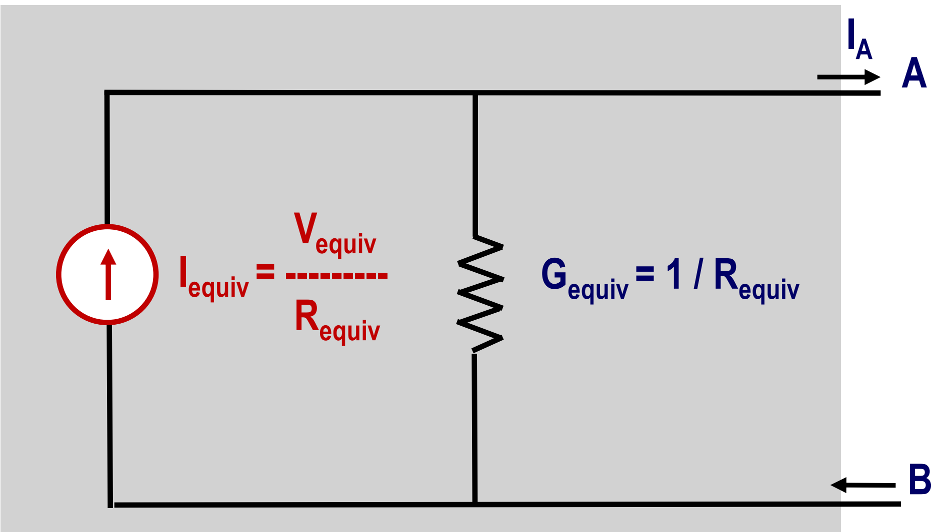
Circuit Analysis

Determination of Norton Equivalent Circuit Parameters

- Divide V_{equiv} by R_{equiv} find I_{equiv},
- Replace V_{equiv} by I_{equiv},
- Calculate G_{equiv} as the reciprocal of R_{equiv},
- Connect G_{equiv} in parallel with I_{equiv}

Thevenin Equivalent Circuit	*Norton Equivalent Circuit*





Circuit Analysis

Example

Question

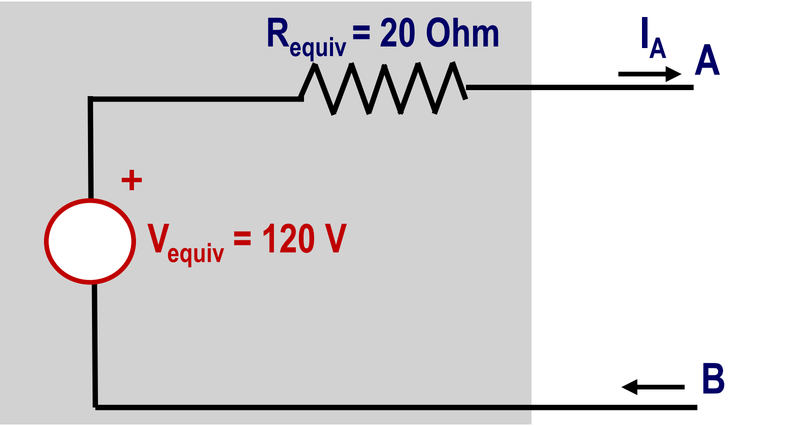
Determine the Norton Equivalent of the Thevenin Equivalent Circuit shown on the RHS

Solution

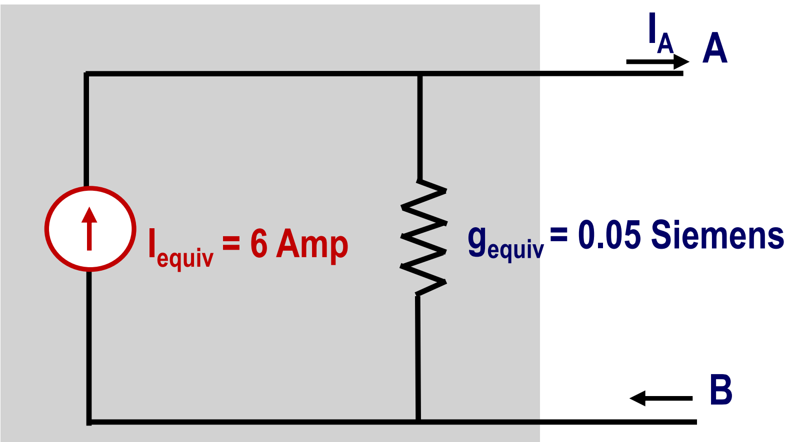
$I_{equiv.} = 120 / 20 = 6 \text{ Amp}$

$G_{equiv.} = 1/R_{equiv.} = 1/20 = 0.05 \text{ Siemens}$

Thevenin Equivalent Circuit



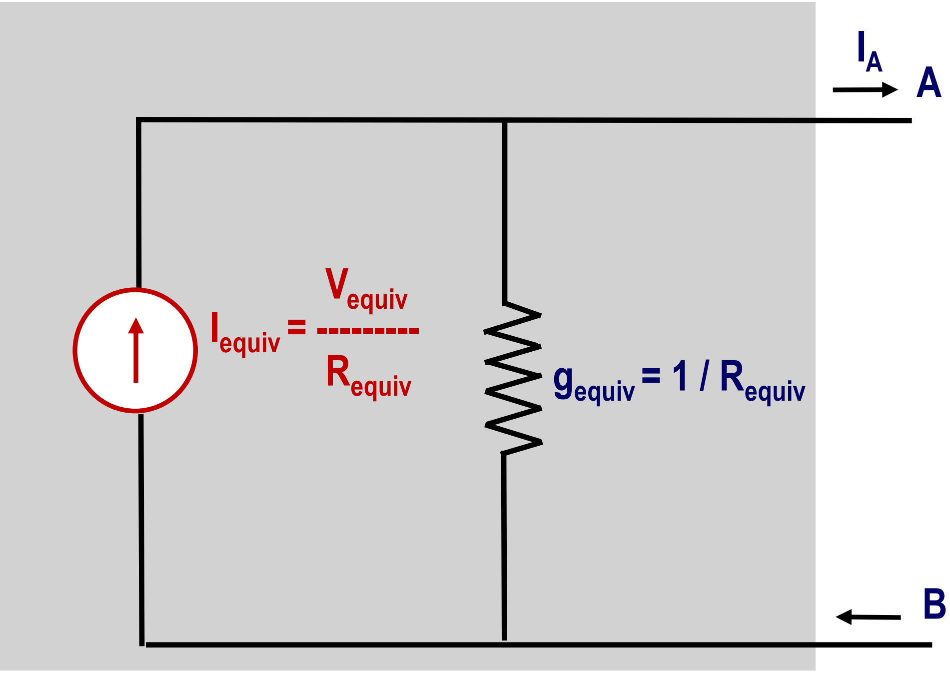
Norton Equivalent Circuit

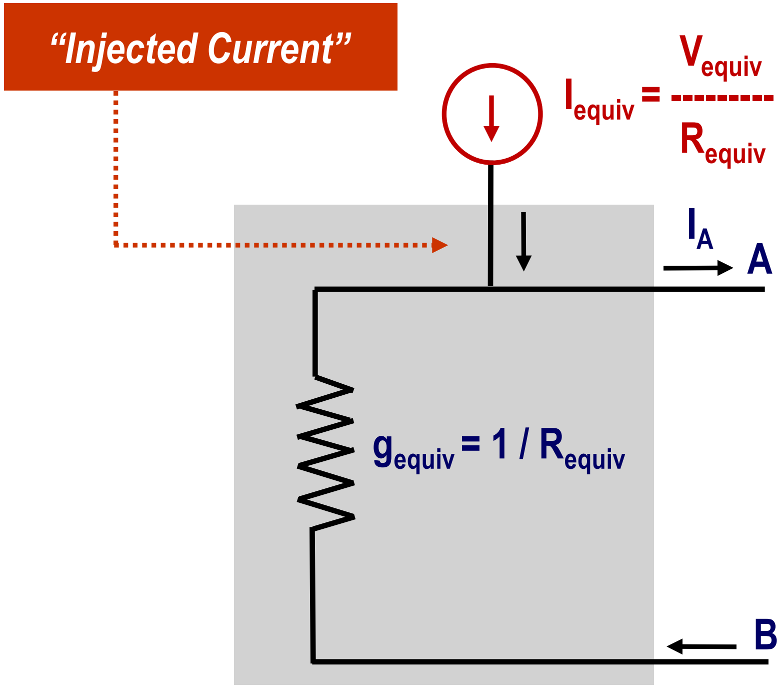


Current Injection Model

Norton equivalent current may be regarded as a current injected from outside, i.e. from the ground node, to the circuit at node A

Norton Equivalent Circuit	*Current Injection Model*



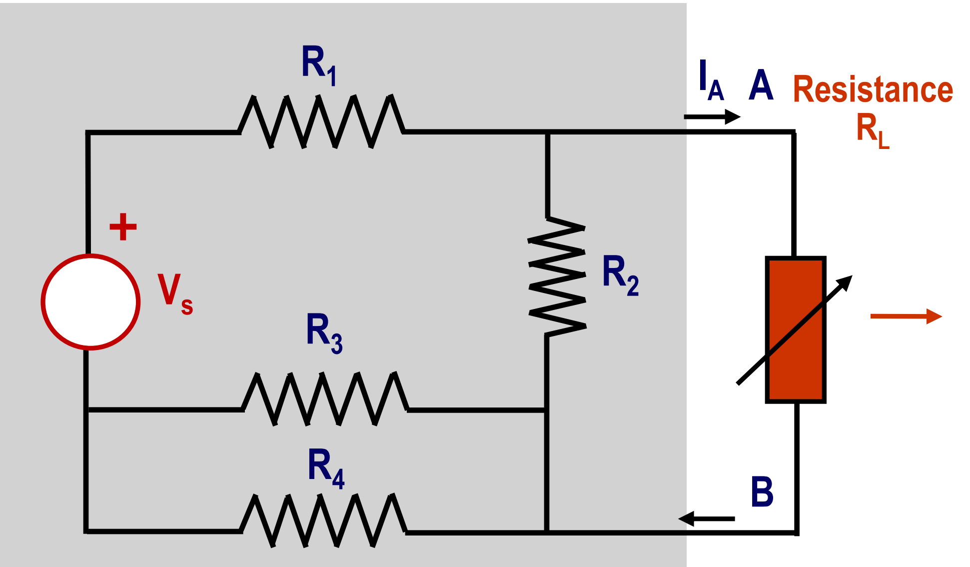


Maximum Power Transfer Condition

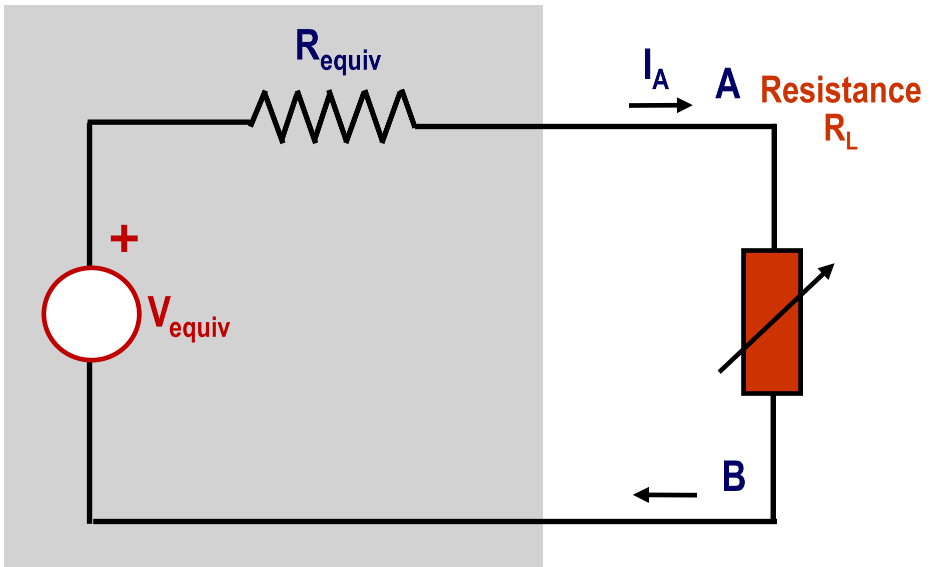
Question:
Determine the value of the resistance in the following circuit in order to transfer maximum power from the source side to the load side

Solution: First simplify the circuit to its Thevenin Equivalent Form as shown on the RHS

Given Circuit



Thevenin Equivalent Circuit



Maximum Power Transfer Condition

Solution: Two Extreme Cases;

Case – 1 (Resistance is short circuited)

$R_L = 0$

In this case the load power will be zero since;

$$P = R_L \times I^2$$
$$= R_L \times (V_{equiv} / (R_{equiv} + 0))^2$$
$$= 0 \times (V_{equiv} / R_{equiv})^2 = 0$$

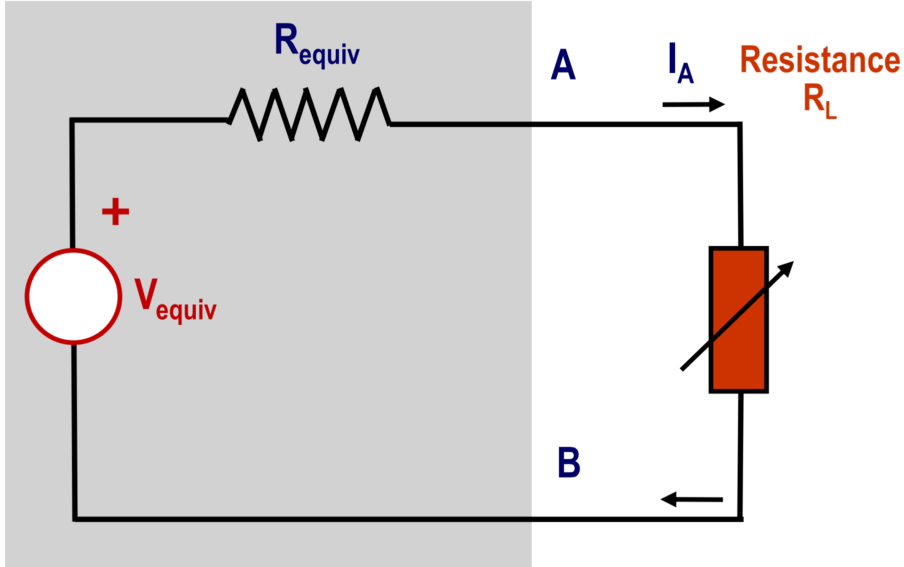
Case – 2 (Resistance is open circuited)

$R_L = \infty$

In this case the load power will again tend to be zero since;

$$P = \infty \times I^2$$
$$= \infty \times (V_{equiv} / (\infty + R_{equiv}))^2 = 0$$

Thevenin Equivalent Circuit

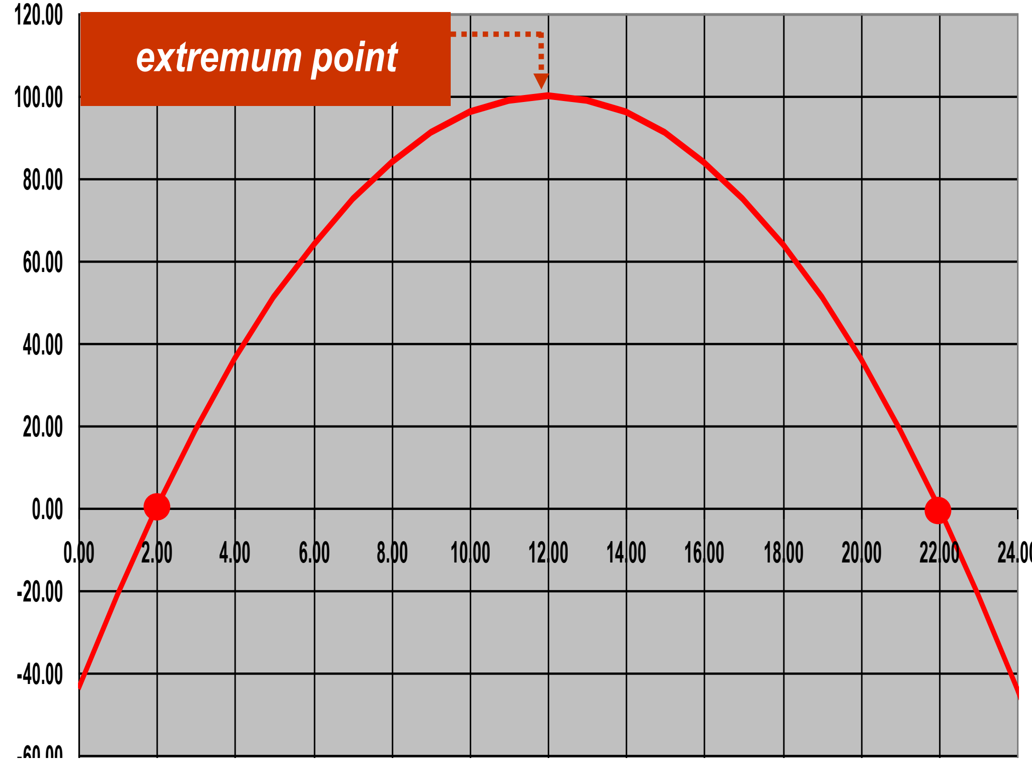


Mathematical Fact

Mathematical Fact

A function passing through zero at two distinct points possesses at least one extremum point in the region enclosed by these points

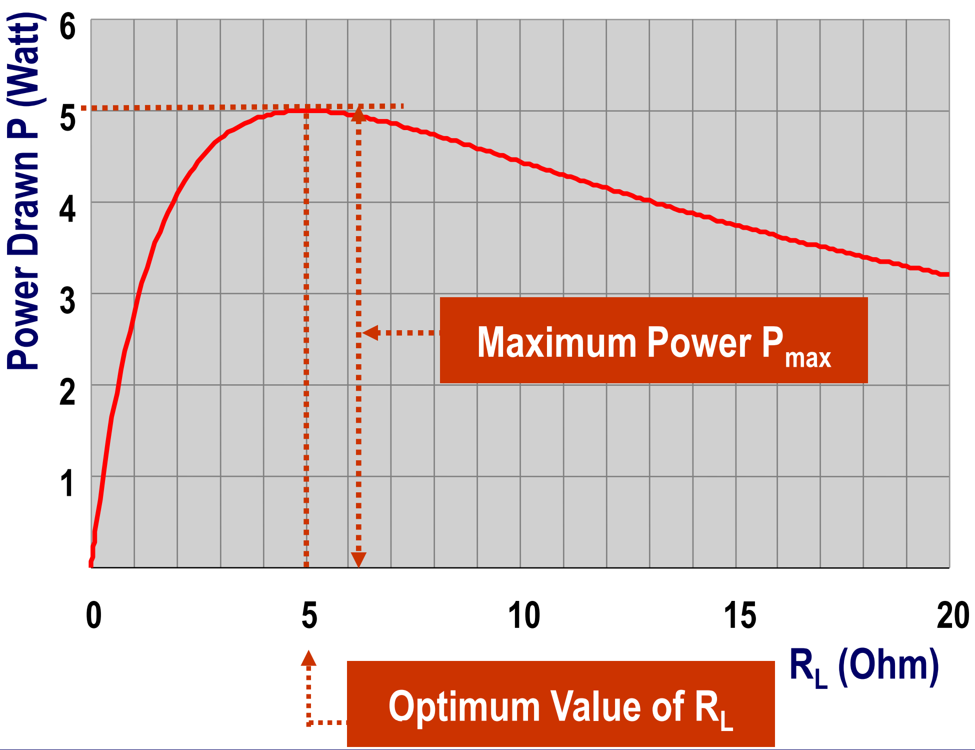
Graphical Illustration



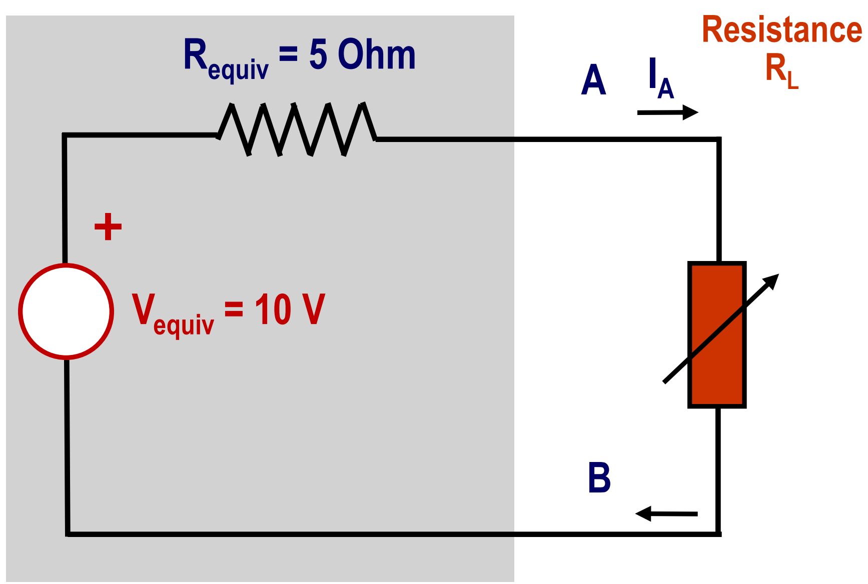
Maximum Power Transfer Condition

Graphical Representation

$$P = R_L \times I^2 = R_L \times (V_{equiv} / (R_{equiv} + R_L))^2$$



Thevenin Equivalent Circuit



Maximum Power Transfer Condition

Solution: Then maximize; $P = R_L I^2$

$P = R_L I^2$

$I^2 = (V_{eq} / R_{total})^2 = (V_{eq} / (R_{eq} + R_L))^2$

Hence,

$P = R_L \; (V_{eq} / (R_{eq} + R_L))^2$

$\;\; = V_{eq}^2 R_L / (R_{eq} + R_L)^2$

Now, maximize P wrt R_L, by differentiating P with respect to R_L

$dP / dR_L = 0$

$d/d \, (R_L \;\; V_{eq}^2 R_L / (R_{eq} + R_L)^2) = 0$

$V_{eq}^2 [(R_{eq} + R_L)^2 - 2 (R_{eq} + R_L) R_L] / denom^2 = 0$

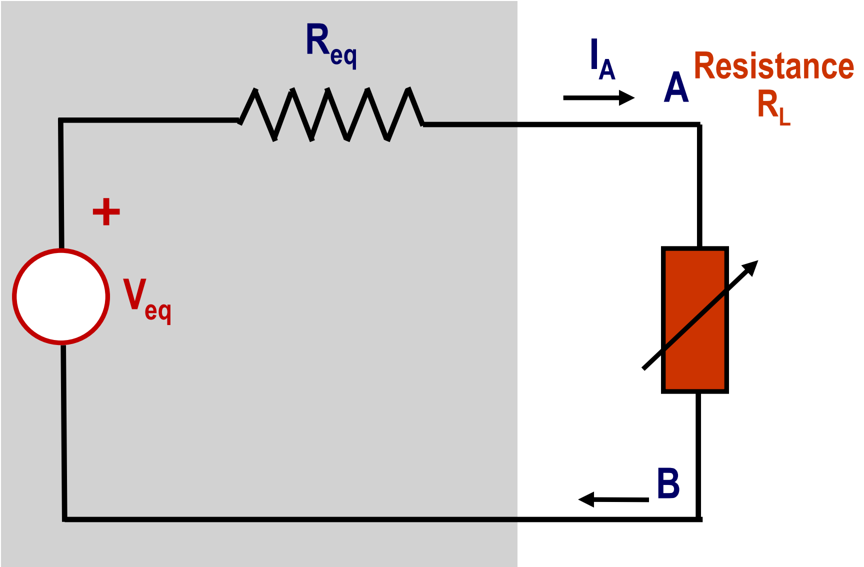
where, $denom = (R_{eq} + R_L)^2$

or

$(R_{eq} + R_L)^2 - 2 (R_{eq} + R_L) R_L = 0$

$\longrightarrow \quad R_{eq} = R_L$

Thevenin Equivalent Circuit



Conclusion:

For maximum power transfer, load resistance R_L must be equal to the Thevenin Equivalent Resistance of the simplified circuit

$R_{eq} = R_L$

Maximum Power Transfer Condition

Why do we need Maximum Power ?

Maximum power means maximum performance and maximum benefit by using the same equipment, and investment,

in other words, maximum speed, or maximum force, or maximum heating, or maximum illumination or maximum performance by using the same equipment, the same weight, and the same investment

Shanghai Maglev Train (World's Fastest Train)

Node (Junction)

Definition

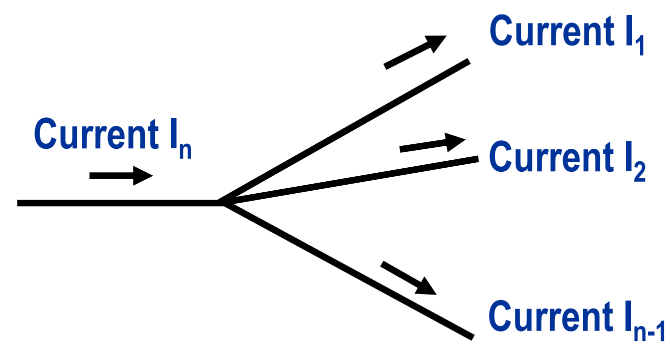
A node is a point at which two or more branches are connected

Basic Rule

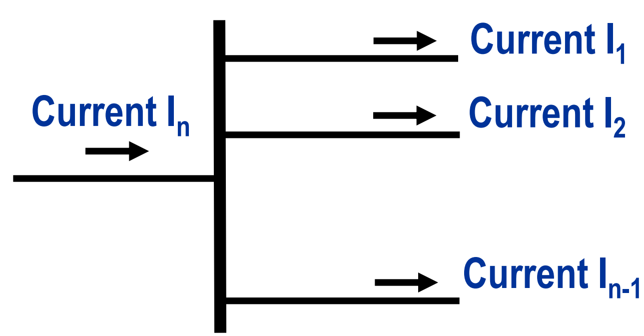
Currents entering a node obey Kirchoff's Current Law (KCL)

$$\sum_{i=1}^{i=n} I_i = 0$$

Circuit Representation of Node (Junction)



Power System Representation of Node (Junction)



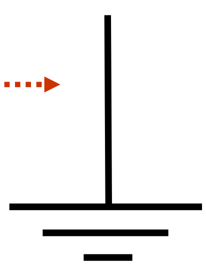
Ground Node (Earth Point)

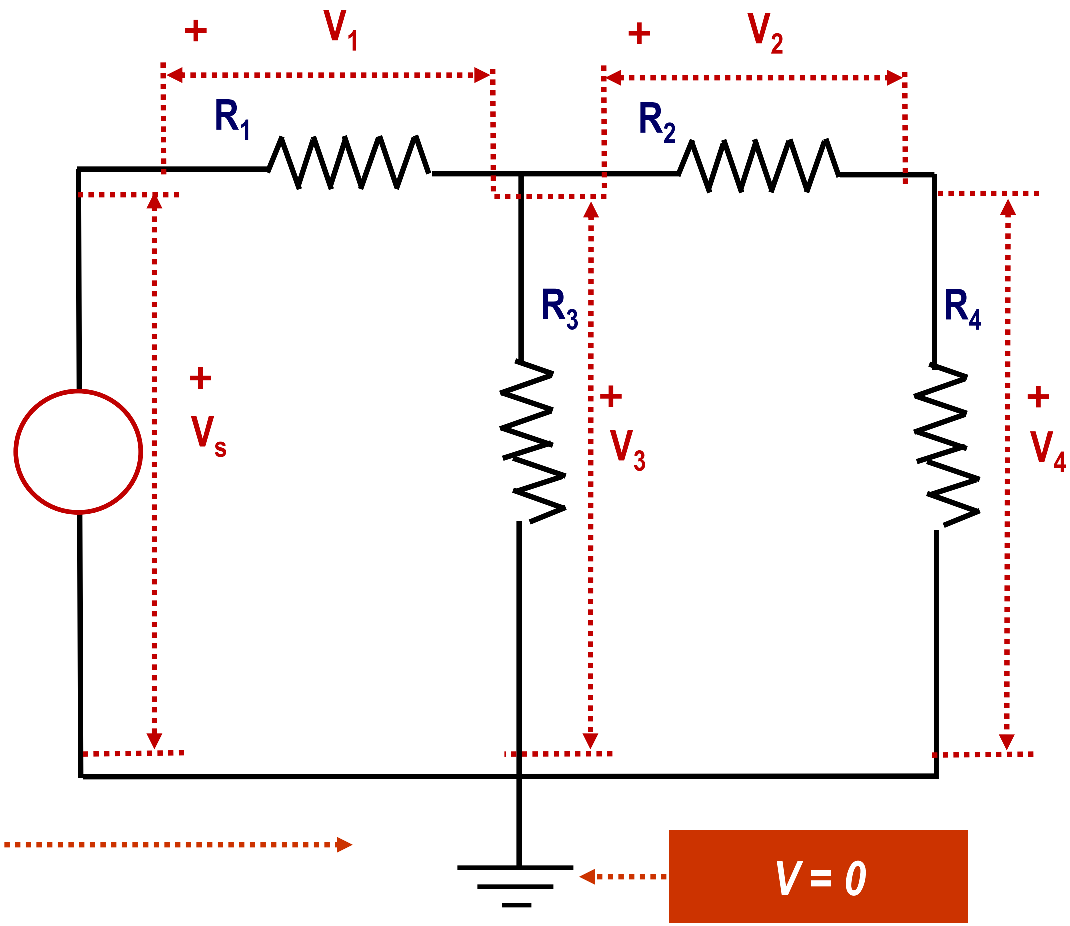
Definition

Ground Node is the point (junction) at which the voltage is assumed to be zero

All other voltages takes their references with respect to this ground node

Representation	*Ground Node*





Ground Node (Earth Point)

Definition

Ground Node is the point (junction) at which the voltage is assumed to be zero

All other voltages takes their references with respect to this ground node

Ground Node (Black Terminal)

Measured Node (Red Terminal)

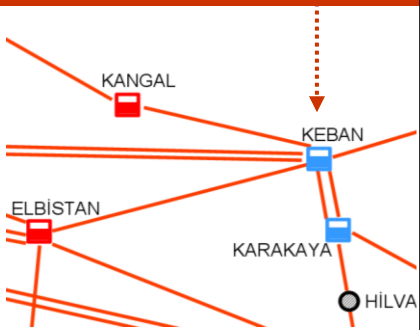
METU

Circuit Analysis

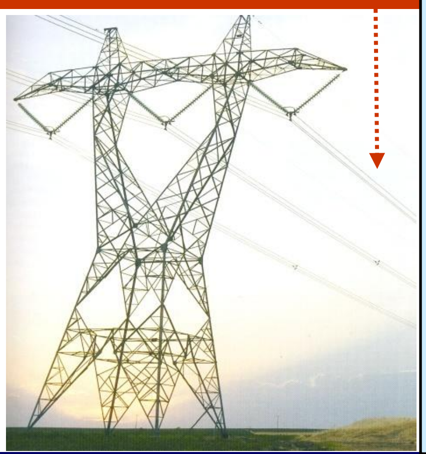
What do we mean by Solution of an Electrical System?

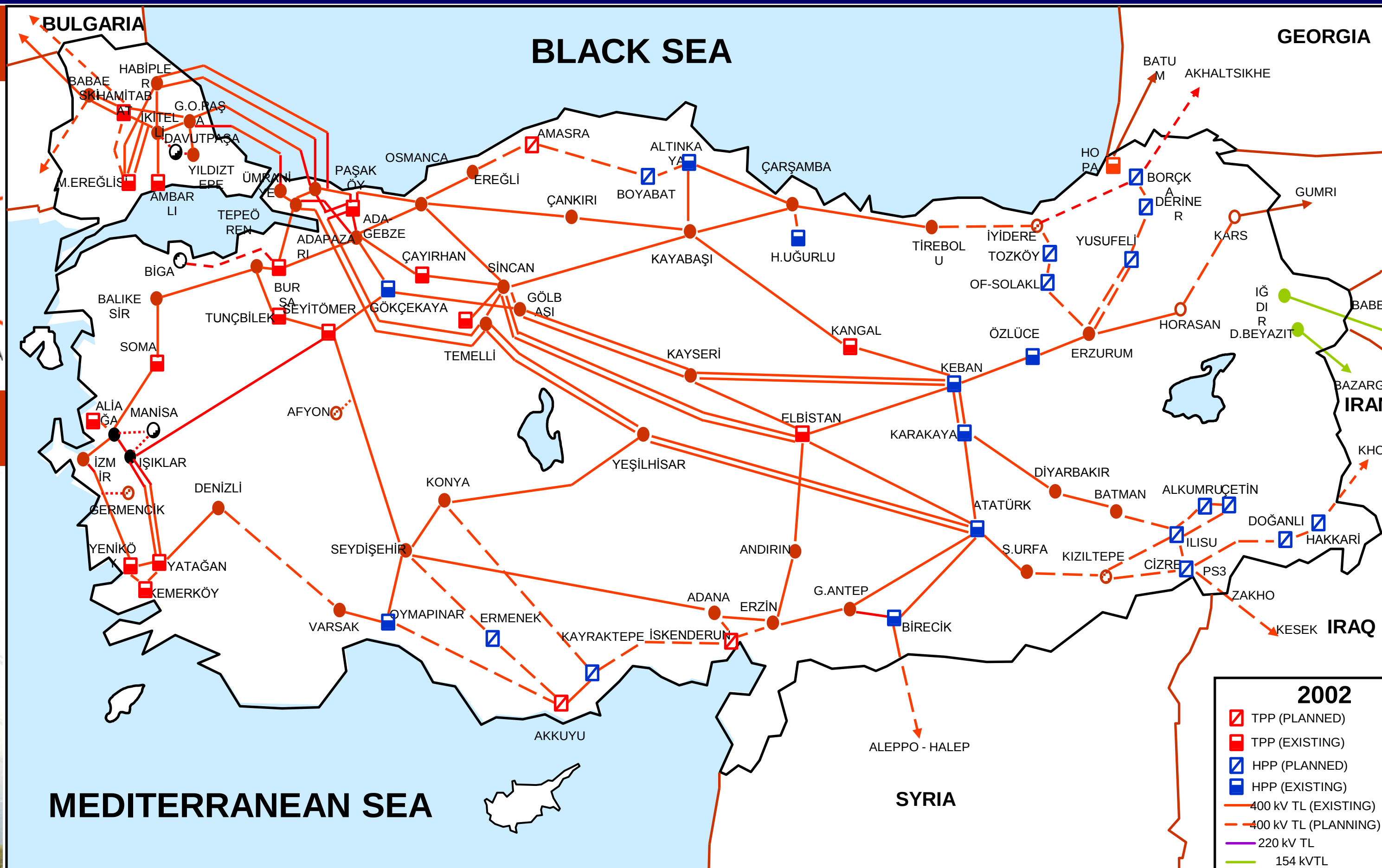
Solution of an electrical system means calculation of all node voltages

Node



Line





Node Voltage Method

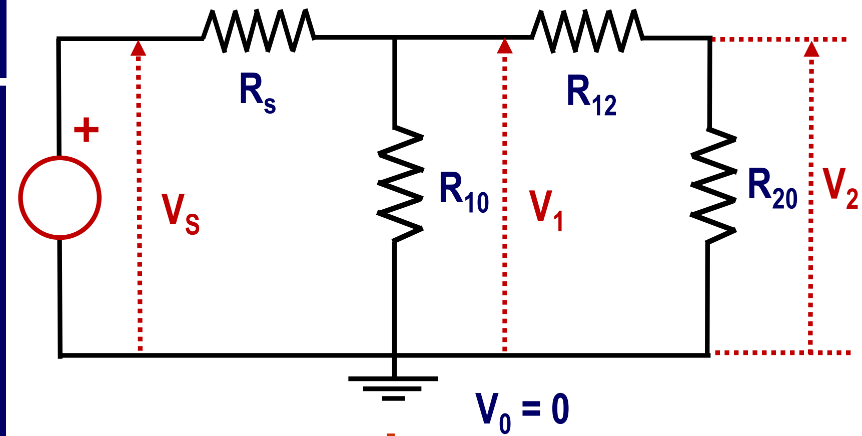
Procedure

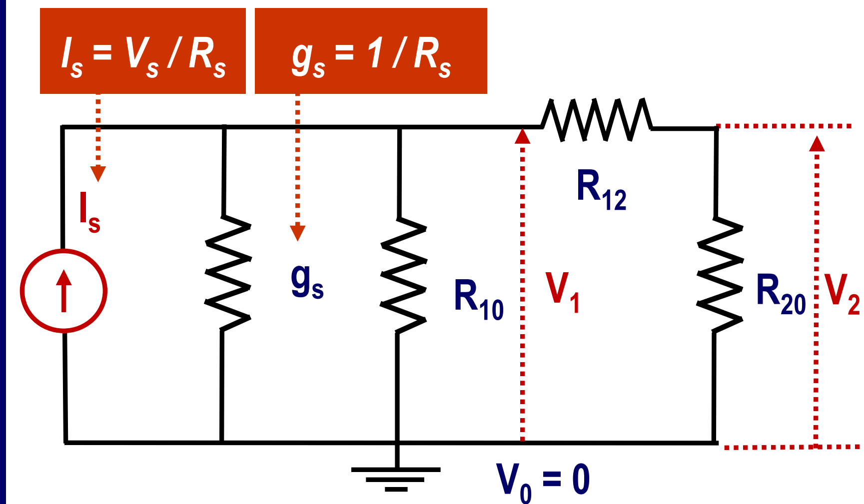
1. Select one of the nodes in the system as the reference (usually the ground node), where voltage is assumed to be zero,
2. Convert Thevenin Equivalent circuits into Norton Equivalent circuits by;
   - Converting source resistances in series with the voltage sources to admittances in parallel with the current sources (injected currents)

     $g_s = 1 / R_s$

   - Converting voltage sources to equivalent current sources, i.e. to equivalent current sources in parallel with admittances,

     $I_s = V_s / R_s = V_s \, g_s$



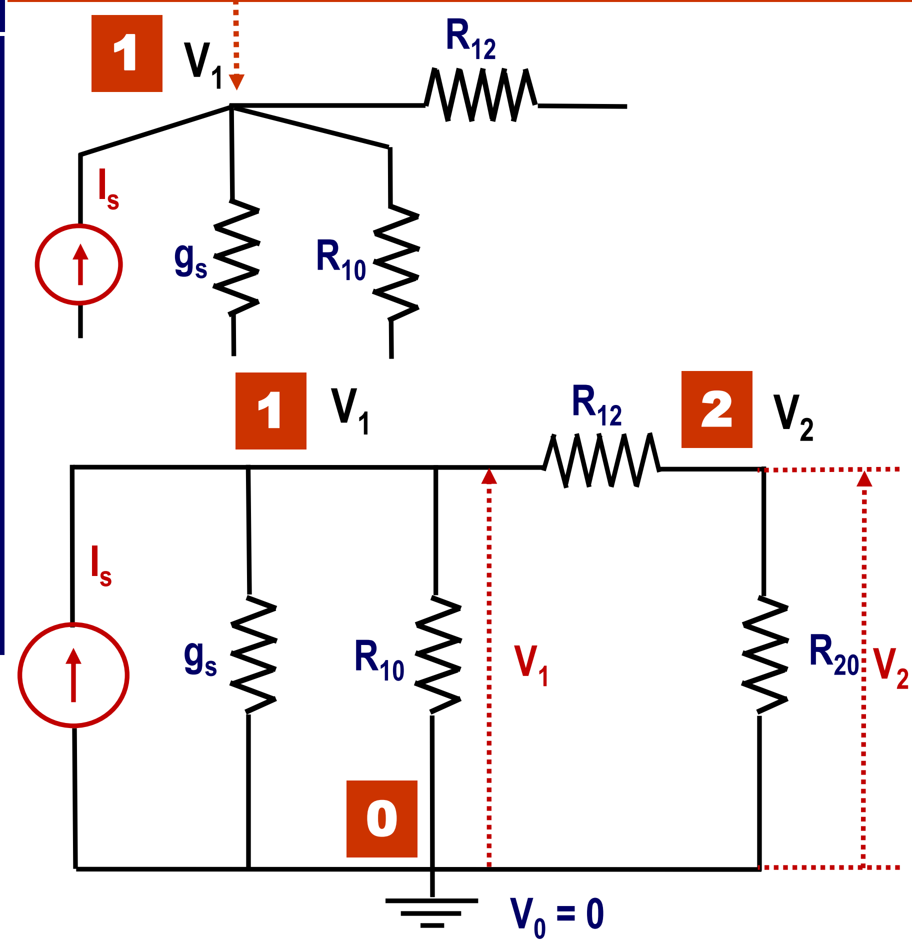


Node Voltage Method

Procedure (Continued)

3. Assign number to each node,
4. Assign zero to ground node, as node the number,
5. Assign voltages V_1, ... V_{n-1} to all nodes except the ground (reference),
6. Set the voltage at the ground node to zero, i.e. $V_0 = 0$,

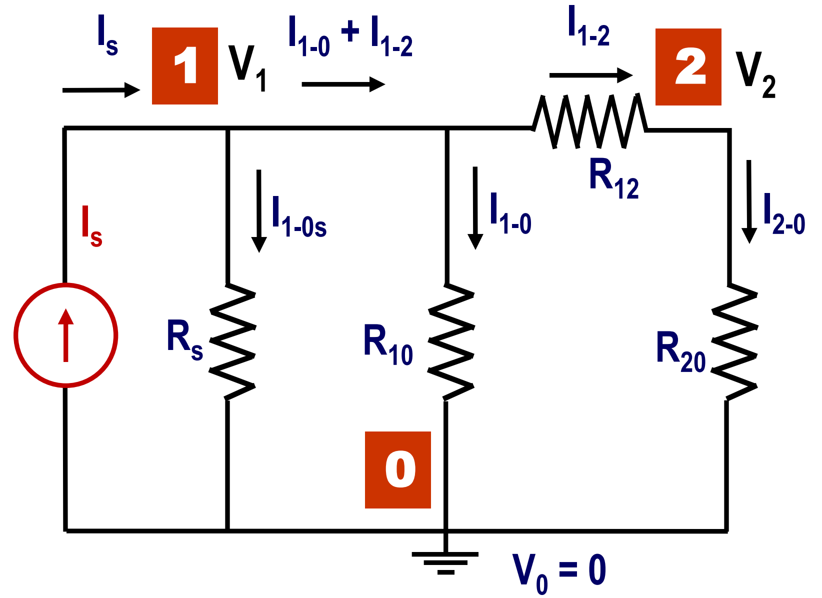
Please note that all these points form a single node



Node Voltage Method

Procedure (Continued)

7. Assign current directions in all branches. <u>(Define the direction of currents in the branches connected to the ground node as always flowing towards the ground),</u>
8. Write-down branch currents in terms of the node numbers at the sending and receiving ends, where the sending and receiving ends are defined with respect to the current directions as defined above and as shown below,



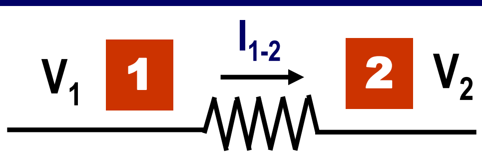
$$I_{1\text{-}2} = (V_1 - V_2)\ / R_{12} = (V_1 - V_2)\, g_{12}$$

$$I_{1\text{-}0} = (V_1 - V_0)\ / R_{10} = V_1\, g_{10}$$

$$I_{1\text{-}0s} = (V_1 - V_0) / R_S = V_1\, g_s$$

$$I_{2\text{-}0} = (V_2 - V_0) / R_{20} = V_2\, g_{20}$$

Sending end

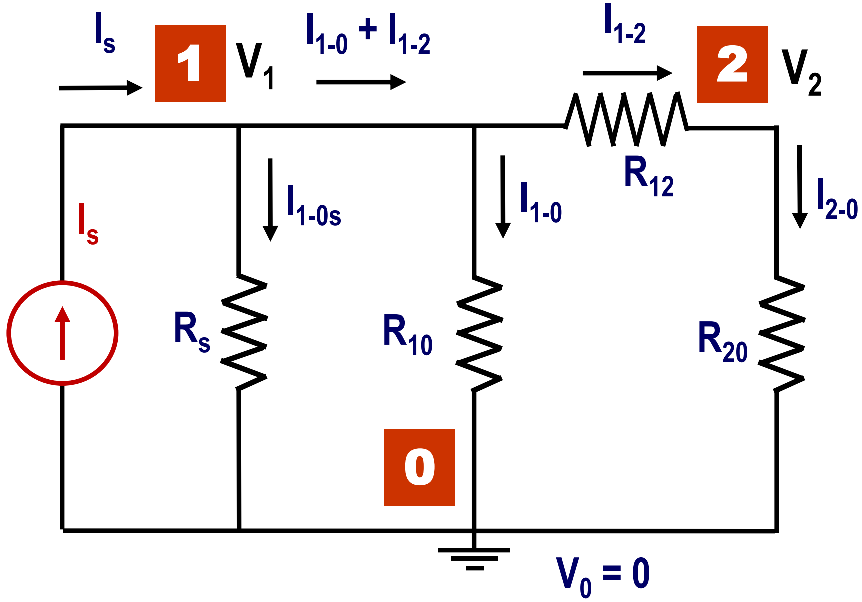


Receiving end

Node Voltage Method

Procedure (Continued)

9. Express branch currents in terms of the voltages at the sending and receiving ends by using Ohm's Law, except those flowing in the current sources (They are already known)



$$I_{1\text{-}2} = (V_1 - V_2)\ / R_{12} = (V_1 - V_2)\, g_{12}$$

$$I_{1\text{-}0} = (V_1 - V_0)\ / R_{10} = V_1\, g_{10}$$

$$I_{1\text{-}0s} = (V_1 - V_0) / R_S = V_1\, g_s$$

$$I_{2\text{-}0} = (V_2 - V_0) / R_{20} = V_2\, g_{20}$$

Node Voltage Method

Procedure (Continued)

Total no. of equations = N - 1

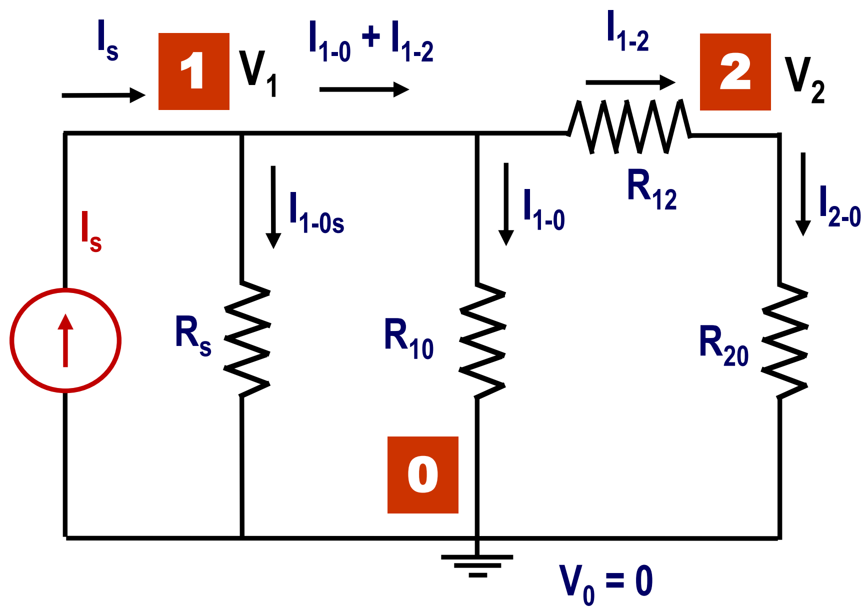
10. Write down KCL at all nodes except the ground (reference) node.
    <u>(Do not write KCL equation for the ground node !)</u>

Number of nodes = N = 3
Number of equations = N-1 = 2

Please note that there are only two unknown voltages, i.e. V_1 and V_2 Hence, KCL equations must be written only at these nodes, i.e. at nodes 1 and node 2

$$I_s = I_{1\text{-}0s} + I_{1\text{-}0} + I_{1\text{-}2}$$

$$I_{1\text{-}2} = I_{2\text{-}0}$$



Node Voltage Method

Procedure (Continued)

$$I_s = I_{1\text{-}0s} + I_{1\text{-}0} + I_{1\text{-}2}$$

$$I_{1\text{-}2} = I_{2\text{-}0}$$

11. Now, substitute the voltage terms into the above equations;

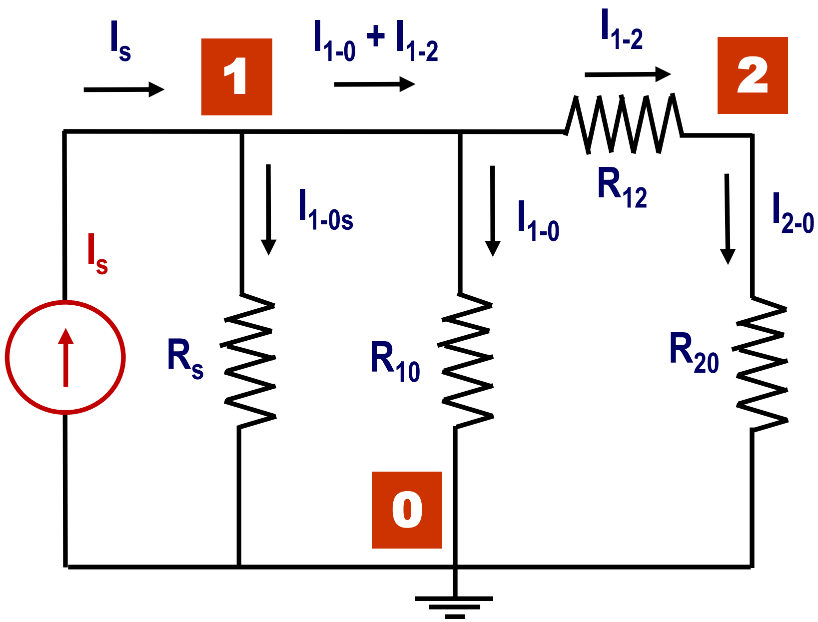
$$I_s = V_s \,/\, R_s = V_s \; g_s$$

$$I_{1\text{-}0s} = (V_1 - V_0) \,/\, R_S = V_1 \, g_s$$

$$I_{1\text{-}0} = (V_1 - V_0) \,/\, R_{10} = V_1 \, g_{10}$$

$$I_{1\text{-}2} = (V_1 - V_2) \,/\, R_{12} = (V_1 - V_2) \, g_{12}$$

$$I_{2\text{-}0} = (V_2 - V_0) \,/\, R_{20} = V_2 \, g_{20}$$



Node Voltage Method

Procedure (Continued)

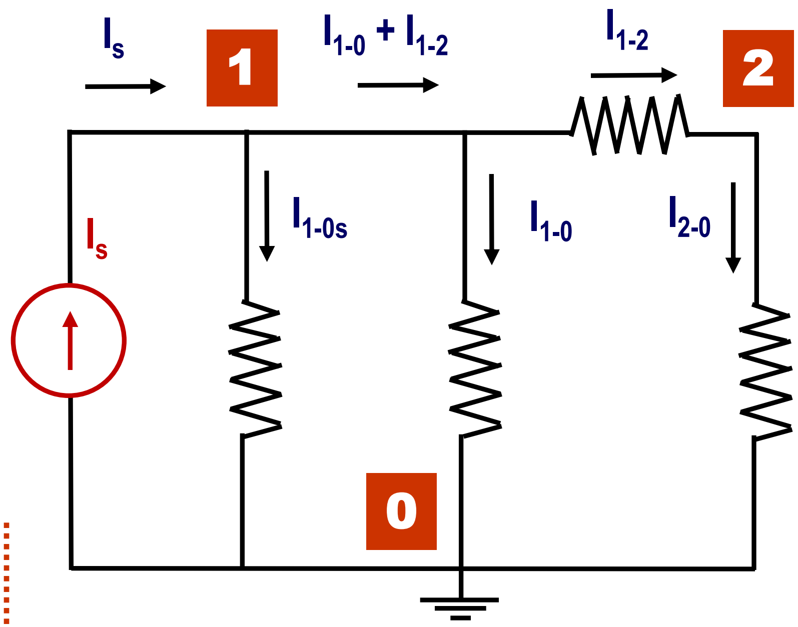
Obtain the following Nodal Equations;

$$I_s = I_{1\text{-}0s} + I_{1\text{-}0} + I_{1\text{-}2}$$

$$I_{1\text{-}2} = I_{2\text{-}0}$$

$$V_S\, g_s = V_1\, g_s + V_1\, g_{10} + (V_1 - V_2)\, g_{12}$$

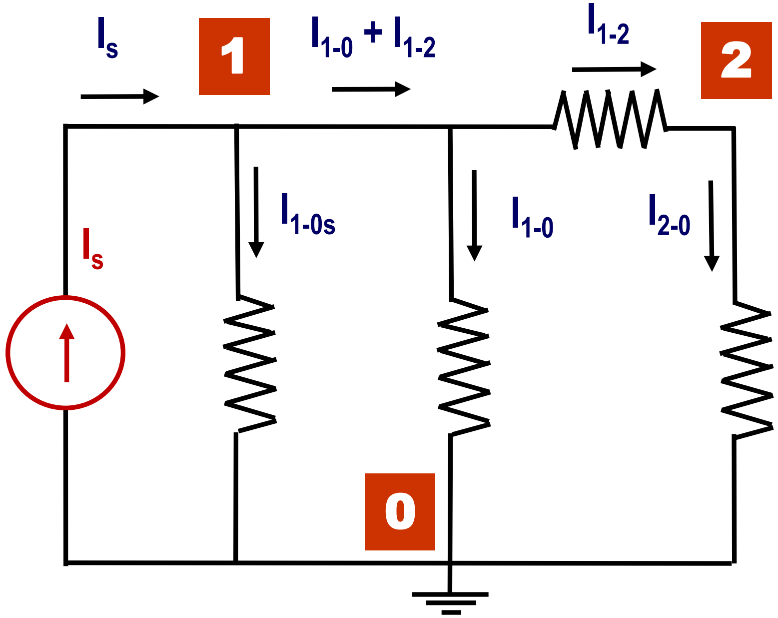
$$(V_1 - V_2)\, g_{12} = V_2\, g_{20}$$



Nodal Equations (Two equations vs two unknowns)
V_1 and V_2 are unknows, all other are knowns

Node Voltage Method

Procedure (Continued)



or, rearranging;

$$V_1\, g_s + V_1\, g_{10} + (V_1 - V_2)\, g_{12} = V_s\, g_s$$

$$-(V_1 - V_2)\, g_{12} + V_2\, g_{20} = 0$$

or **Nodal Equations**

$$\begin{bmatrix} g_s + g_{10} + g_{12} & -g_{12} \\ -g_{12} & g_{12} + g_{20} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_s\, g_s \\ 0 \end{bmatrix}$$

Node Voltage Method

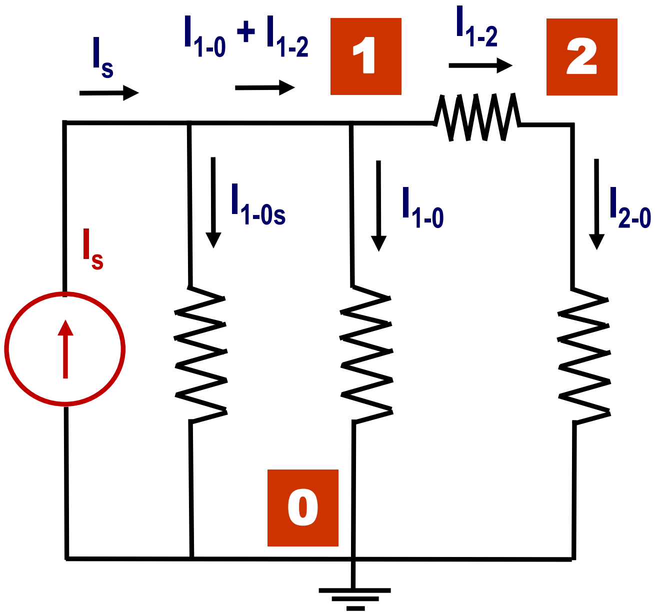
Procedure (Continued)

$$\begin{bmatrix} g_s + g_{10} + g_{12} & -g_{12} \\ -g_{12} & g_{12} + g_{20} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_s\, g_s \\ 0 \end{bmatrix}$$

Nodal admittance matrix

Voltage (Unknown) vector

Injected Current (Known) (RHS) vector



9. Solve the resulting nodal equations for node voltages

A Simple Rule for Forming Nodal Admittance Matrix

Rule

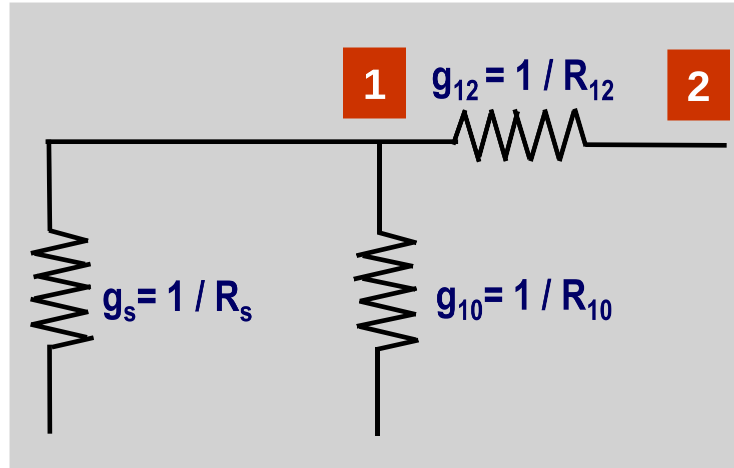
$$\begin{array}{cc} 1 & 2 \\ \left[\begin{array}{c:c} g_s + g_{10} + g_{12} & -g_{12} \\ \hdashline -g_{12} & g_{12} + g_{20} \end{array}\right] \end{array}$$

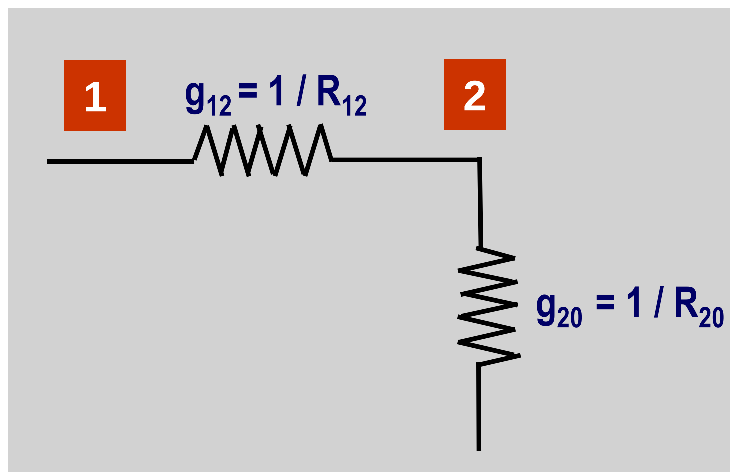
Symmetrical

- Find the admittances of the branches in the circuit by calculating the inverse of resistances;

$$g_{12} = 1 / R_{12}$$

- Put the summation of admittances of those branches connected to the i-th node to the i-th diagonal element of the nodal admittance matrix
- Put the negative of the admittance of the branch connected between the nodes i and j to the i - jth and j-ith element of the nodal admittance matrix



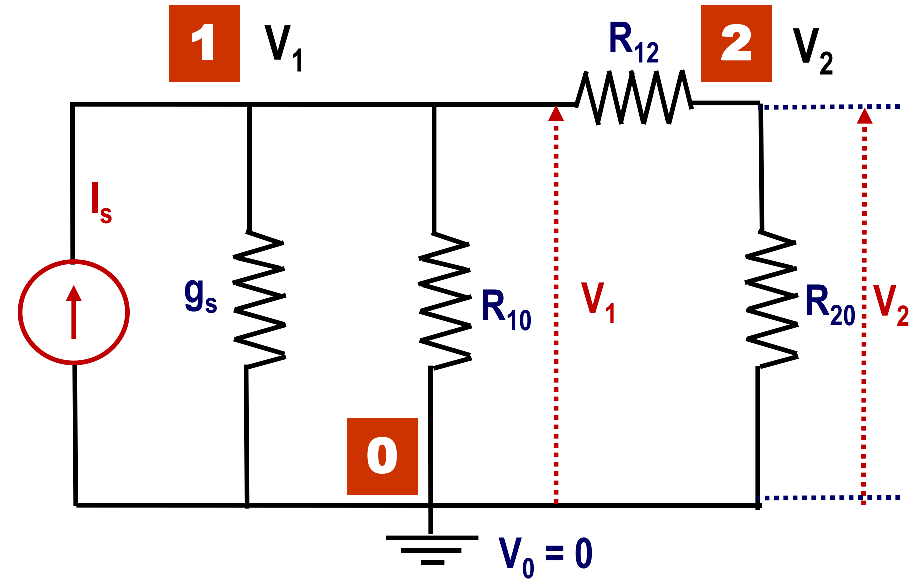


A Simple Rule for Forming Node Voltage Vector

Rules

$$\begin{array}{c} 1 \\ 2 \end{array} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

- Write down the unknown node voltages in this vector in sequence starting from 1 to n-1 (i.e. exclude the voltage of the reference node. Voltage of the reference node is assumed to be zero)



A Simple Rule for Forming Current Injection Matrix

Rules

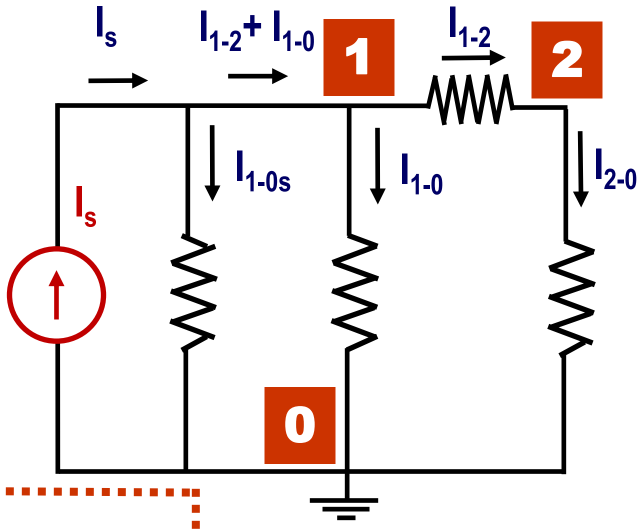
$$\begin{array}{c} 1 \\ 2 \end{array} \begin{bmatrix} I_{1\text{-injected}} \\ 0 \end{bmatrix} = \begin{bmatrix} V_S\, g_s \\ 0 \end{bmatrix}$$

- Write down the injected currents in this vector,
- Write down = $\begin{cases} V_S\, g_s & \text{if there is an injection,} \\ 0 & \text{otherwise} \end{cases}$

Solution of Nodal Equations

Procedure

$$\begin{bmatrix} g_s + g_{10} + g_{12} & -g_{12} \\ -g_{12} & g_{12} + g_{20} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_s\, g_s \\ 0 \end{bmatrix}$$



Methods for finding the node voltages

Substitution Method

Write down the above equations as a set of linear equations and solve them by using the known *"Substitution and Elimination Technique"* seen in the high school

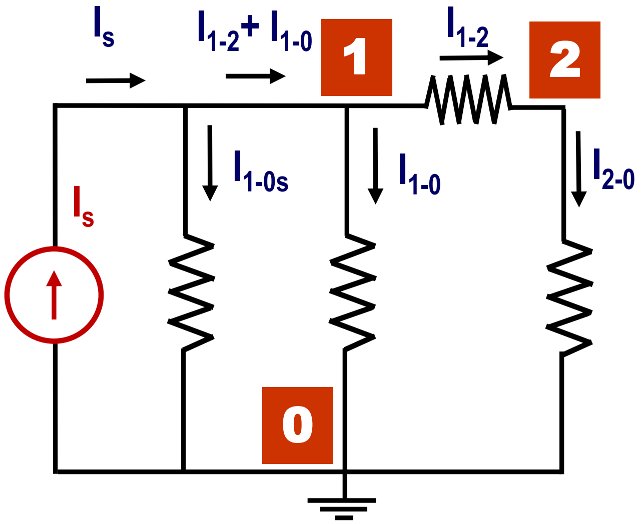
Matrix Methods

Read the next page

Software Packages

Use computer

Solution of Nodal Equations



Procedure (Continued)

$$\begin{bmatrix} g_s + g_{10} + g_{12} & -g_{12} \\ -g_{12} & g_{12} + g_{20} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_S\, g_s \\ 0 \end{bmatrix}$$

<u>To find the node voltages</u>

<u>Matrix Methods</u>

- Invert the nodal admittance matrix,
- Multiply the RHS vector by this inverse

Calculation of Inverse of a 2x2 Matrix

Procedure (Continued)

<u>To find the inverse of a 2 x 2 matrix</u>

1. First calculate the determinant of the given matrix;

   Determinant $= a_{11} \times a_{22} - a_{21} \times a_{12}$
   $= d$

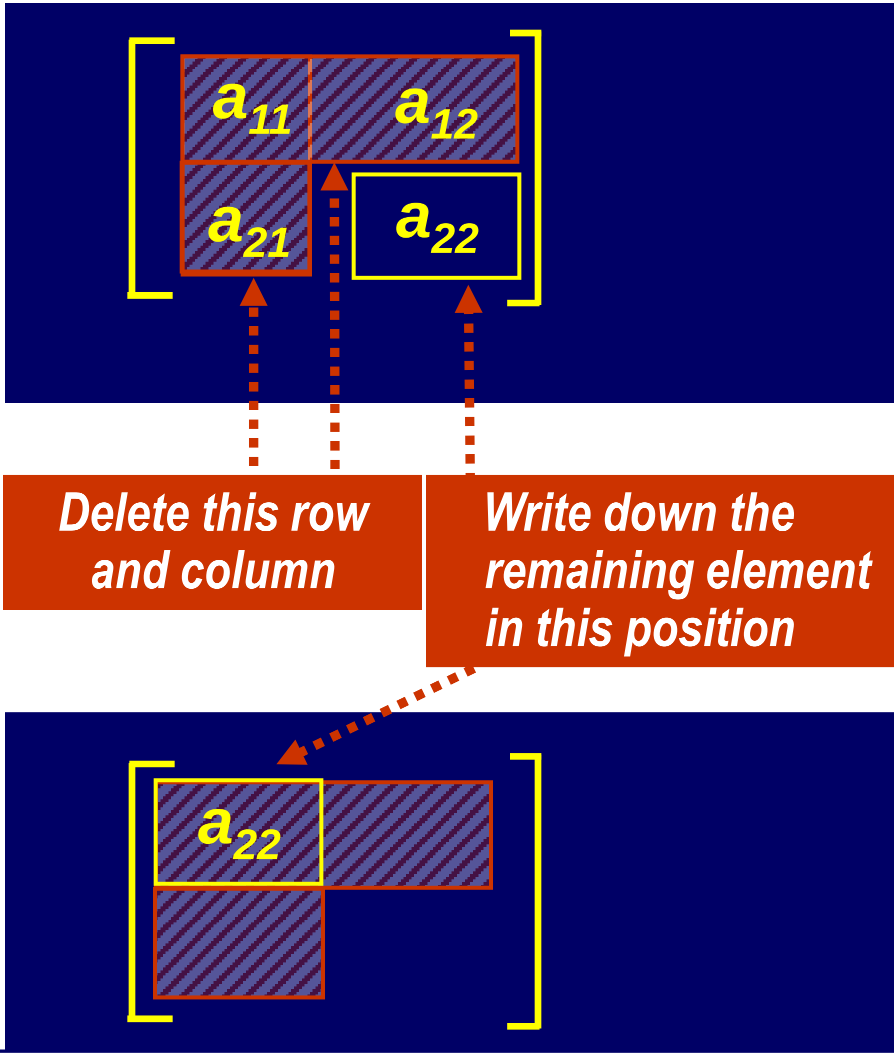
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Calculation of Inverse of a 2x2 Matrix

Procedure (Continued)

To find the inverse of 2 x 2 matrix

2. Then, calculate the co-factor matrix. To calculate the a_{11} element of the co-factor matrix;
   - Delete the 1st row and 1st column of the matrix,
   - Write down the remaining element a_{22} in the diagonal position: 2,2, where the deleted row and column intercepts,

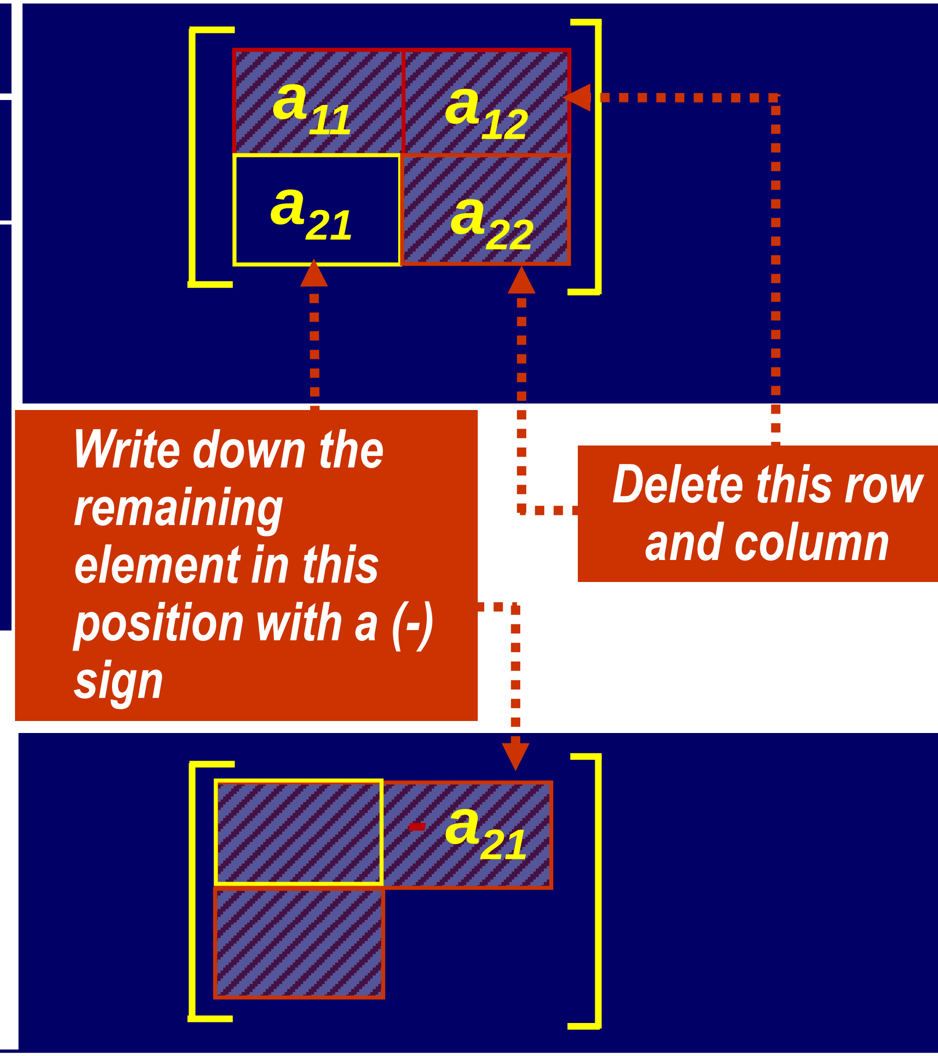


Calculation of Inverse of a 2x2 Matrix

Procedure (Continued)

To find the inverse of 2 x 2 matrix

- Perform the sam procedure for the next element a_{12} in the matrix
- Repeat this procedure for all elements in A.



Calculation of Inverse of a 2x2 Matrix

Procedure (Continued)

To find the inverse of 2 x 2 matrix

- Set the sign of the i-jth element in the co-factor matrix such that;
  - ✓ sign = $\begin{cases} +1 & \text{when } i + j \text{ is even,} \\ -1 & \text{otherwise} \end{cases}$

$$\begin{bmatrix} a_{11} & -a_{21} \\ -a_{12} & a_{22} \end{bmatrix}$$

- Transpose the resulting matrix

$$\begin{bmatrix} a_{11} & -a_{21} \\ -a_{12} & a_{22} \end{bmatrix}^T$$

Calculation of Inverse of a 2x2 Matrix

Procedure (Continued)

To find the inverse of 2 x 2 matrix

3. Finally, divide the resulting transposed co-factor matrix by the determinant

$$\frac{1}{d}\begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}$$

Determinant

RESULT: Inverse of the given matrix

$$\begin{bmatrix} a_{11}/d & -a_{12}/d \\ -a_{21}/d & a_{22}/d \end{bmatrix}$$

Example

Find the Inverse of the Matrix given on the RHS

1. First calculate the determinant of the given matrix;

   Determinant = 1 x 6 – 2 x 4 = -2

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$$

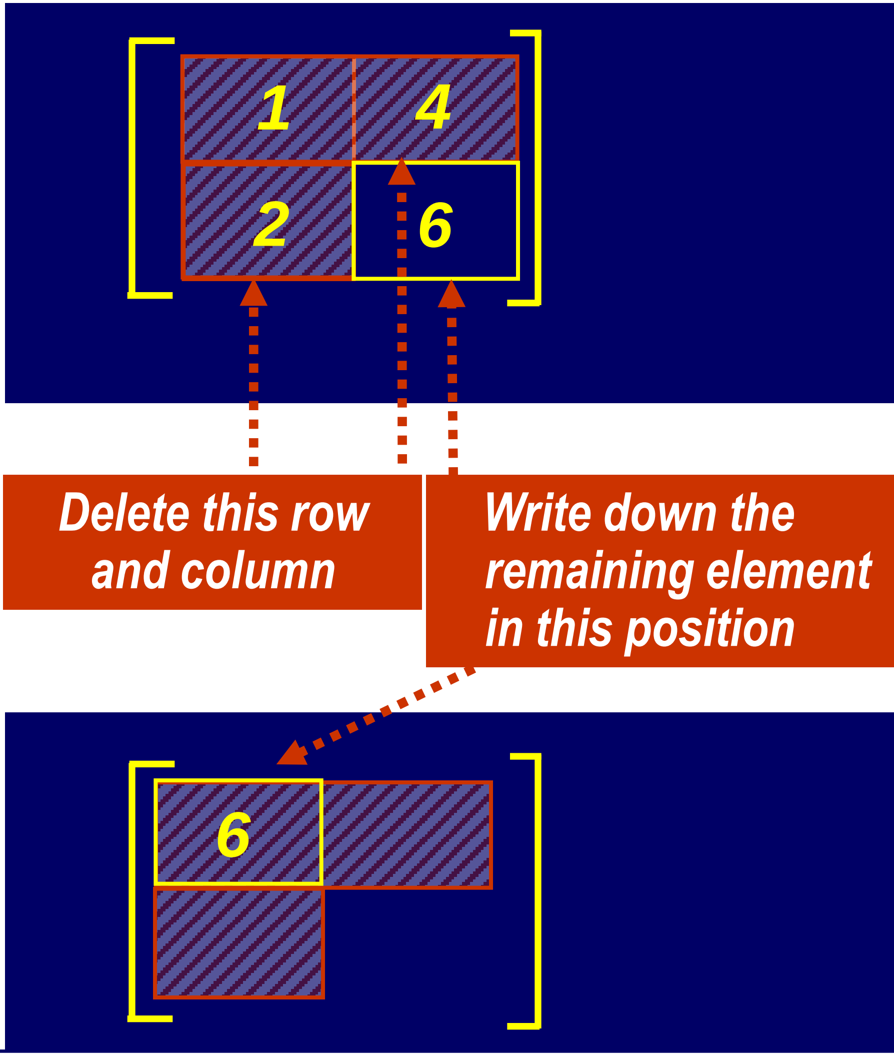
$$\begin{bmatrix} 1 \;\; + & - \;\; 4 \\ 2 & 6 \end{bmatrix}$$

Example (Continued)

Procedure (Continued)

To find the inverse of 2 x 2 matrix

2. Then, calculate the co-factor matrix. To calculate the a_{11} element of the co-factor matrix;
   - Delete the 1st row and 1st column of the matrix,
   - Write down the remaining element a_{22} in the diagonal position: 2,2, where the deleted row and column intercepts,

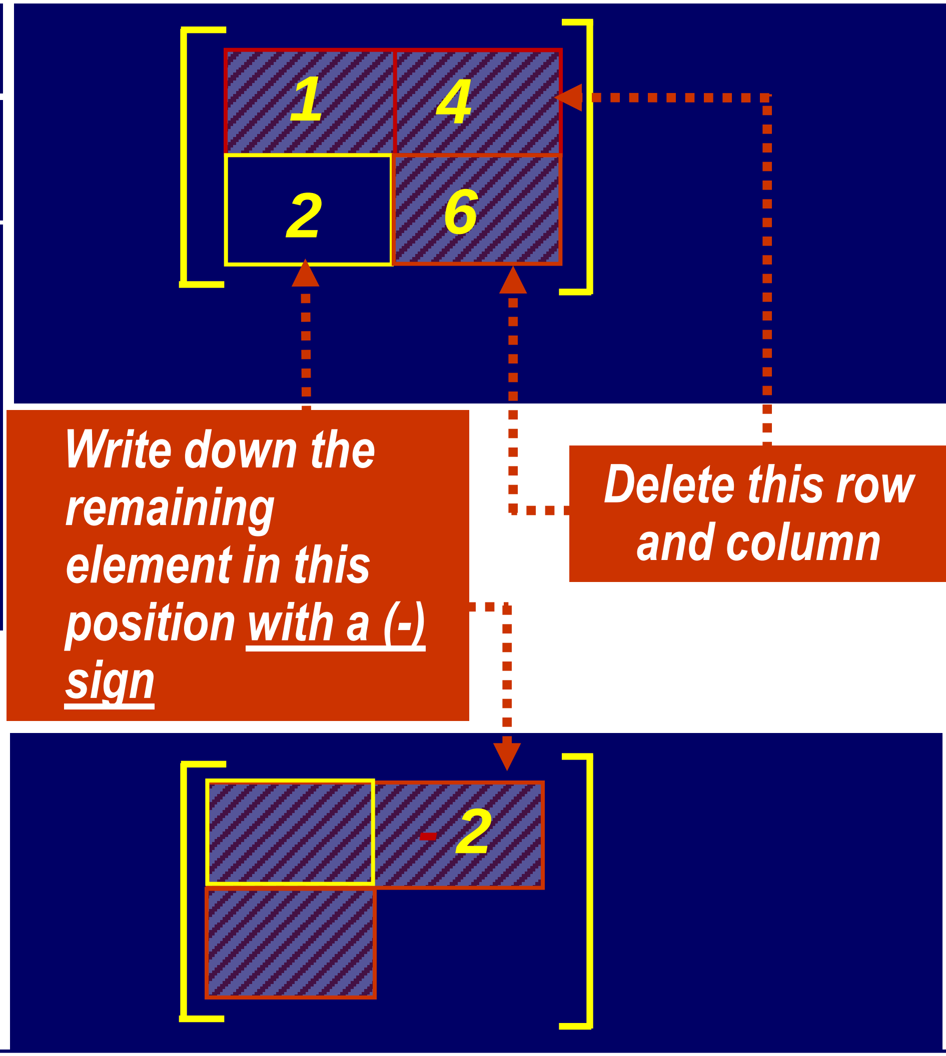


Example (Continued)

Procedure (Continued)

To find the inverse of 2 x 2 matrix

- Perform the sam procedure for the next element a_{12} in the matrix
- Repeat this procedure for all elements in A.



Example (Continued)

Procedure (Continued)

<u>To find the inverse of 2 x 2 matrix</u>

- Set the sign of the i-jth element in the co-factor matrix such that;
  - ✓ sign = $\begin{cases} +1 & \text{when } i + j \text{ is even,} \\ -1 & \text{otherwise} \end{cases}$

$$\begin{bmatrix} 6 & -2 \\ -4 & 1 \end{bmatrix}$$

- Transpose the resulting matrix

$$\begin{bmatrix} 6 & -2 \\ -4 & 1 \end{bmatrix}^T$$

Example (Continued)

Procedure (Continued)

<u>To find the inverse of 2 x 2 matrix</u>

3. Finally, divide the resulting transposed co-factor matrix by the determinant

$$\frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -2 & 1 \end{bmatrix}$$

Determinant

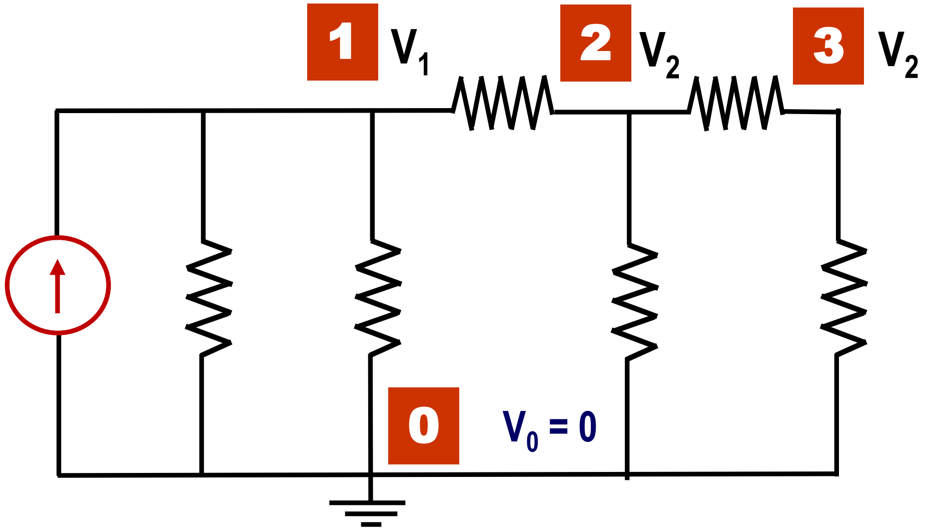
<u>RESULT</u>: Inverse of the given matrix

$$\begin{bmatrix} -3 & 2 \\ 1 & -1/2 \end{bmatrix}$$

Solution of Large - Size Systems

Procedure

Suppose that we want to solve the three-bus system shown on the RHS for node voltages



Nodal equations for this system may be written as follows

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_{S1}\, g_{s1} \\ V_{S2}\, g_{s2} \\ V_{S3}\, g_{s3} \end{bmatrix}$$

$$G \quad V = I$$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

Hence, we must find the inverse of the coefficient matrix G

To find the inverse of 3 x 3 matrix

1. First calculate the determinant of the matrix;
   - For that purpose, first augment the given matrix by the "first two" columns of the same matrix from the RHS
   - Then, multiply the terms on the main diagonal

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

a_{11} x a_{22} x a_{22}

a_{12} x a_{23} x a_{31}

a_{13} x a_{21} x a_{32}

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

- Then, multiply the terms on the other (cross) diagonal

$$\left[\begin{array}{ccc:cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$a_{13} \times a_{22} \times a_{31}$

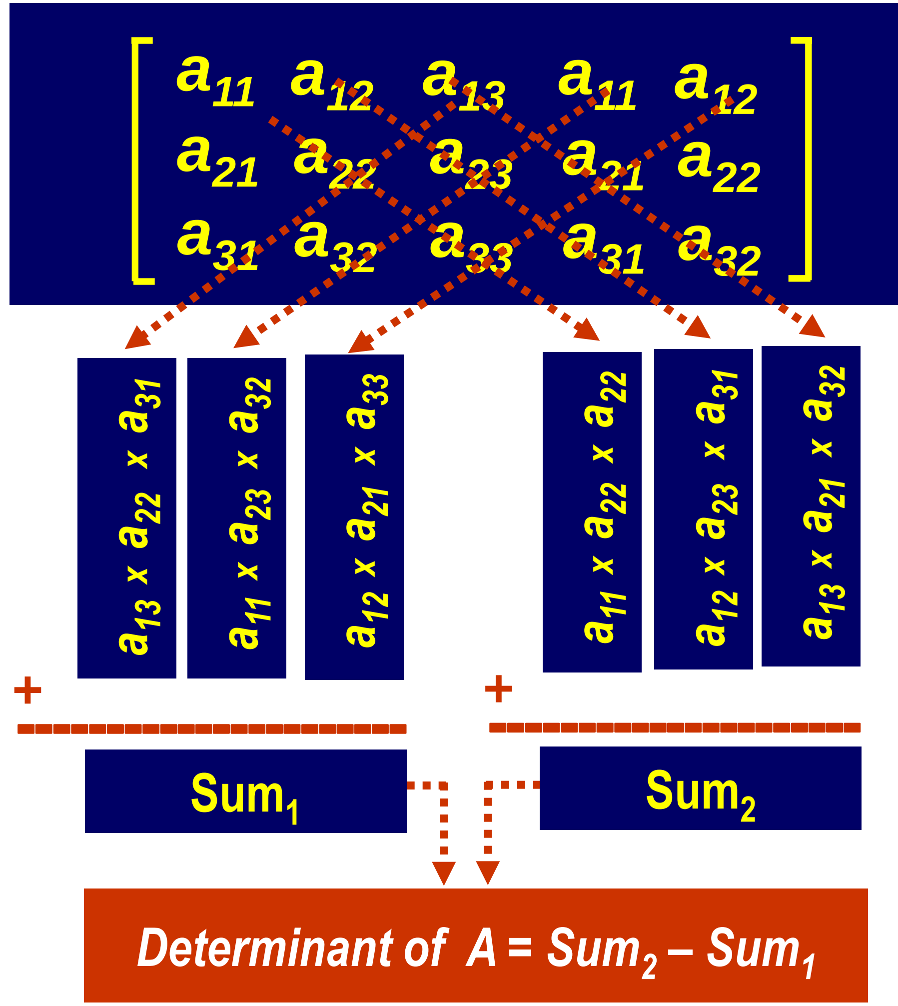
$a_{11} \times a_{23} \times a_{32}$

$a_{12} \times a_{21} \times a_{33}$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

- Then, subtract the latter three multiplications from those found in the former



Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

1. Then, calculate the co-factor matrix. To calculate the a_{ij}-th element of the co-factor matrix;
   - Delete the i the row and j the column of the matrix,
   - Calculate the determinant c_{ij} of the remaining 2 x 2 submatrix by using the method given earlier for 2 x 2 matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{a_{22}} & \boxed{a_{23}} \\ a_{31} & \boxed{a_{32}} & \boxed{a_{33}} \end{bmatrix}$$

Delete this row and column

Calculate the determinant of the resulting 2 x 2 submatrix

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

- Repeat this procedure for all elements in the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Delete this row and column

Calculate the determinant of the resulting 2 x 2 submatrix

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

- Write down these determinants in the corresponding locations,
- Set the sign of these elements such that;

✓ sign = $\begin{cases} +1 & \text{when } i + j \text{ is even,} \\ -1 & \text{otherwise} \end{cases}$

$$\begin{bmatrix} c_{11} & -c_{12} & c_{13} \\ -c_{21} & c_{22} & -c_{23} \\ c_{31} & -c_{32} & c_{33} \end{bmatrix}$$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

2. Then, transpose the resulting co-factor matrix

$$\begin{bmatrix} c_{11} & -c_{12} & c_{13} \\ -c_{21} & c_{22} & -c_{23} \\ c_{31} & -c_{32} & c_{33} \end{bmatrix}^T$$

$$\begin{bmatrix} c_{11} & -c_{21} & c_{31} \\ -c_{12} & c_{22} & -c_{32} \\ c_{13} & -c_{23} & c_{33} \end{bmatrix}$$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

3. Finally, divide the resulting transposed co-factor matrix by the determinant

$$\frac{1}{d}\begin{bmatrix} c_{11} & -c_{21} & c_{31} \\ -c_{12} & c_{22} & -c_{32} \\ c_{13} & -c_{23} & c_{33} \end{bmatrix}$$

Determinant

$$\begin{bmatrix} c_{11}/d & -c_{21}/d & c_{31}/d \\ -c_{12}/d & c_{22}/d & -c_{32}/d \\ c_{13}/d & -c_{23}/d & c_{33}/d \end{bmatrix}$$

Example

Example

Find the inverse of the coefficient matrix given on the RHS

1. First calculate the determinant of the matrix;
   - For that purpose, first augment the given matrix by the "first two" columns of the same matrix from the RHS
   - Then, multiply the terms on the main diagonal

$$\left[\begin{array}{ccc:cc} 1 & 2 & 4 & 1 & 2 \\ 2 & 8 & -2 & 2 & 8 \\ 4 & -2 & 2 & 4 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc:cc} 1 & 2 & 4 & 1 & 2 \\ 2 & 8 & -2 & 2 & 8 \\ 4 & -2 & 2 & 4 & -2 \end{array}\right]$$

16	-16	-16

Example

Procedure (Continued)

- Then, multiply the terms on the other (cross) diagonal

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 2 & 8 & -2 & 2 & 8 \\ 4 & -2 & 2 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 2 & 8 & -2 & 2 & 8 \\ 4 & -2 & 2 & 4 & -2 \end{bmatrix}$$

128   4   8

Example

Procedure (Continued)

- Then, subtract the latter three multiplications from those found in the former

$$\left[\begin{array}{ccc|cc} 1 & 2 & 4 & 1 & 2 \\ 2 & 8 & -2 & 2 & 8 \\ 4 & -2 & 2 & 4 & -2 \end{array}\right]$$

128	4	8

+

140

16	-16	-16

+

-16

Determinant of A = -16 - 140 = -156

Example

Procedure (Continued)

1. Then, calculate the co-factor matrix. To calculate the a_{ij} -th element of the co-factor matrix;
   - Delete the i the row and j the column of the matrix,
   - Calculate the determinant c_{ij} of the remaining 2 x 2 submatrix by using the method given earlier for 2 x 2 matrices

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 2 \end{bmatrix}$$

Delete this row and column

Calculate the determinant of the resulting 2 x 2 submatrix:

2 x 8 – 2 x 2 = 12

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

- Repeat this procedure for all elements in the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 2 \end{bmatrix}$$

Delete this row and column

Calculate the determinant of the resulting 2 x 2 submatrix:

2 x 2 - 4 x (-2) = 12

$$\begin{bmatrix} 2 & -2 \\ 4 & 2 \end{bmatrix}$$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

- Form the co-factor matrix as shown on the RHS
- Transpose the co-factor matrix (It will not change since it is symmetrical)

$$\begin{bmatrix} 12 & -12 & -36 \\ -12 & -14 & 10 \\ -36 & 10 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 12 & -12 & -36 \\ -12 & -14 & 10 \\ -36 & 10 & 4 \end{bmatrix}$$

Calculation of Inverse of a 3x3 Matrix

Procedure (Continued)

3. Finally, divide the resulting transposed co-factor matrix by the determinant

$$= \frac{1}{-156}\begin{bmatrix} 12 & -12 & -36 \\ -12 & -14 & 10 \\ -36 & 10 & 4 \end{bmatrix}$$

Determinant

$$= \begin{bmatrix} -12/156 & 12/156 & 36/156 \\ 12/156 & 14/156 & -10/156 \\ 36/156 & -10/156 & -4/156 \end{bmatrix}$$

Solution Step

Procedure (Continued)

These elements are zero for nodes with no current injection

Final step of the solution procedure is the multiplying the RHS vector with the inverse of the nodal admittance matrix

$$\begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}^{-1} \begin{bmatrix} V_{S1}\, g_{s1} \\ V_{S2}\, g_{s2} \\ V_{S3}\, g_{s3} \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$G^{-1} \quad I \quad = \quad V$$

Nodal Analysis with Pure Voltage Sources

Procedure (Continued)

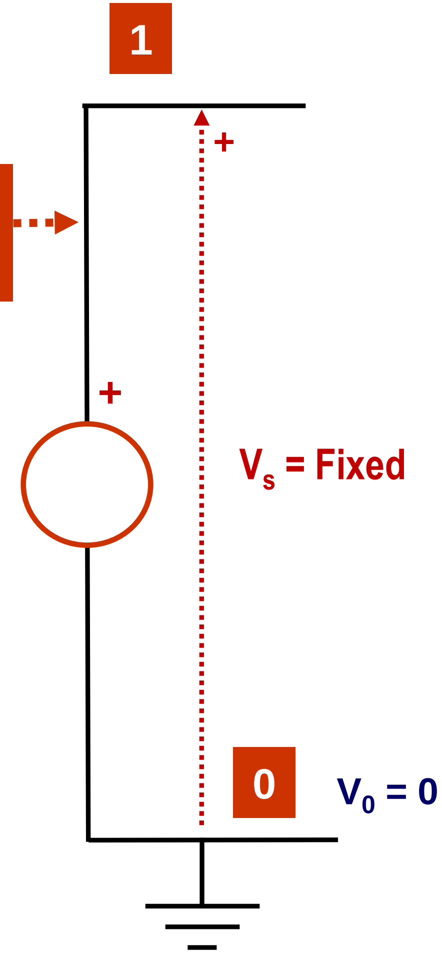
Sometimes we may encounter a voltage source with no series resistance, called; *"Pure Voltage Source"*

A pure voltage source connecting a node to ground means that the voltage is fixed at this node, *(i.e. it is no longer unknown)*

A pure voltage source with no series resistance creates problem in the solution procedure, since it cannot be converted to an equivalent Norton Equivalent Circuit, i.e.

$$V_s / R_s = V_s / 0 = \infty$$

Please note that there is no resistance here



Nodal Analysis with Pure Voltage Sources

Procedure (Continued)

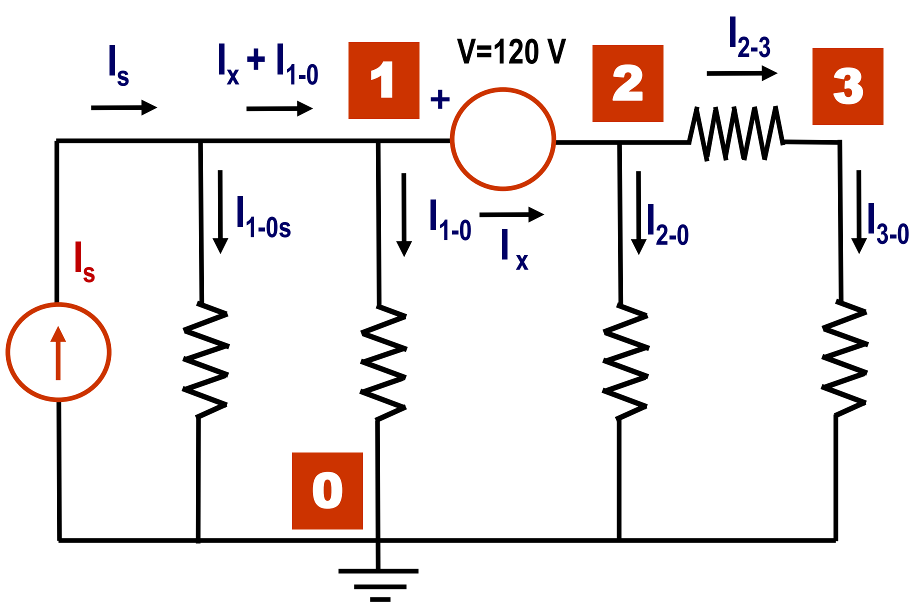
Sometimes we may encounter a ***"Pure Voltage Source"*** connecting two nodes other than ground.

This means that the voltage difference between these nodes is fixed

A pure voltage source with no series resistance creates problem in the solution procedure, since it cannot be converted to an equivalent Norton Equivalent Circuit, i.e.

$$V_s / R_s = V_s / 0 = \infty$$

$V_1 - V_2 = 120\ V = Fixed$

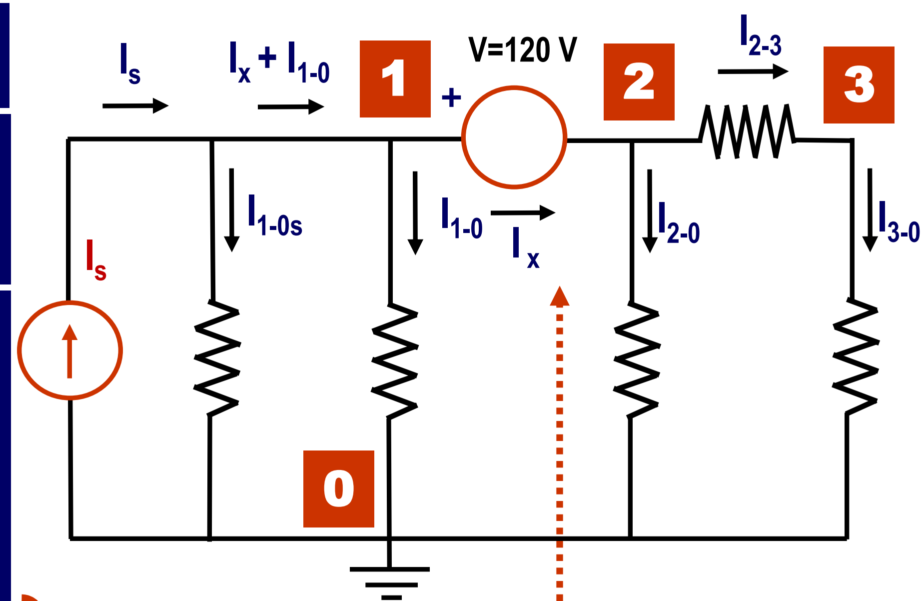


Nodal Analysis with Pure Voltage Sources

Procedure (Continued)

In this case, the circuit can be solved as follows

1. Define the current flowing in this voltage source as I_x
2. Define this current as a new variable,
3. Write down KCL at each node, except the reference node,
4. Write down the equation for the voltage difference between the terminals of this pure voltage source



n-1 equations

$V_1 - V_2 = 120$ V

k equations

+

n-1 + k equations

Nodal Analysis with Pure Voltage Sources

Resulting Nodal Equations

1

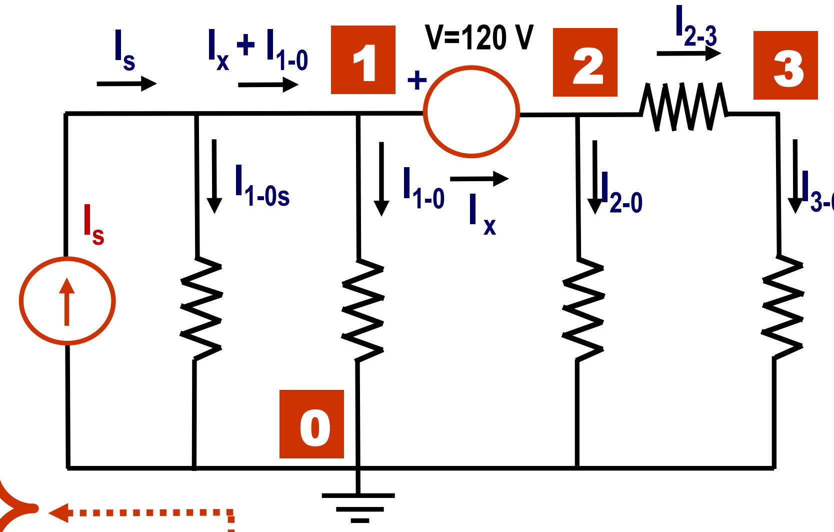
$$\sum_{i=1}^{i=n-1} I_i \ (\text{including } I_X) = 0$$

$$I_s = I_{1\text{-}0s} + I_{1\text{-}0} + I_x$$

2

$$\sum_{i=1}^{i=n-1} I_i \ (\text{including } I_X) = 0$$

$$I_x = I_{2\text{-}0} + I_{2\text{-}3}$$



(n – 1 + k = n) equations vs n unknowns
k = 1, hence n -1 + k = n

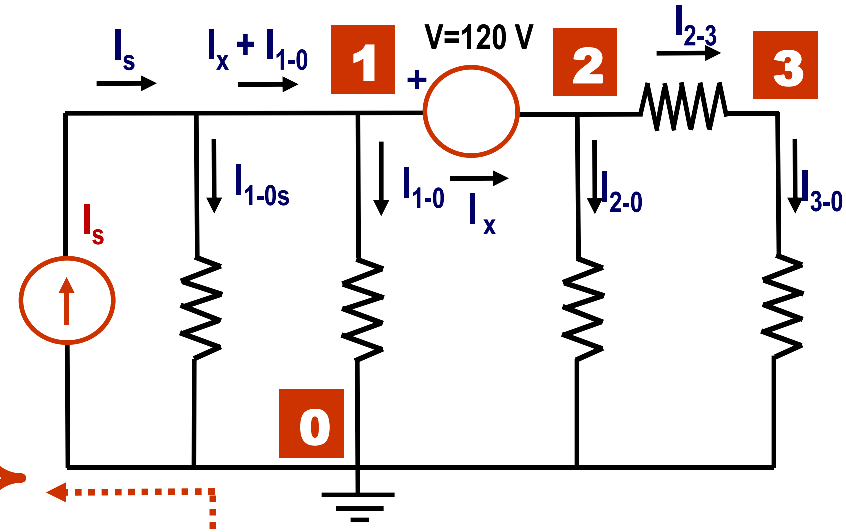
Nodal Analysis with Pure Voltage Sources

Resulting Nodal Equations

3

$$\sum_{i=1}^{i=n-1} I_i = 0$$

$$I_{2-3} = I_{3-0}$$



and finally, writing down the equation for voltage difference across the pure voltage source

4

$$V_1 - V_2 = 120\ V$$

$n = 4$, $k = 1$
$n - 1 + k = 4$
4 equations vs 4 unknowns

This equation spoils the symmetry of the nodal admittance matrix

Nodal Analysis with Pure Voltage Sources

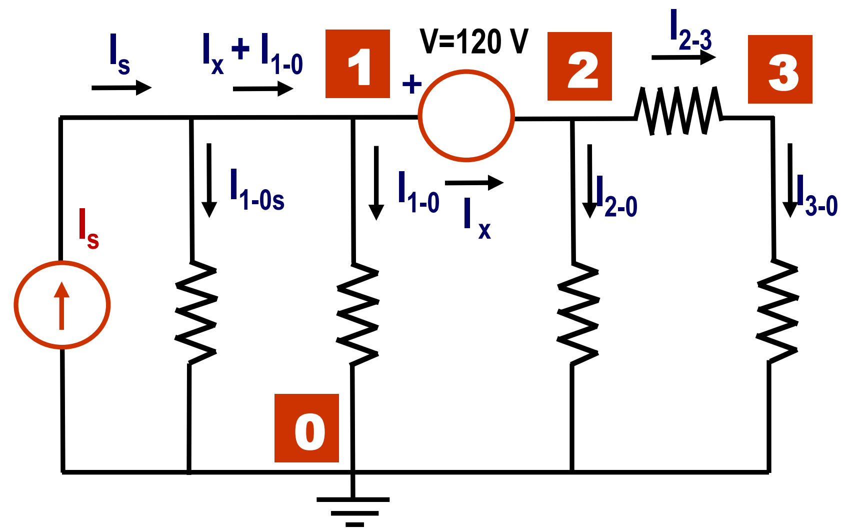
Resulting Equations

1 $V_1 g_s + V_1 g_{10} + I_x - I_s = 0$

2 $V_2 g_{20} + (V_2 - V_3)\, g_{23} - I_x = 0$

3 $V_3 g_{30} + (V_3 - V_2)\, g_{23} = 0$

Extra Equation $V_1 - V_2 = 120 \text{ Volts}$

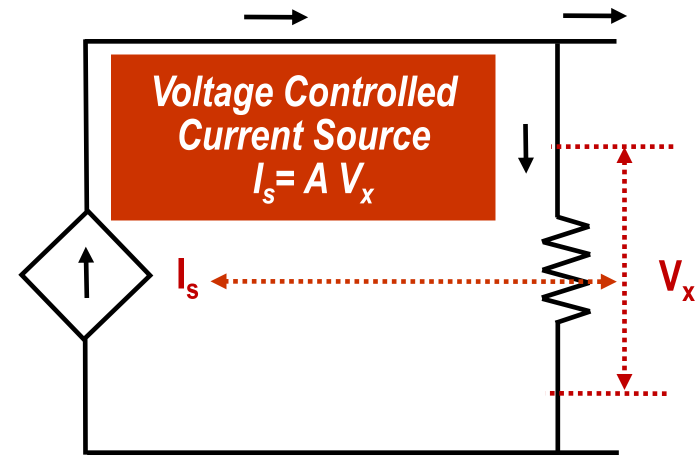


$$
\begin{bmatrix} g_s + g_{10} & & & 1 \\ & g_{20} + g_{23} & -g_{23} & -1 \\ & -g_{23} & g_{23} + g_{30} & \\ 1 & -1 & & \end{bmatrix}
\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_x \end{bmatrix}
=
\begin{bmatrix} I_S \\ 0 \\ 0 \\ 120 \end{bmatrix}
$$

Nodal Analysis with Controlled Sources

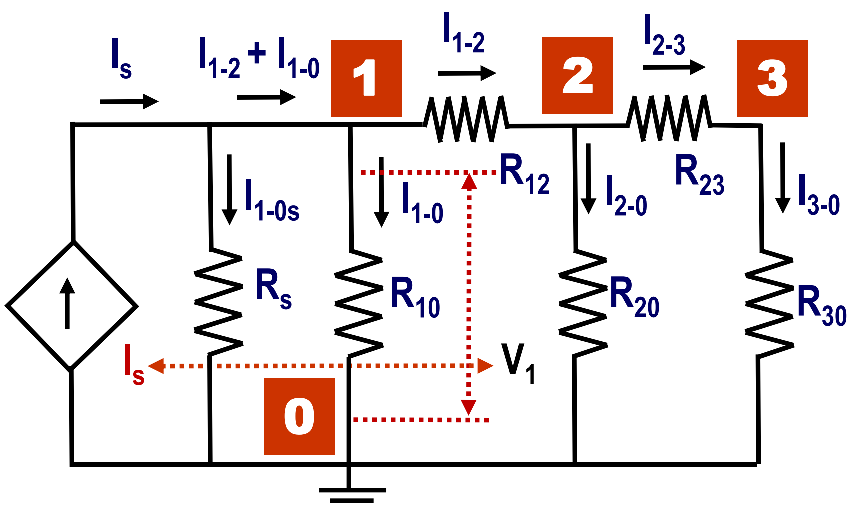
Nodal Analysis with Voltage Controlled Current Sources

Voltage Controlled Current Source: $I_s = A\ V_x$



Procedure

- Write down the expression for the current provided by the controlled current source in terms of the node voltage depended: $I_s = A\ V_1$
- Include this current in the summation when writing KCL for the node that controlled current source is connected,
- Solve the resulting nodal equations for node voltages



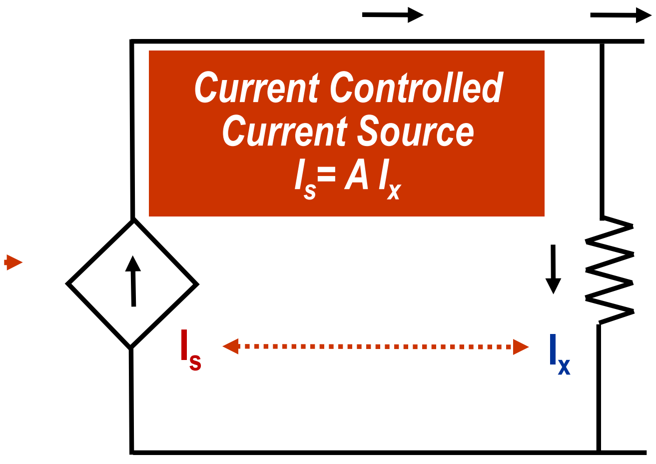
Nodal Analysis with Controlled Sources

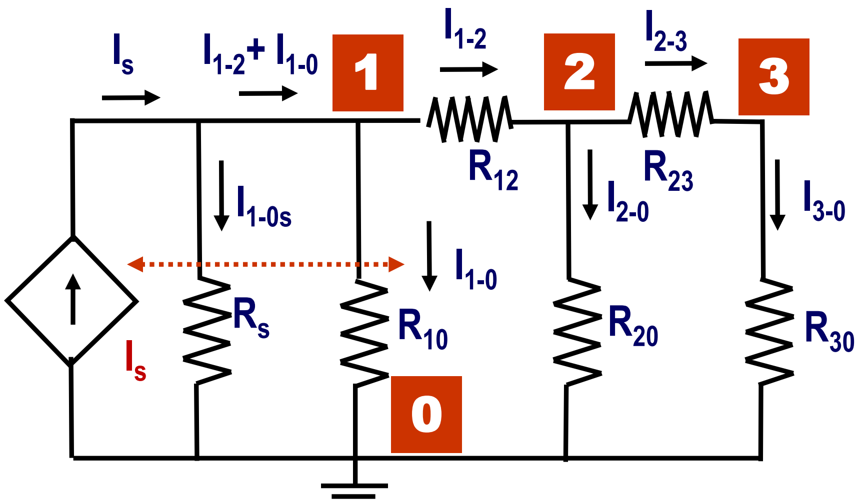
Nodal Analysis with Current Controlled Current Sources

Current Controlled Current Source: $I_s = A\, I_x$

Procedure

- Write down the expression for the current provided by the controlled current source in terms of current depended: $I_s = A\, I_{1\text{-}0}$
- Express the depended current, $I_{1\text{-}0}$ and hence I_S in terms of node voltages;
  $I_S = A\,(V_1 - V_0)\,/\,R_{1\text{-}0} = A\,V_1\,g_{1\text{-}0}$
- Include this current in the summation when writing KCL for node that controlled current is injected,
- Solve the resulting nodal equations for node voltages

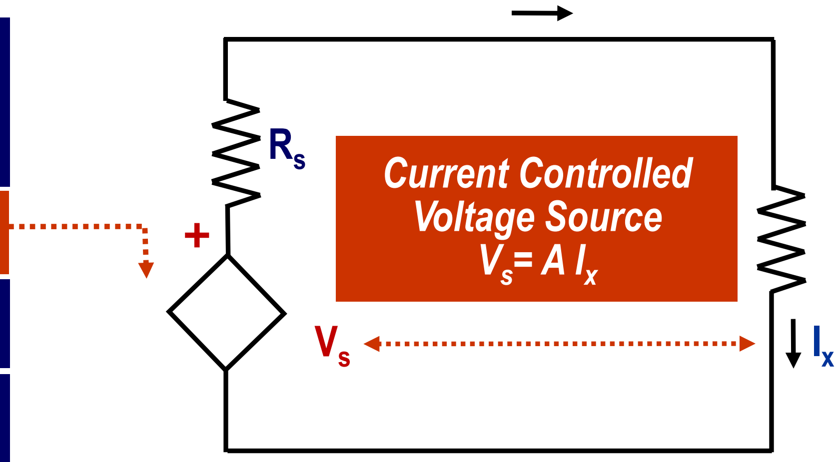




Nodal Analysis with Controlled Sources

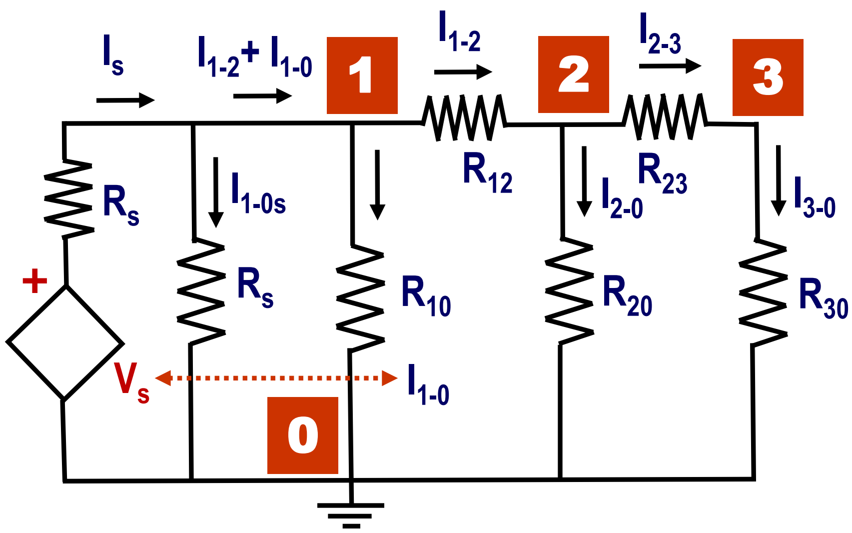
Nodal Analysis with Current Controlled Voltage Sources

Current Controlled Voltage Source: $V_s = A\, i_x$



Procedure

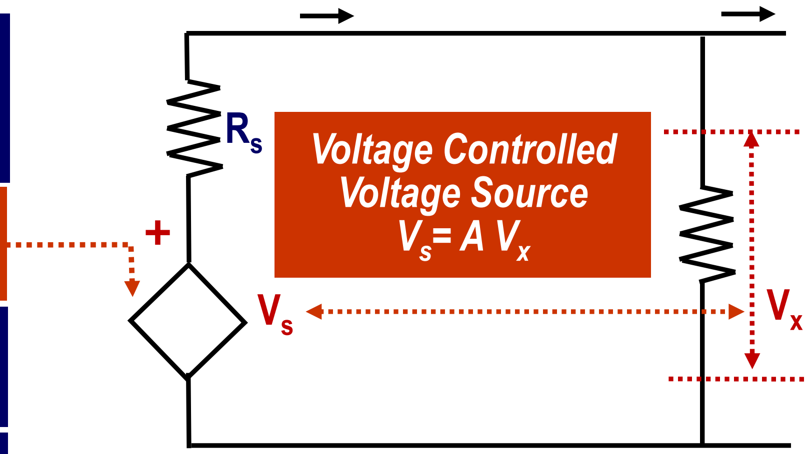
- Write down the expression for the controlled voltage in terms of the current depended: $V_s = A\, I_{1\text{-}0}$
- Express the depended current, $I_{1\text{-}0}$ and hence V_S in terms of the node voltages,
- Convert the resulting voltage source V_S to equivalent Norton current source,
- Include this current in the summation when writing KCL for node that that controlled current is injected,
- Solve the resulting nodal equations for node voltages



Nodal Analysis with Controlled Sources

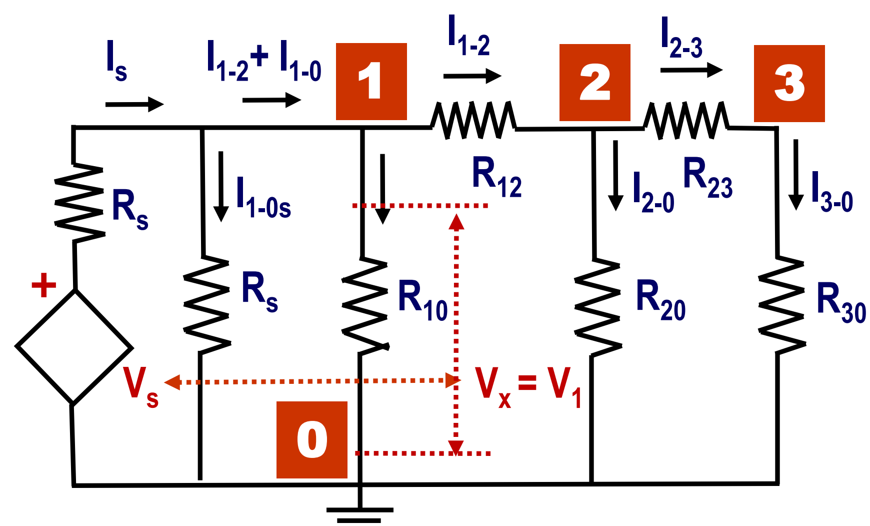
Nodal Analysis with Voltage Controlled Voltage Sources

Voltage Controlled Voltage Source: $V_s = A\,V_x$



Procedure

- Write down the expression for the controlled voltage in terms of the voltage depended: $V_s = A\,V_x = A\,V_1$
- Convert the resulting voltage source V_s to equivalent Norton current source,
- Include this current in the summation when writing KCL for node that controlled current is injected,
- Solve the resulting nodal equations for node voltages



Example

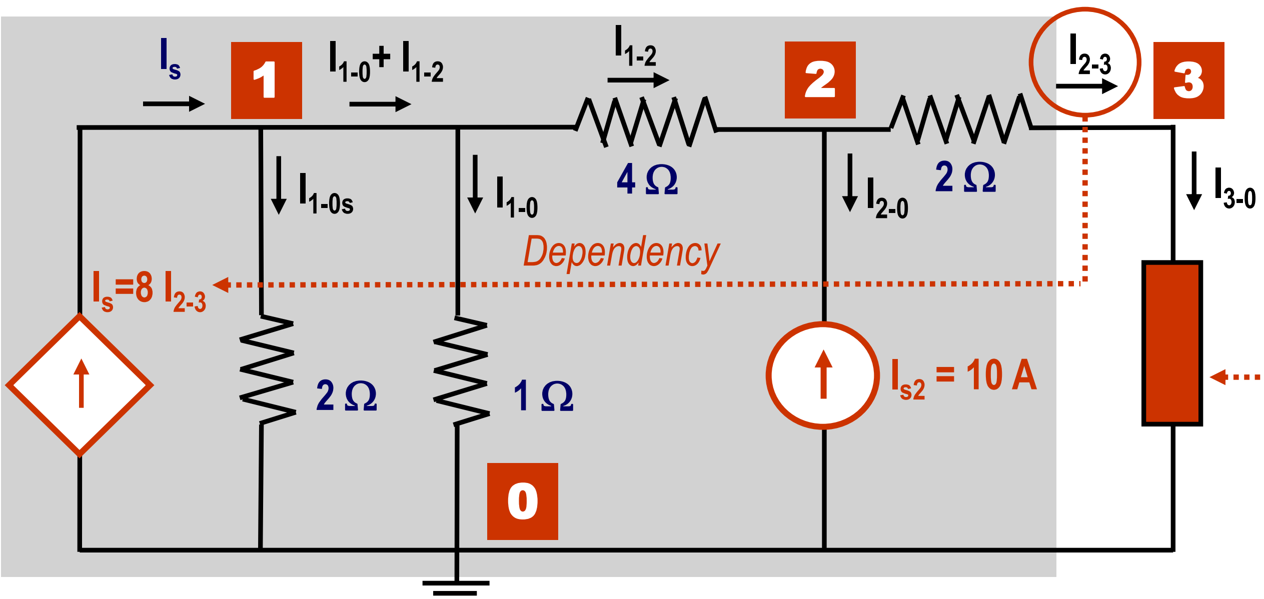
Node Voltage Method with Controlled Current Source

Find the power dissipated in the resistance R_L in the following circuit by using the Node Voltage Method

Please note that current controlled current source in the circuit can NOT be killed for finding the Thevenin Equivalent Circuit

If you do, the result will be INCORRECT !

Hence, simplification by employing Thevenin Equivalent Circuit Method is NOT applicable to this problem



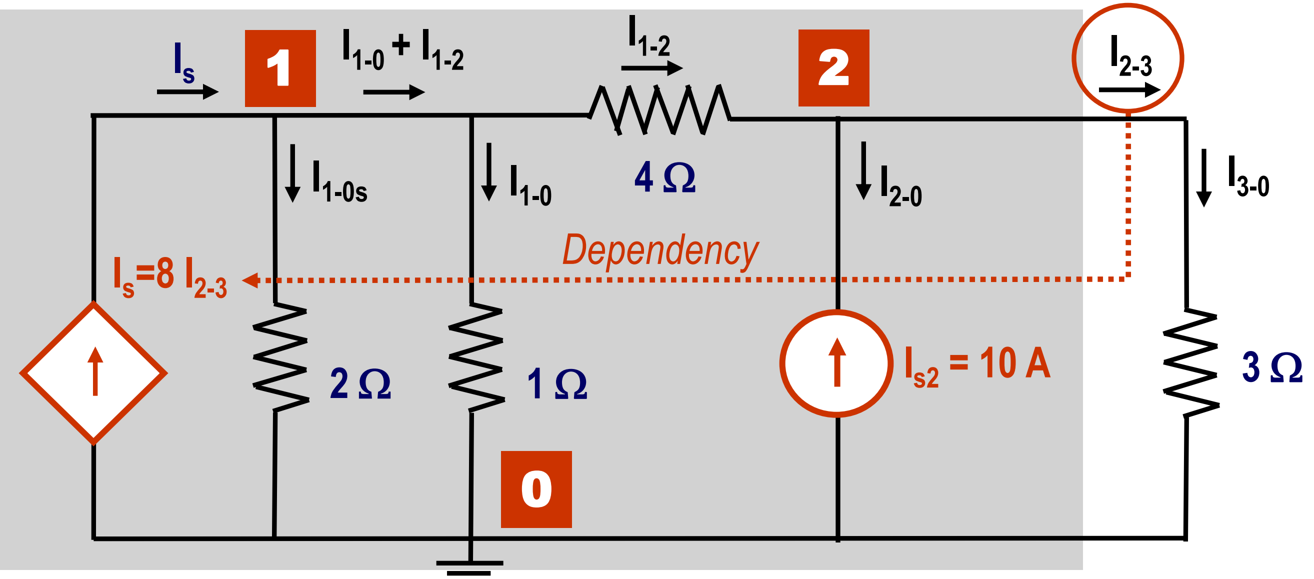
Load Resistance
R_L = 1 Ω

Circuit Analysis

Example (Continued)

Node Voltage Method with Controlled Current Source

The first step of the solution is to combine the resistances R_L and 1 Ohm yielding a 3 Ohm resistance, thus eliminating the third node



Example (Continued)

Node Voltage Method with Controlled Current Source

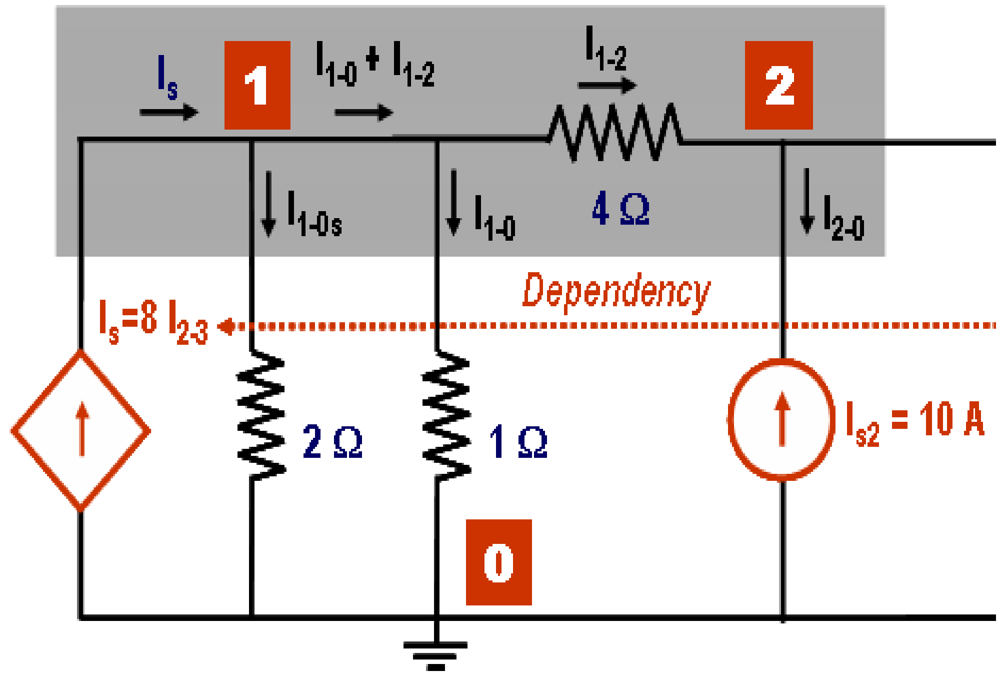
Now write down KCL equation at Node-1

$$8\,I_{2\text{-}3} - I_{1\text{-}0s} - I_{1\text{-}0} - I_{1\text{-}2} = 0$$

$$I_{1\text{-}0s} = V_1 / 2\,\Omega$$

$$I_{1\text{-}0} = V_1 / 1\,\Omega$$

$$I_{1\text{-}2} = (V_1 - V_2) / 4\,\Omega$$



Equation - 1

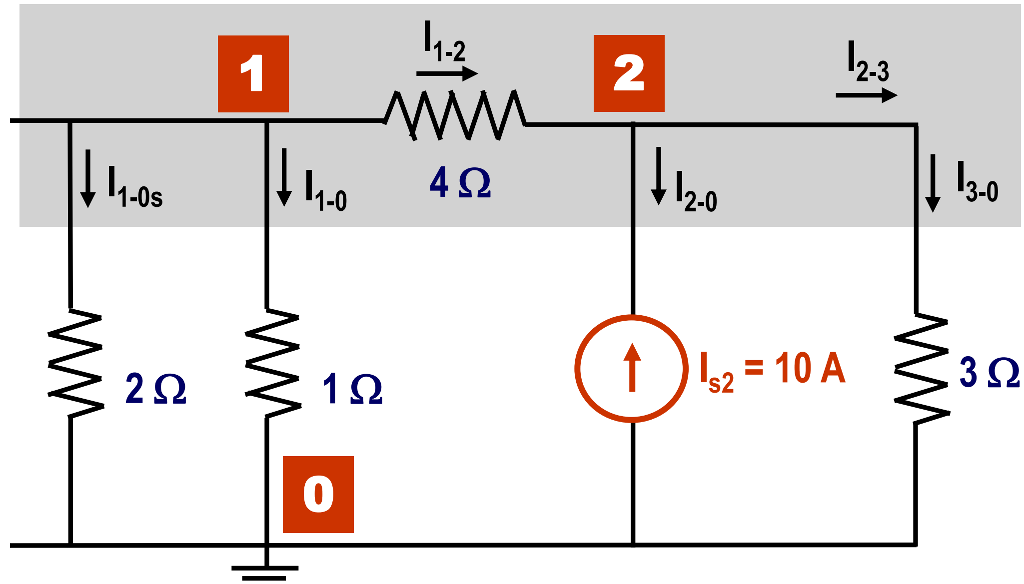
$$8\,V_2 / 3 - V_1 / 2 - V_1 / 1 - (V_1 - V_2) / 4 = 0$$

Example (Continued)

Node Voltage Method with Controlled Current Source

Now, write down KCL equation at Node-2

$$I_{1\text{-}2} - I_{2\text{-}0} - I_{2\text{-}3} = 0$$
$$I_{1\text{-}2} = (V_1 - V_2) / 4\ \Omega$$
$$I_{2\text{-}0} = -I_{s2} = -10\ A$$
$$I_{2\text{-}3} = I_{3\text{-}0} = V_2 / 3\ \Omega$$



Equation - 2

$$(V_1 - V_2) / 4 - (-10\ Amp) - V_2 / 3 = 0$$

Example (Continued)

Node Voltage Method with Controlled Current Source

Reorganizing eq-1 and eq-2 in matrix form

$$\begin{bmatrix} -(1/2 + 1/1 + 1/4) & 8/3 + 1/4 \\ 1/4 & -(1/4 + 1/3) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} -1.75 & 2.9167 \\ 0.25 & -0.5833 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

Solution: V_1 = -100 Volts,
V_2 = -60 Volts

Circuit Analysis

Example (Continued)

Node Voltage Method with Controlled Current Source

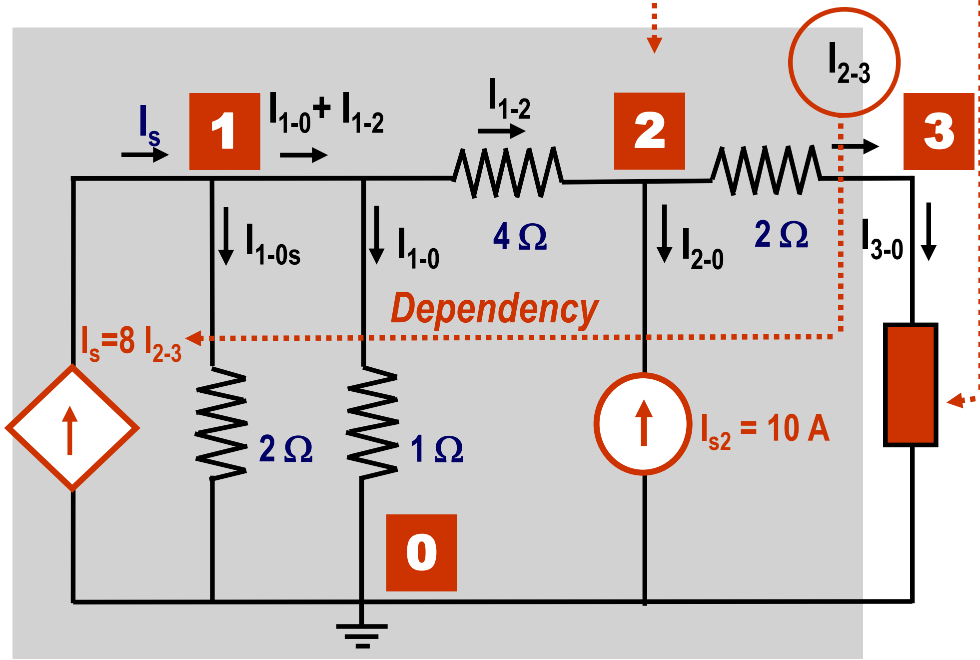
Now, find the power dissipated in R_L

$I_{2\text{-}3} = 60\ V / 3\ \Omega = 20\ A$

$P = R_L\ I^2 = 1 \times 20^2 = \underline{400\ W}$

Load Resistance: $R_L = 1\ \Omega$

$V_2 = -60$ Volts

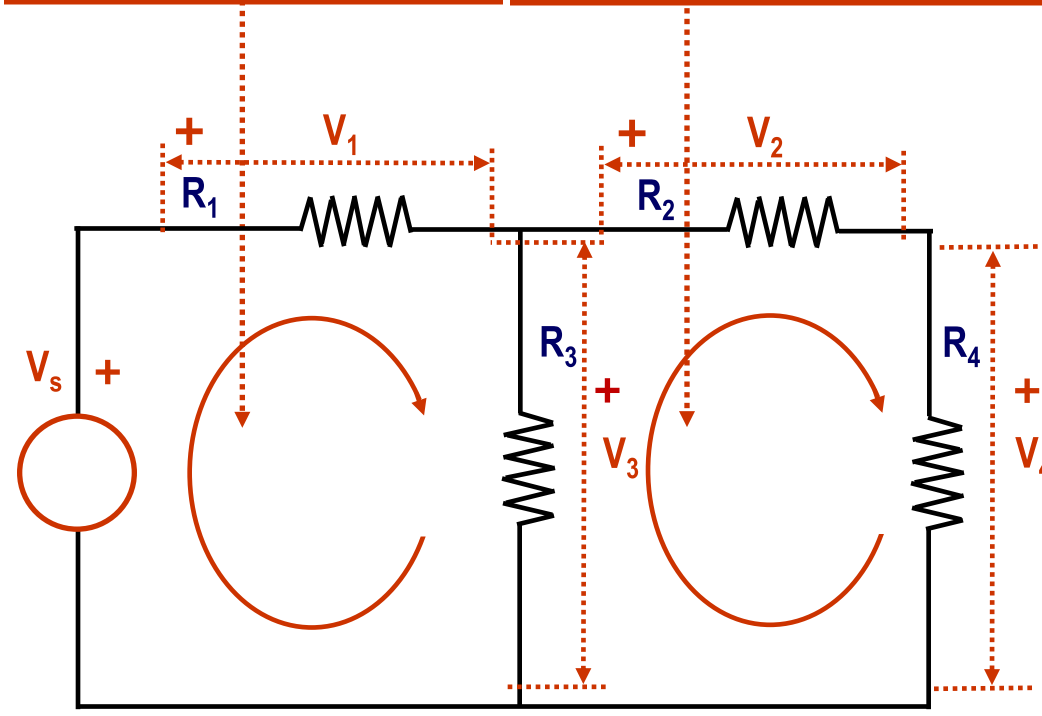


Mesh Current Method

Loop	Loop -1	Loop -2

Definition

A <u>Loop</u> is a closed path of branches followed <u>in clockwise direction</u> that begins from one node and ends again at the same node



Circuit Analysis

Mesh Current Method

Loop	Loop -1	Loop -2

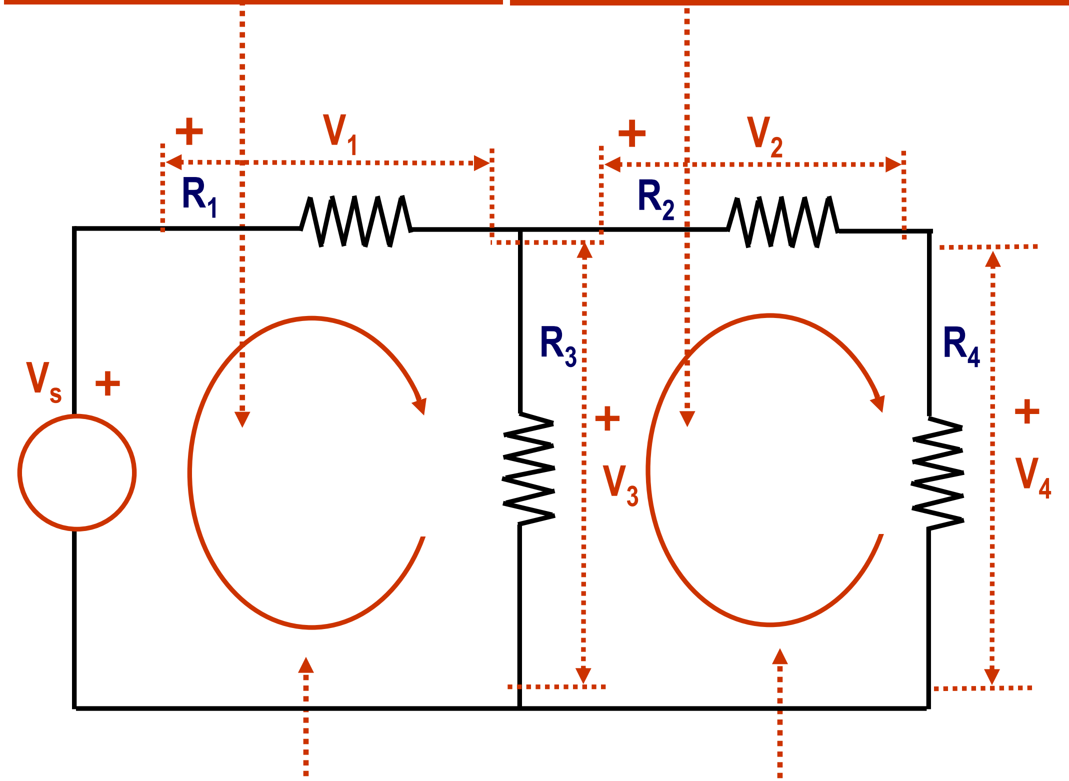
Basic Rule

Loops obey Kirchoff's Voltage Law (KVL)

$$\sum_{i=1}^{i=n} V_i = 0$$

$$V_s - V_1 - V_3 = 0$$

$$V_3 - V_2 - V_4 = 0$$



Define the mesh currents in each mesh flowing ***always in clockwise direction***

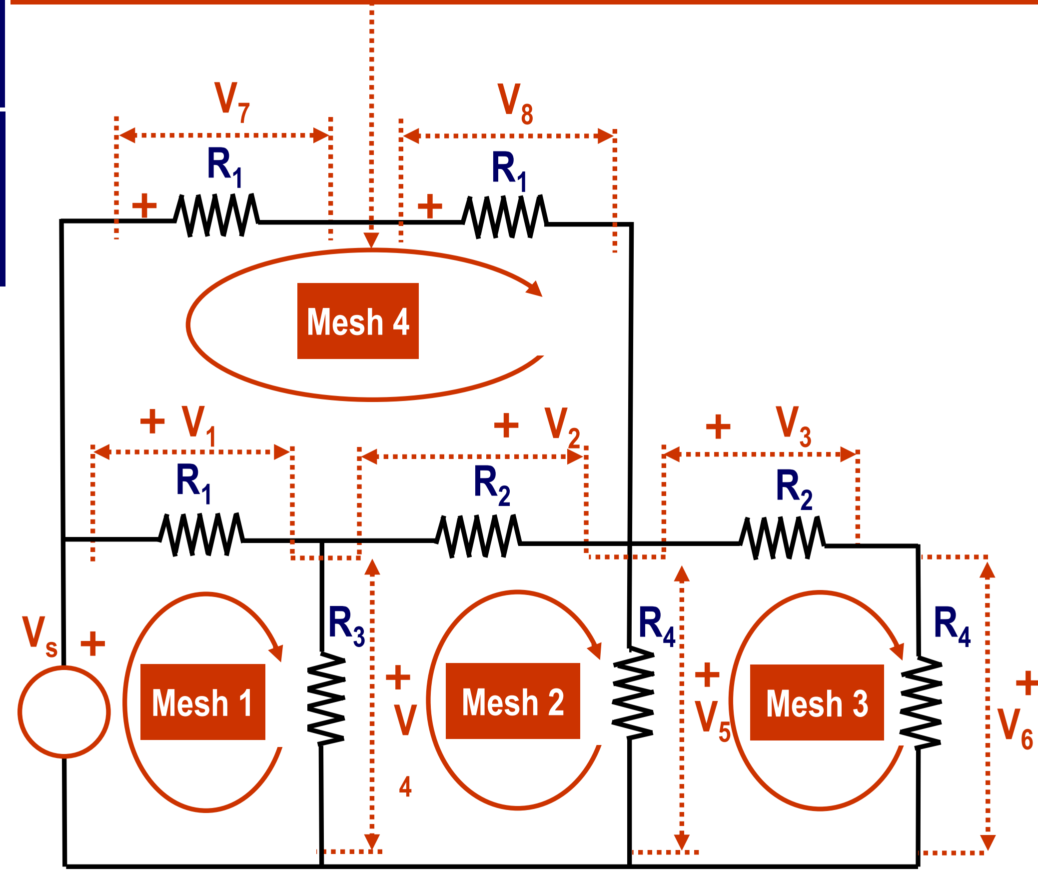
Mesh Current Method

Mesh

Definition

A Mesh is a loop that does not contain any other loop inside

Define the mesh currents in each mesh flowing always in clockwise direction

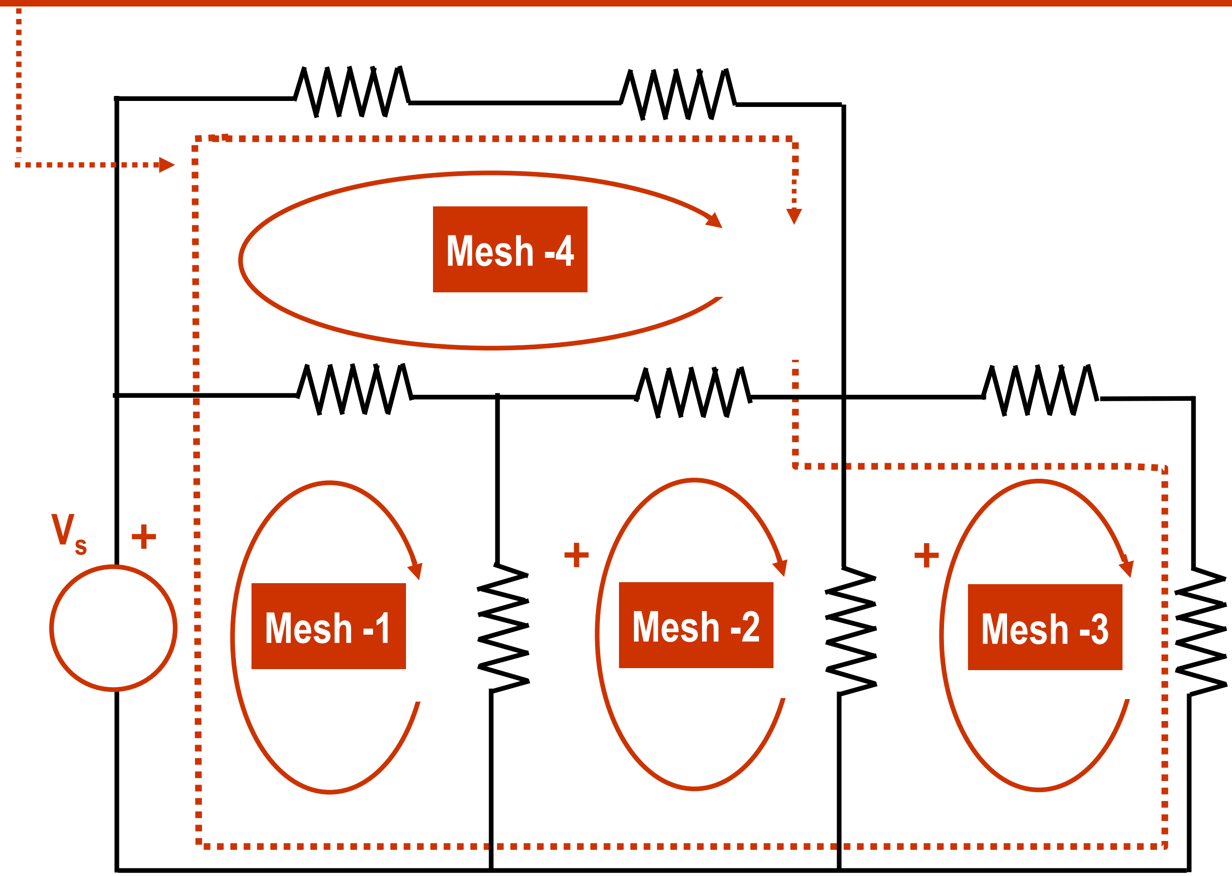


Circuit Analysis

Mesh Current Method

Mesh

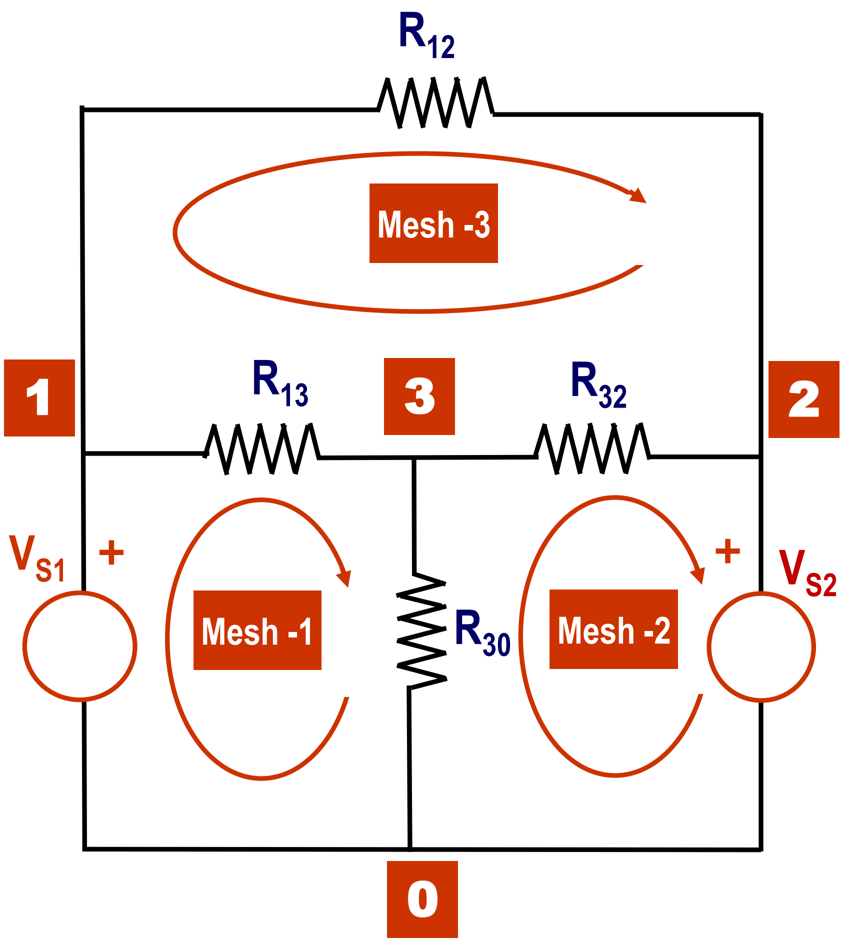
Please note that the path shown by dashed line is NOT a mesh, since it contains some other loops inside



Mesh Current Method

Procedure

1. Determine the meshes and mesh current directions in the circuit by following the rules;

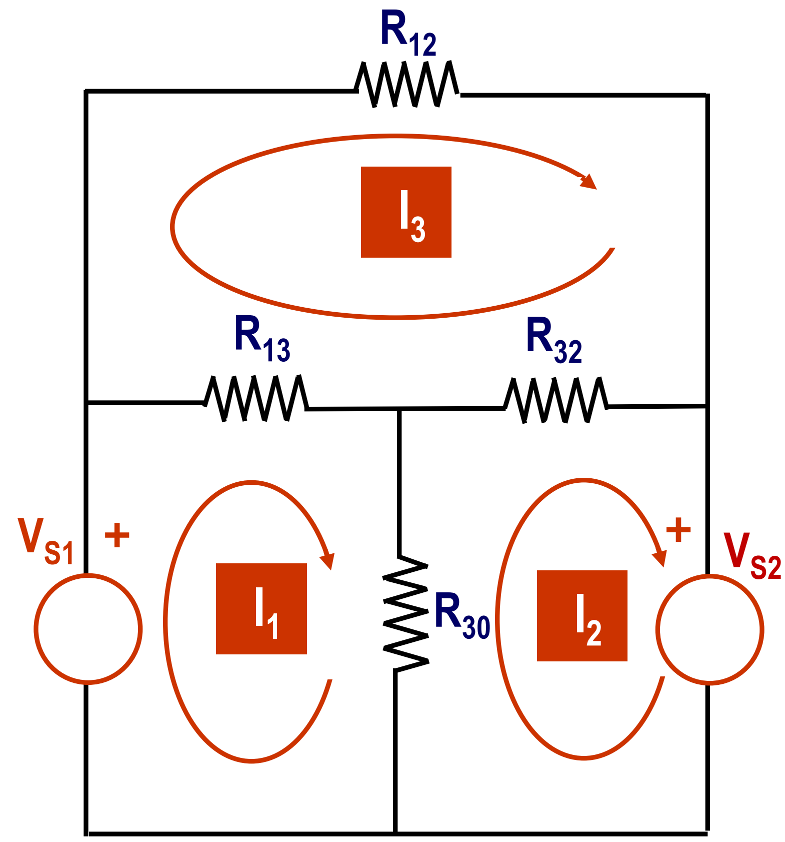


Circuit Analysis

Mesh Current Method

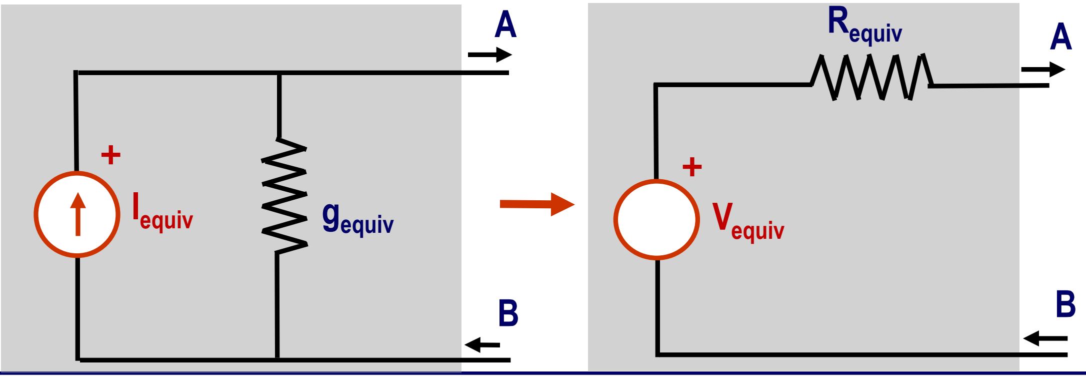
Procedure

2. Define mesh currents in each mesh flowing in the clockwise direction,
3. Convert all current sources with parallel admittances, if any, to equivalent Thevenin voltage sources with series Thevenin equivalent resistances,



Norton Equivalent Circuit

Thevenin Equivalent Circuit

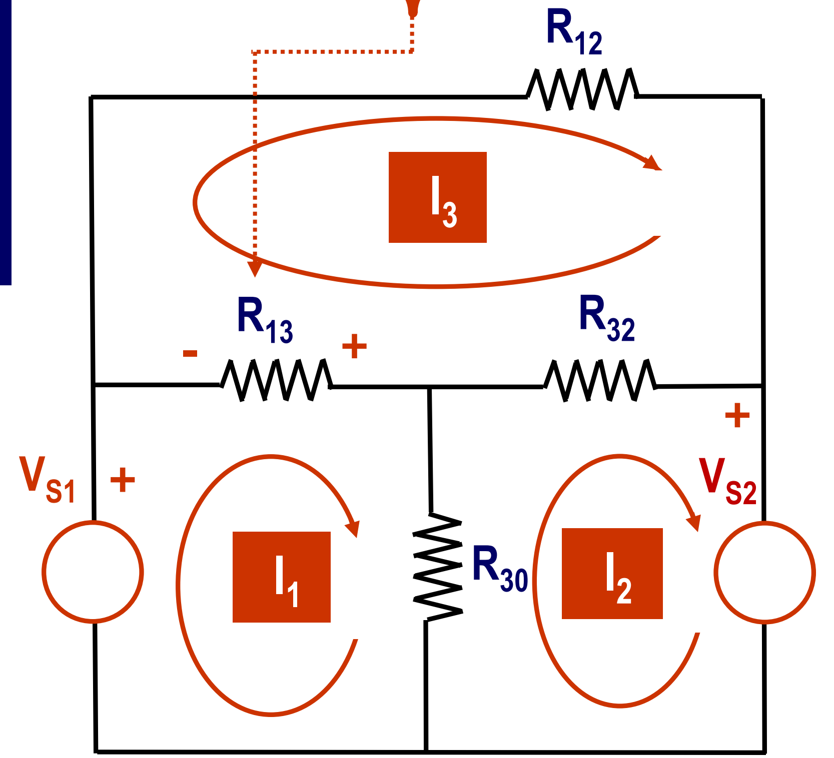


Mesh Current Method

Procedure

4. Write down Kirchoff's Voltage Law (KVL) in each mesh in terms of the source voltages, mesh currents and resistances,
5. Solve the resulting equations

$$R_{13} \times (- I_3) + R_{13} \times (+ I_1) = -R_{13} \times (I_3 - I_1)$$



$$- R_{13} (I_3 - I_1) - R_{12} I_3 - R_{32} (I_3 - I_2) = 0$$

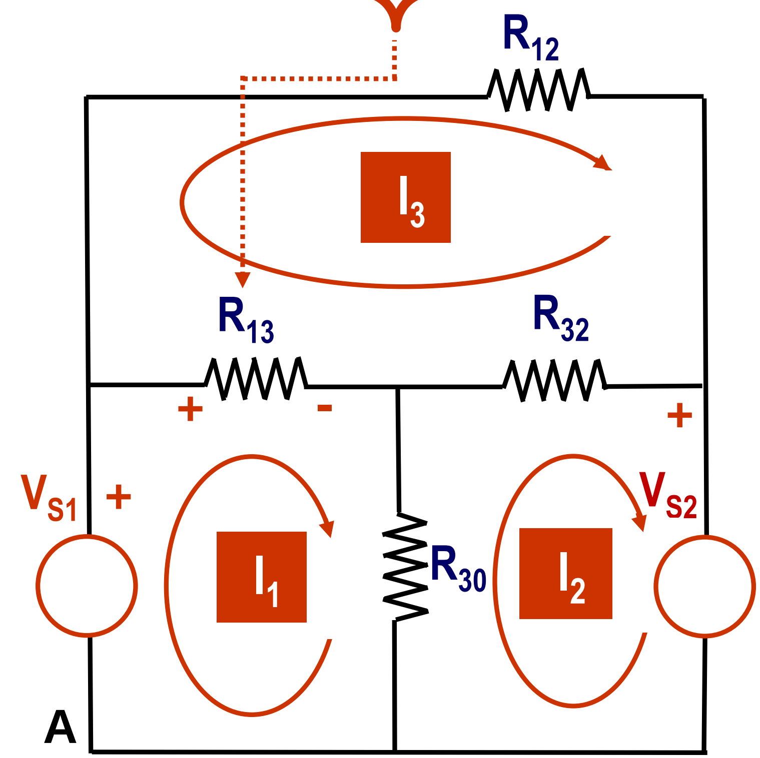
Mesh Current Method

Example

Mesh -1

- Start from a certain point in Mesh-1, if possible from the ground node A and follow a closed path in clockwise direction,
- When you pass over a resistance, for instance, over resistance R_{13};
  - assign "-" sign to the current, if it is in the same direction as your clockwise direction, i.e. I_1,
  - assign "+" sign to the current, if it opposes your clockwise direction, i.e. I_3,
  - sum up the resulting voltage terms in the mesh

$$\underbrace{R_{13}\,x\,(-\,I_1) + R_{13}\,x\,(+\,I_3) = -R_{13}\,x\,(I_1 - I_3)}$$



$$V_{S1} - R_{13}\,(I_1 - I_3) - R_{30}\,(I_1 - I_2) = 0$$

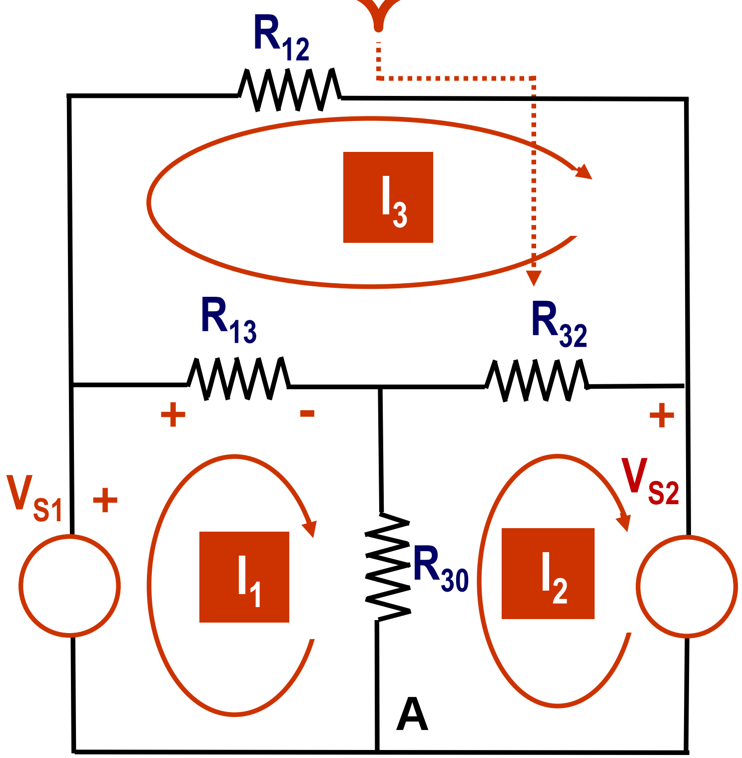
Mesh Current Method

Example

Mesh - 2

- Start from a certain point in Mesh-1, for instance, from point A and follow a path in clockwise direction,
- When you pass over a resistance, for instance, over resistance R_{13};
  - assign "-" sign to the current, if it is in the same direction as your clockwise direction, i.e. I_1,
  - assign "+" sign to the current, if it opposes your clockwise direction, i.e. I_3,
  - sum up the resulting voltage terms in the mesh

$$\underbrace{R_{30} \times (-I_2) + R_{30} \times (+I_1) = -R_{30} \times (I_2 - I_1)}$$



$$-R_{30}\,(I_2 - I_1) - R_{32}\,(I_2 - I_3) - V_{S2} = 0$$

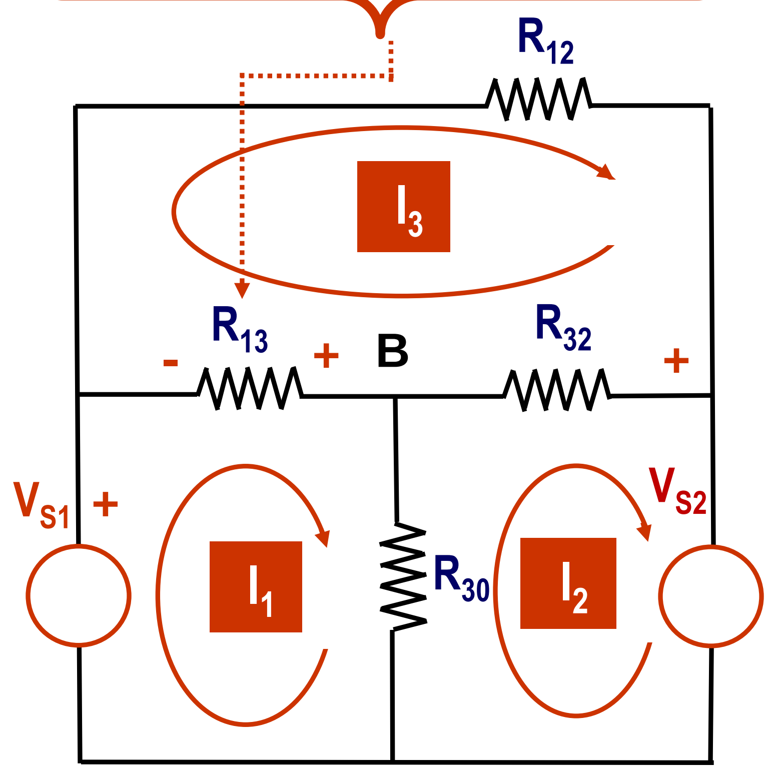
Mesh Current Method

Example

Mesh - 3

- Start from a certain point in Mesh-3, for instance, from point B, and follow a path in clockwise direction,
- When you pass over a resistance, for instance, over resistance R_{13};
  - assign "-" sign to the current, if it is in the same direction as your clockwise direction, i.e. I_3,
  - assign "+" sign to the current, if it opposes your closckwise direction, i.e. I_1,
  - sum up the resulting voltage terms in the mesh

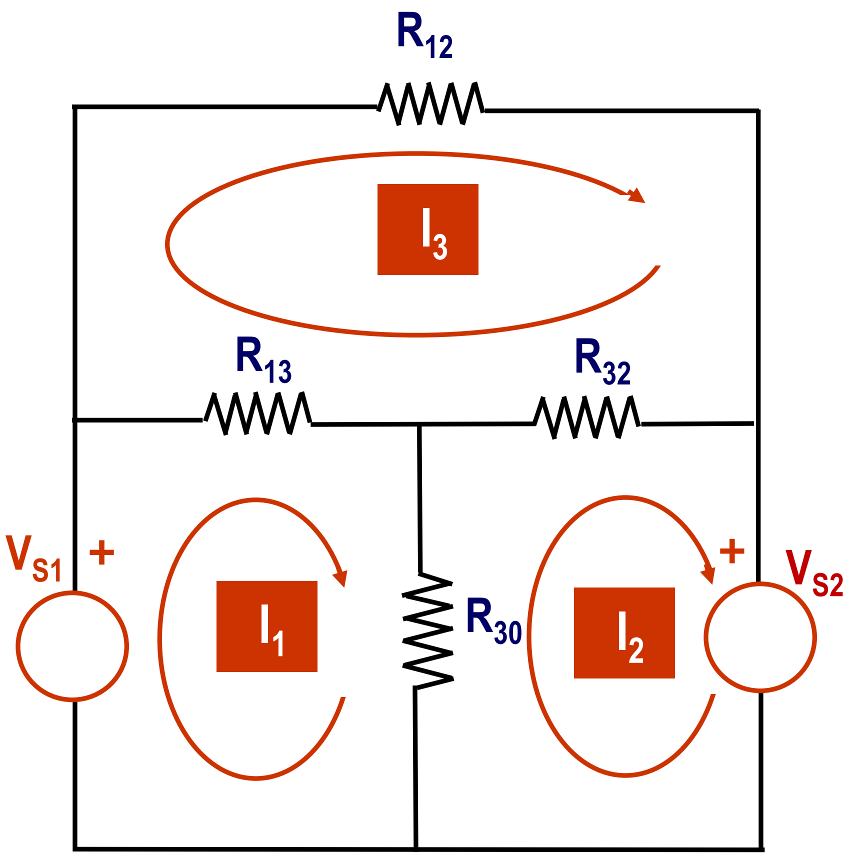
$$\underbrace{R_{13} \, x \, (-I_3) + R_{13} \, x \, (+I_1) = -R_{13} \, x \, (I_3 - I_1)}$$



$$-R_{13}(I_3 - I_1) - R_{12} I_3 - R_{32}(I_3 - I_2) = 0$$

Mesh Current Method

Procedure	
Resulting Mesh Equations	
Mesh -1	$V_{S1} - R_{13}\,(I_1 - I_3) - R_{30}\,(I_1 - I_2) = 0$
Mesh -2	$-V_{S2} - R_{30}\,(I_2 - I_1) - R_{32}\,(I_2 - I_3) = 0$
Mesh -3	$-R_{32}\,(I_3 - I_2) - R_{13}\,(I_3 - I_1) - R_{12}\, I_3 = 0$



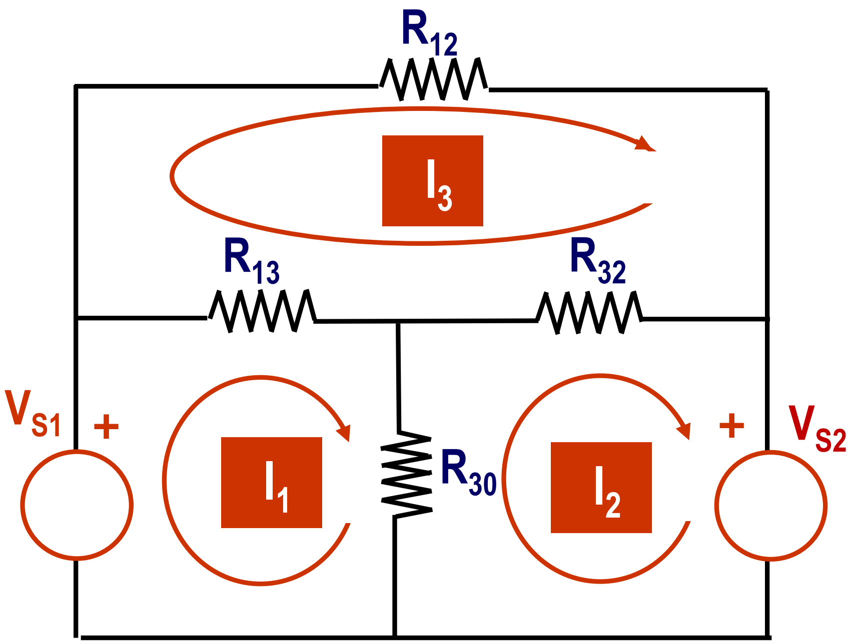
Mesh Equations in Matrix Form

Resulting Mesh Equations

Mesh -1 $V_{S1} - R_{13}\,(I_1 - I_3) - R_{30}\,(I_1 - I_2) = 0$

Mesh -2 $-V_{S2} - R_{30}\,(I_2 - I_1) - R_{32}\,(I_2 - I_3) = 0$

Mesh -3 $R_{32}\,(I_3 - I_2) + R_{13}\,(I_3 - I_1) + R_{12}\,I_3 = 0$



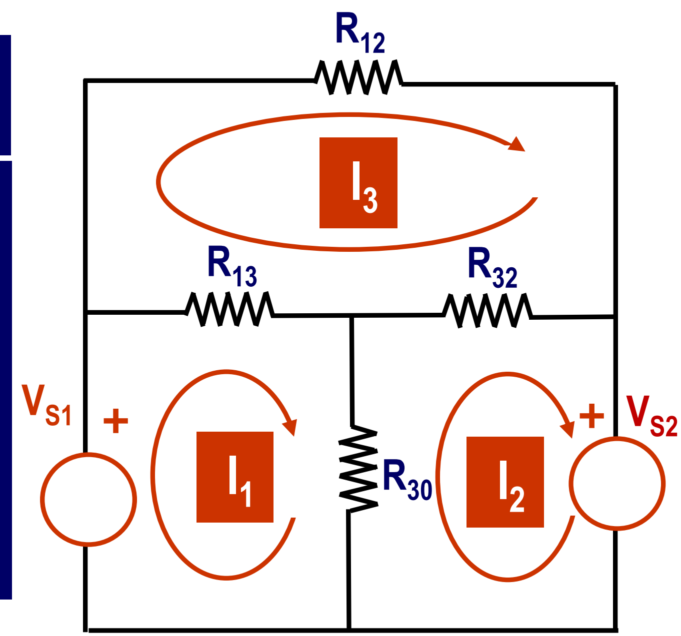
$$\begin{bmatrix} R_{13}+R_{30} & -R_{30} & -R_{13} \\ -R_{30} & R_{32}+R_{30} & -R_{32} \\ -R_{13} & -R_{32} & R_{32}+R_{13}+R_{12} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_{S1} \\ -V_{S2} \\ 0 \end{bmatrix}$$

Mesh Current Method

Procedure

$$
\begin{array}{ccc} \text{Mesh-1} & \text{Mesh-2} & \text{Mesh-3} \end{array}
$$

$$
\begin{bmatrix} R_{13}+R_{30} & -R_{30} & -R_{13} \\ -R_{30} & R_{32}+R_{30} & -R_{32} \\ -R_{13} & -R_{32} & R_{32}+R_{13}+R_{12} \end{bmatrix}
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}
=
\begin{bmatrix} V_{S1} \\ -V_{S2} \\ 0 \end{bmatrix}
$$



Mesh Resistance Matrix

Current (Unknowns) Vector

Voltage (Knowns) (RHS) Vector

Solution Step

Procedure

These elements are zero for meshes with no voltage source

Final step is the solution of the nodal equations by multiplying the Voltage (RHS) Vector with the inverse of the Mesh Resistance (coefficient) matrix

$$\begin{bmatrix} R_{13}+R_{30} & -R_{30} & -R_{13} \\ -R_{30} & R_{32}+R_{30} & -R_{32} \\ -R_{13} & -R_{32} & R_{32}+R_{13}+R_{12} \end{bmatrix}^{-1} \begin{bmatrix} V_{S1} \\ -V_{S2} \\ 0 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

$$R^{-1} \quad V = I$$

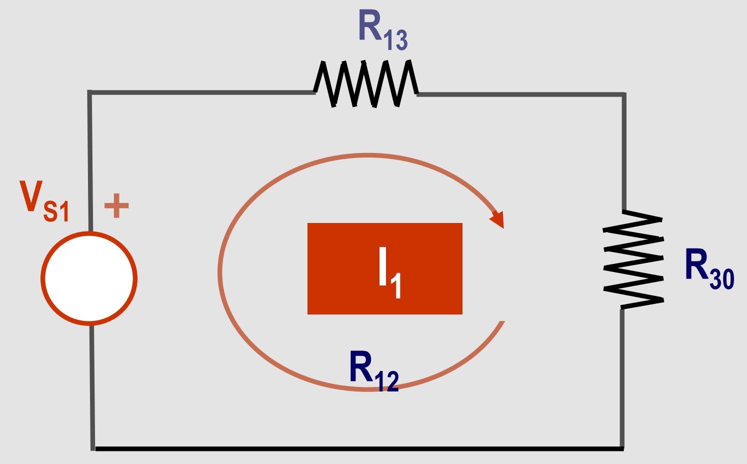
Rules for Forming Mesh Resistance Matrix

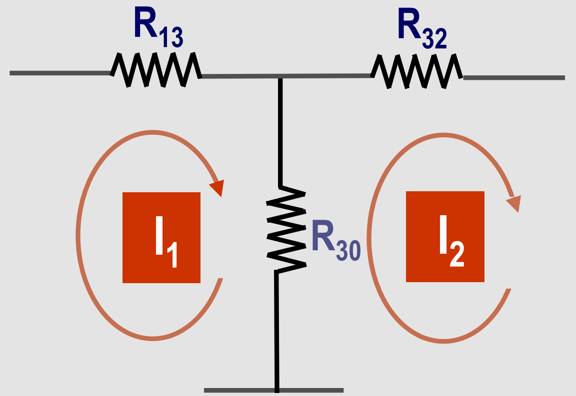
Rules

$$
\begin{array}{ccc}
\text{Mesh-1} & \text{Mesh-2} & \text{Mesh-3} \\
\end{array}
$$

$$
\begin{bmatrix}
R_{13}+R_{30} & -R_{30} & -R_{13} \\
-R_{30} & R_{32}+R_{30} & -R_{32} \\
-R_{13} & -R_{32} & R_{32}+R_{13}+R_{12}
\end{bmatrix}
$$

- Put the summation of the resistances of branches in the i^{th} mesh path to the i^{th} diagonal location in the mesh resistance matrix,
- Put the negative of the resistance of branch which is common to both i^{th} and j^{th} meshes to the $(i-j)^{th}$ location of the mesh resistance matrix



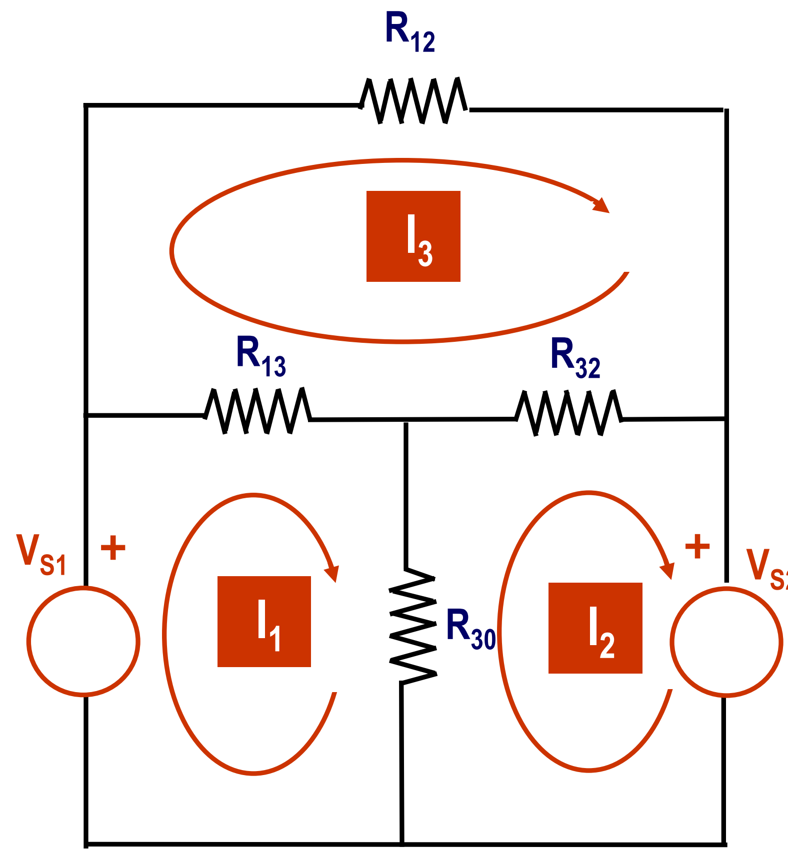


Rules for Forming the Unknown (Mesh Current) Vector

Rules

1	I_1
2	I_2
3	I_3

- Write down the mesh currents in this vector in a sequence starting from 1 to n-1 (i.e. for all meshes in the circuit)



Rules for Forming the known (RHS) Vector

Rules

$$\begin{array}{c} 1 \\ 2 \\ \\ \end{array} \begin{bmatrix} V_{S1} \\ -V_{S2} \\ 0 \end{bmatrix}$$

- Write down the source voltages in meshes in this vector,
- i-th element in this vector = $\begin{cases} V_{Si1} + V_{Si2} + \dots & \text{(Sum of the voltage sources in the mesh)} \\ 0 & \text{otherwise} \end{cases}$

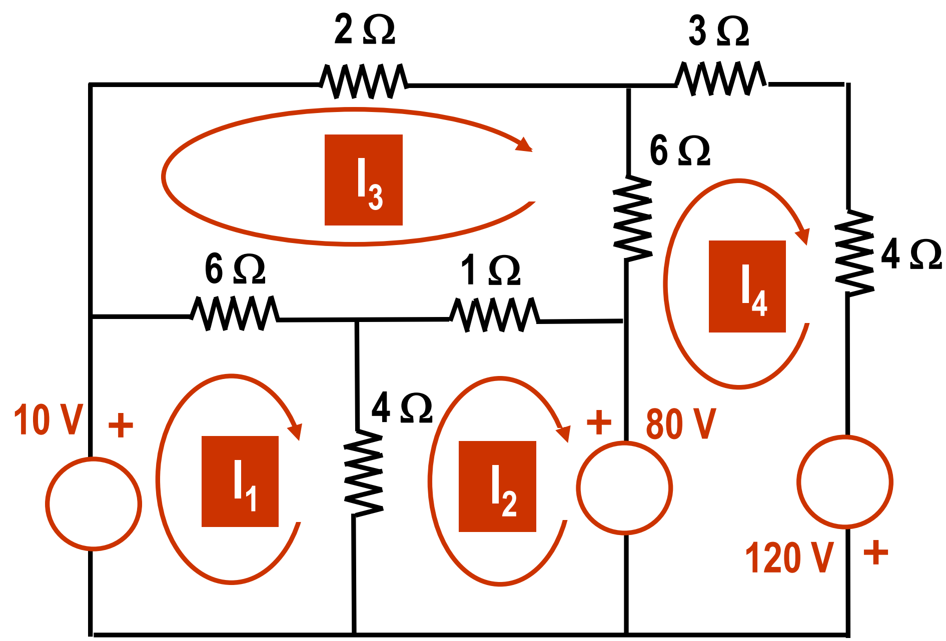
10 kW Turbine

Circuit Analysis

Example - 1

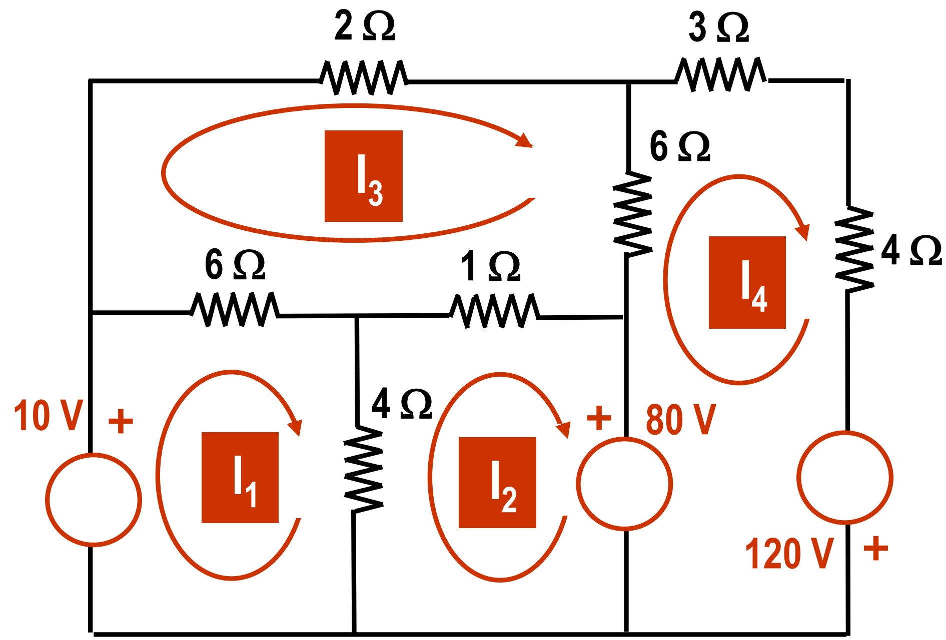
Example

Write down the mesh equations for the circuit shown on the right hand side



Mesh -1	$10 - 6\,(I_1 - I_3) - 4\,(I_1 - I_2) = 0$
Mesh -2	$-\,4\,(I_2 - I_1) - 1\,(I_2 - I_3) - 80 = 0$
Mesh -3	$-\,2\,I_3 - 6\,(I_3 - I_4) - 1\,(I_3 - I_2) - 6\,(I_3 - I_1) = 0$
Mesh -4	$+80 - 6\,(I_4 - I_3) - 7\,I_4 + 120 = 0$

Example - 1



Example

These equations may then be written in matrix form as follows

$$\begin{bmatrix} 10 & -4 & -6 & \\ -4 & 5 & -1 & \\ -6 & -1 & 15 & -6 \\ & & -6 & 13 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 10 \\ -80 \\ 0 \\ 200 \end{bmatrix}$$

Example - 2

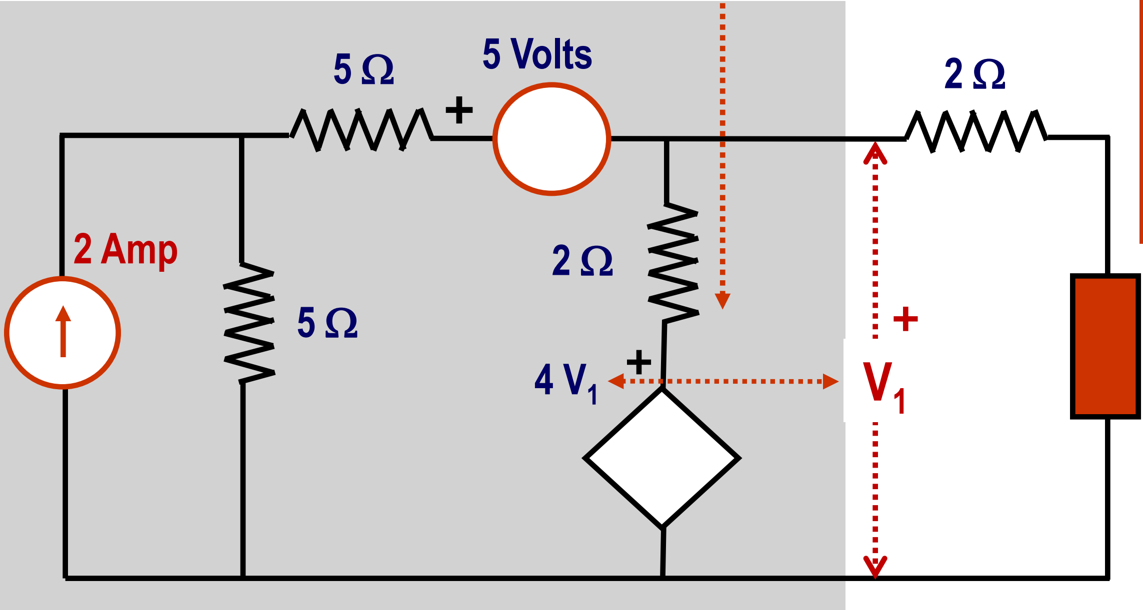
Mesh Current Method

Find the power dissipated in the load resistance R_L by using the Mesh Current Method

Please note that voltage controlled voltage source in the following figure can NOT be killed for finding the Thevenin Equivalent Circuit.

If you do, the result will be INCORRECT !

Hence, Thevenin Equivalencing Method is NOT applicable



Load Resistance: R_L = 1 Ω

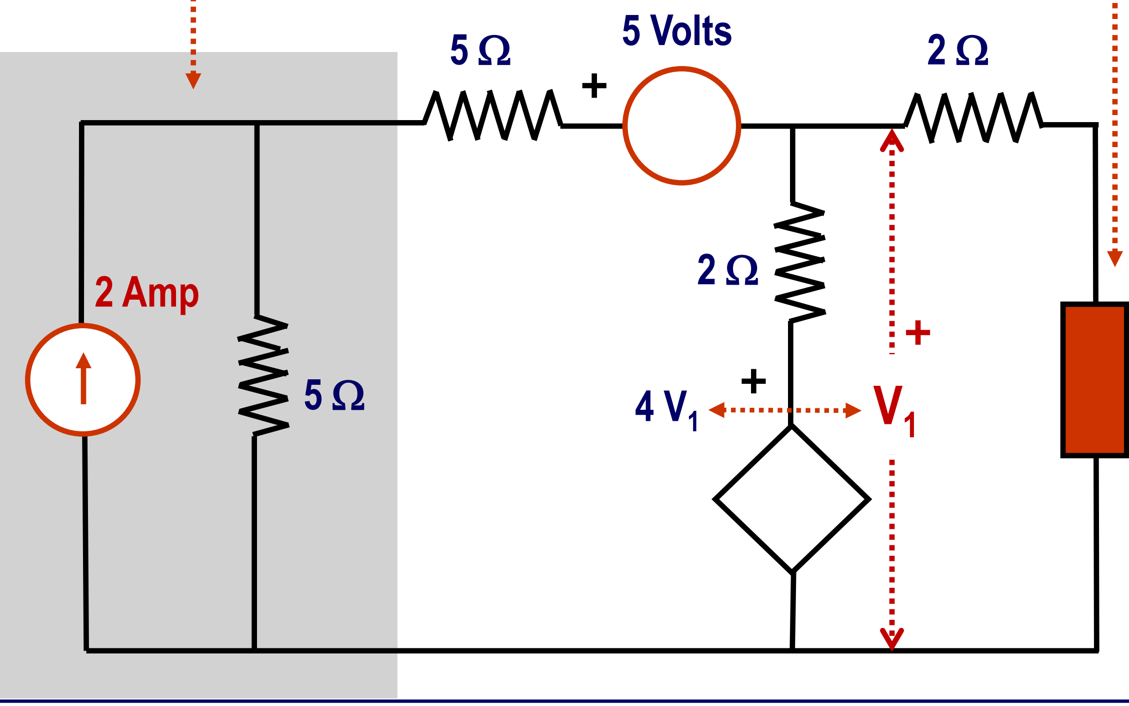
Example - 2 (Continued)

Mesh Current Method

Northon Equivalent Circuit

Load Resistance: 1 Ω

To simplify the circuit, first convert the Northon Equivalent Circuit shown in the shaded area to Thevenin Equivalent Circuit



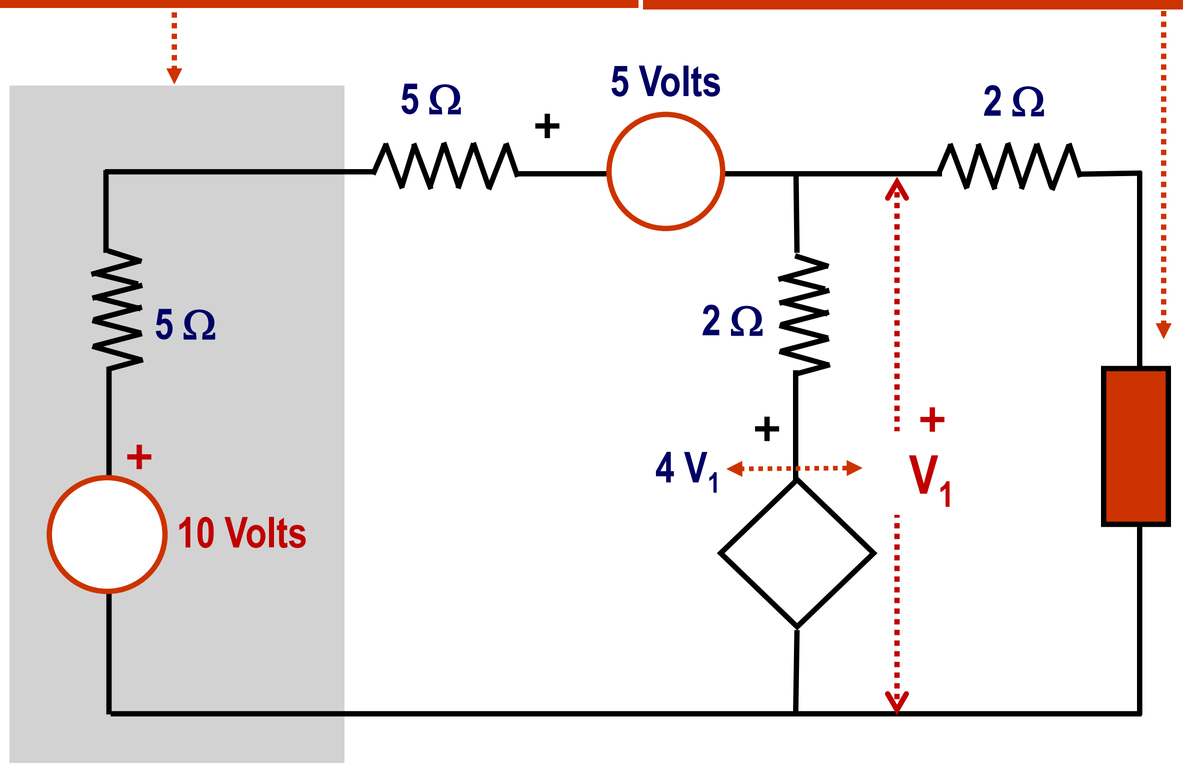
Example - 2 (Continued)

Mesh Current Method

Thevenin Equivalent Circuit

Load Resistance: 1 Ω

To simplify the circuit, further, combine the voltage sources and the 5 Ohm resistances



Circuit Analysis

Example - 2 (Continued)

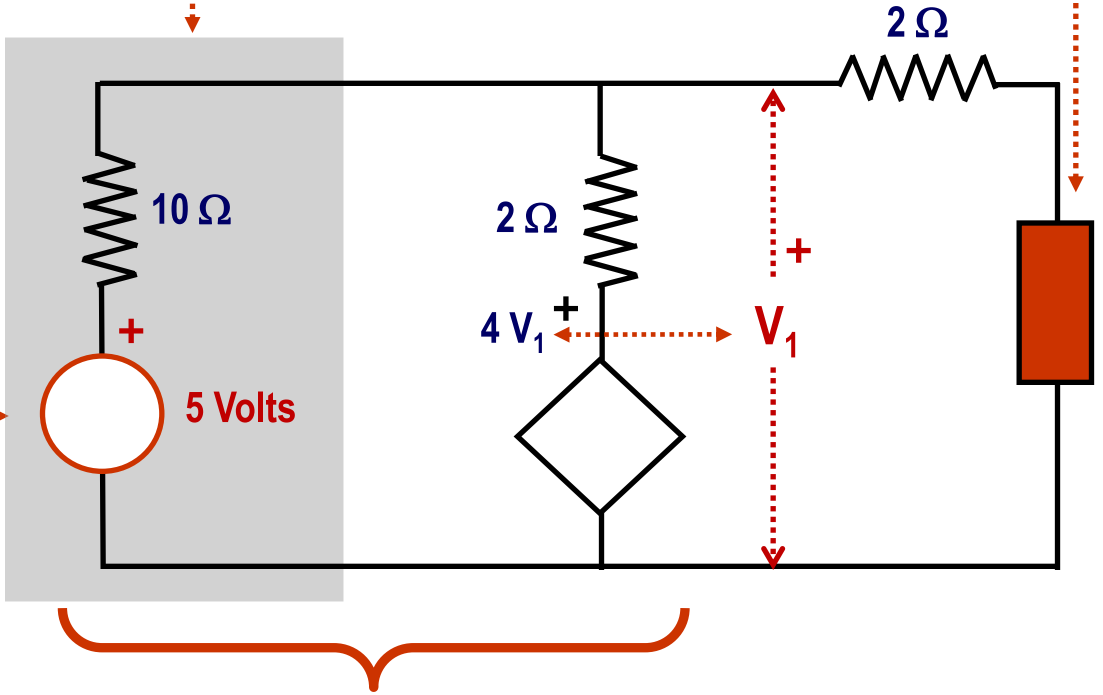
Mesh Current Method

Now, write down the Mesh Equations

Thevenin Equivalent Circuit

Load Resistance: 1 Ω

Please note that 10 - 5 = 5 Volts voltage source remains here



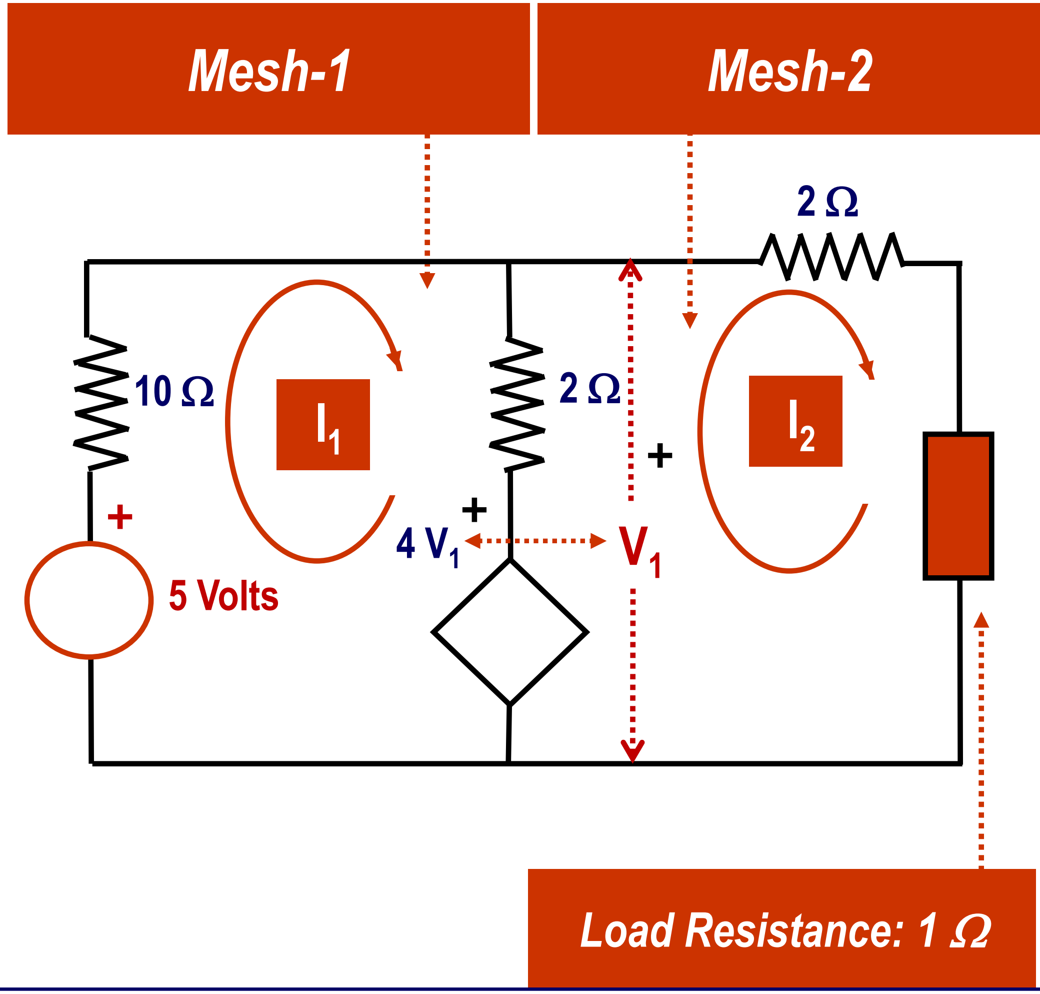
This part can not be further simplified, by employing Thevenin Equivalencing Method, since it contains a controlled voltage source

Example - 2 (Continued)

Mesh Current Method

Further simplifying the circuit

Mesh -1	$5 - 10\, I_1 - 2\,(I_1 - I_2) - 4\, V_1 = 0$
Mesh -2	$4V_1 - 2\,(I_2 - I_1) - (2+1)\, I_2 = 0$
Extra	$V_1 = (2+1)\, I_2 = 3\, I_2$



Circuit Analysis

Example - 2 (Continued)

Mesh Current Method

Substituting the Extra Equation into Mesh-1 and Mesh-2 Equations;

Mesh -1	$5 - 12\, I_1 - 10\, I_2 = 0$
Mesh -2	$7\, I_2 + 2\, I_1 = 0$

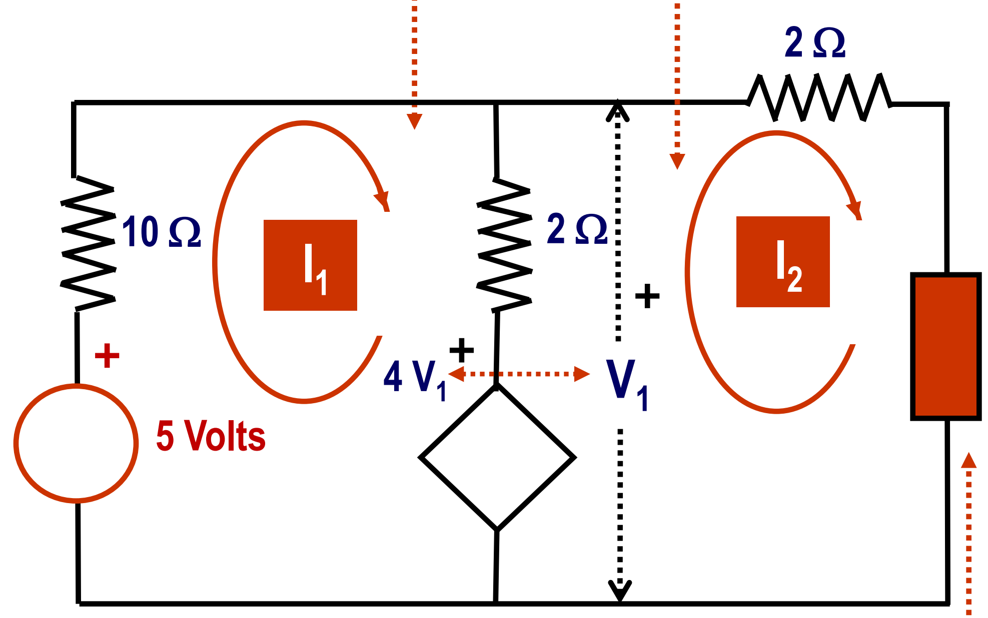
$I_2 = 0.15625$ Amp

$P_{load} = 1 \times I_2^2 = 0.15625^2$

$= 0.02441$ W $=$ <u>24.41 mWatt</u>

Mesh-1

Mesh-2



Load Resistance: 1 Ω

Mesh Analysis with Pure Current Sources

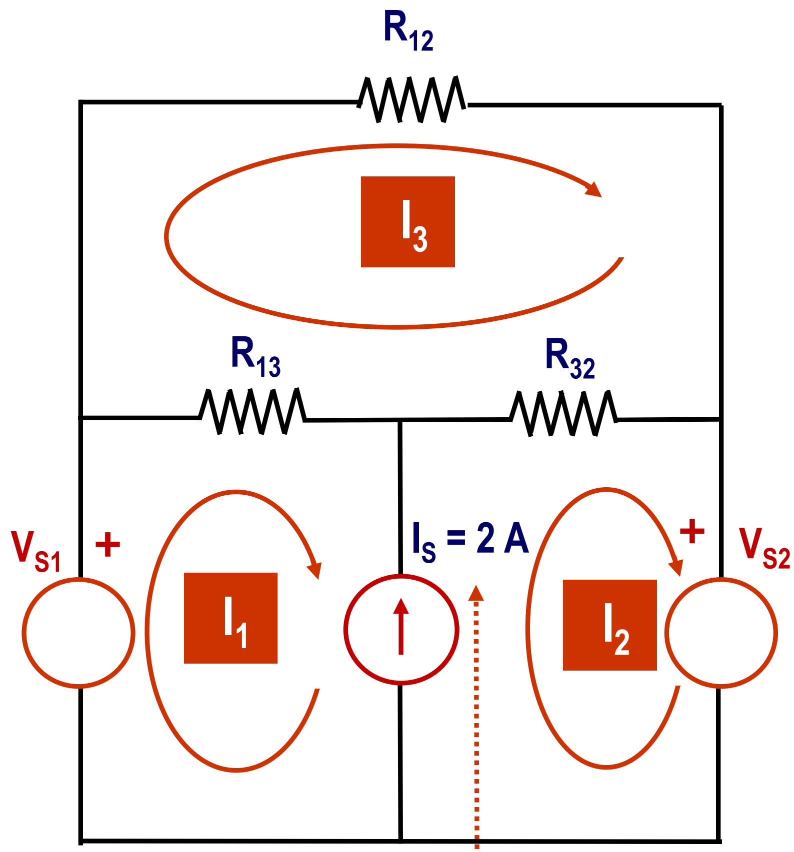
Procedure

Sometimes we may encounter a current source with no parallel admittance, called “Pure Current Source”

A pure current source connecting two nodes without any shunt admittance means that there is fixed difference between the mesh currents involving this current source

A pure current source with no shunt admittance creates problem, since it cannot be converted into an equivalent Thevenin form, i.e.

$$I_s \,/\, g_{equiv} = I_s \,/\, 0 = \infty$$



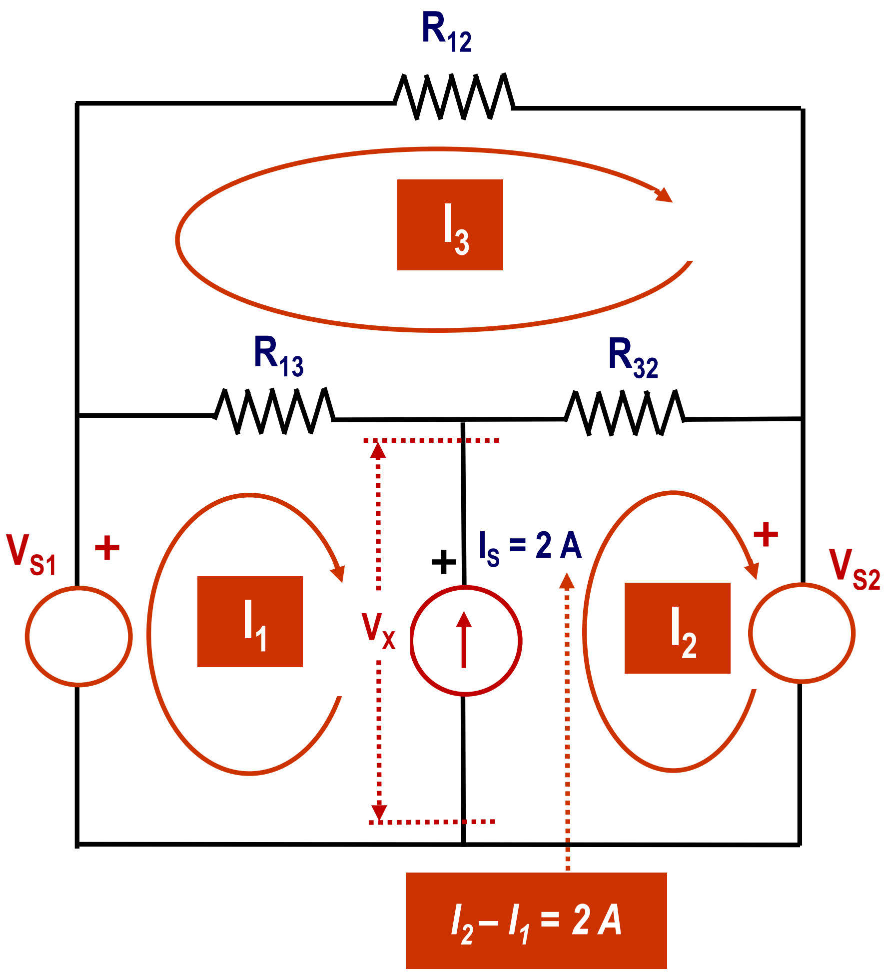
$I_2 - I_1 = 2\ A$

Mesh Analysis with Pure Current Sources

Procedure

The circuit is solved as follows

1. Define the voltage across the pure current source as V_x
2. Define this voltage (V_x) as a new variable,
3. Write down KVL for each mesh,
4. Write down the equation for the current difference between the meshes by using this pure current source



Mesh Analysis with Pure Current Sources

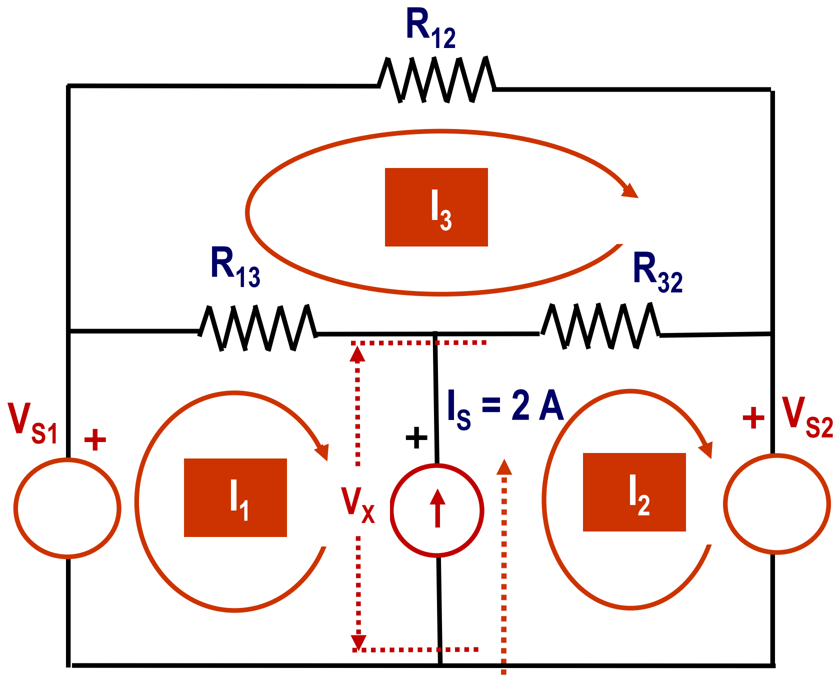
Resulting Mesh Equations

1 $\sum_{i=1}^{i=n} V_i \text{ (including } V_X \text{) } = 0$

2 $\sum_{i=1}^{i=n} V_i \text{ (including } V_X \text{) } = 0$

3 $\sum_{i=1}^{i=n} V_i = 0$

4 $I_2 - I_1 = 2 \text{ Amp}$



$I_2 - I_1 = 2\ A$

(n + 1) Equations vs (n + 1) unknowns

Circuit Analysis

Mesh Analysis with Pure Current Sources

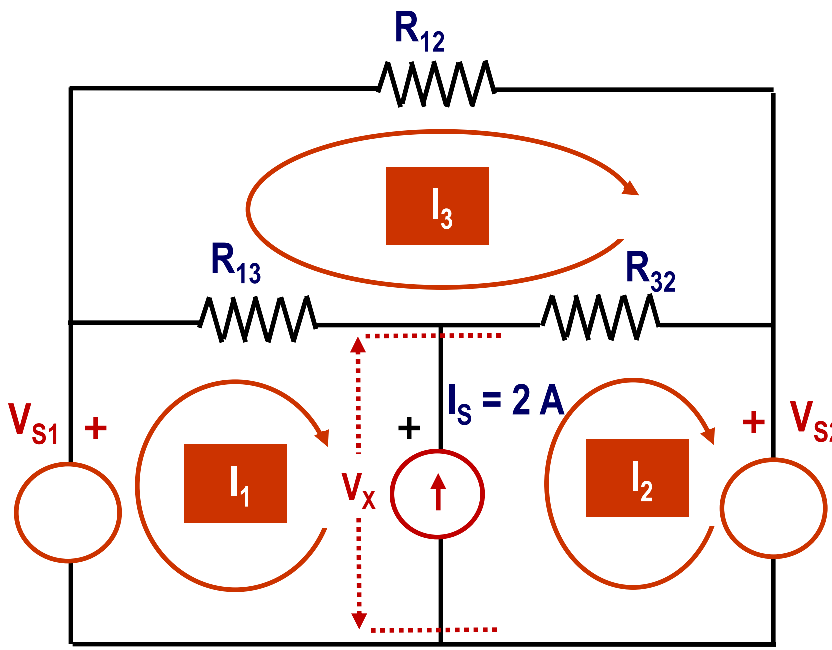
Resulting Equations

Mesh -1 $V_{S1} - R_{13}\,(I_1 - I_3) - V_x = 0$

Mesh -2 $-V_{S2} + V_x - R_{32}\,(I_2 - I_3) = 0$

Mesh -3 $R_{32}\,(I_3 - I_2) + R_{13}\,(I_3 - I_1) + R_{12}\,I_3 = 0$

Extra Equation $I_2 - I_1 = 2\ Amp$



$$\begin{bmatrix} R_{13} & & -R_{13} & 1 \\ & R_{32} & -R_{32} & -1 \\ -R_{13} & -R_{32} & R_{32}+R_{13}+R_{12} & \\ -1 & 1 & & \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_x \end{bmatrix} = \begin{bmatrix} V_{S1} \\ -V_{S2} \\ 0 \\ 2 \end{bmatrix}$$

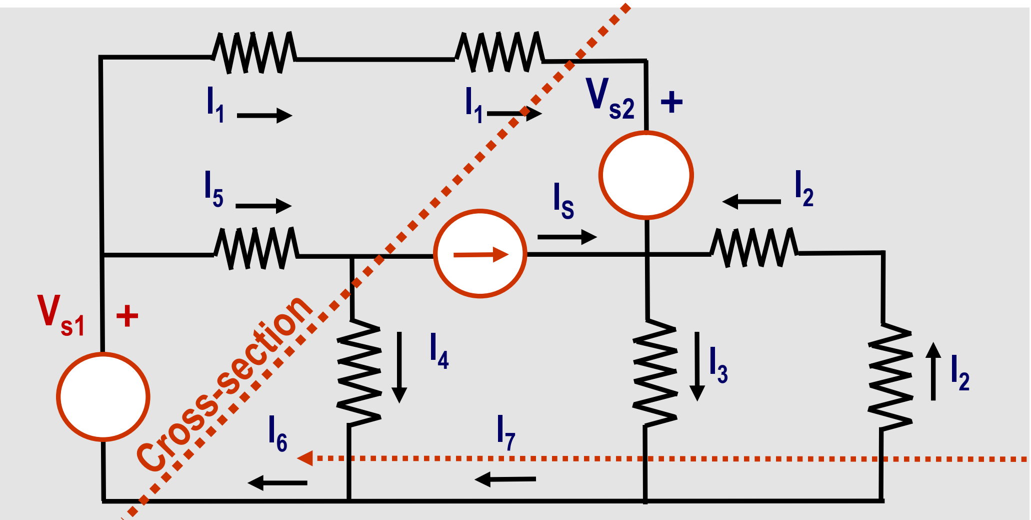
Supernode

Net current flowing <u>through any cross-section</u> in a circuit is zero

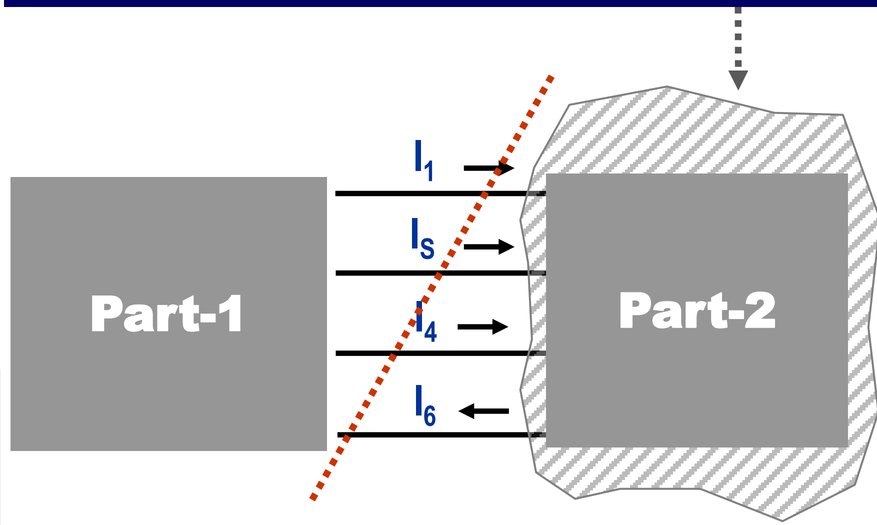
$$I_1 + I_S + I_4 - I_6 = 0$$

or

$$\sum_{i=1}^{i=n} I_i = 0$$



This part (Part-2) may be regarded as a node; "supernode"



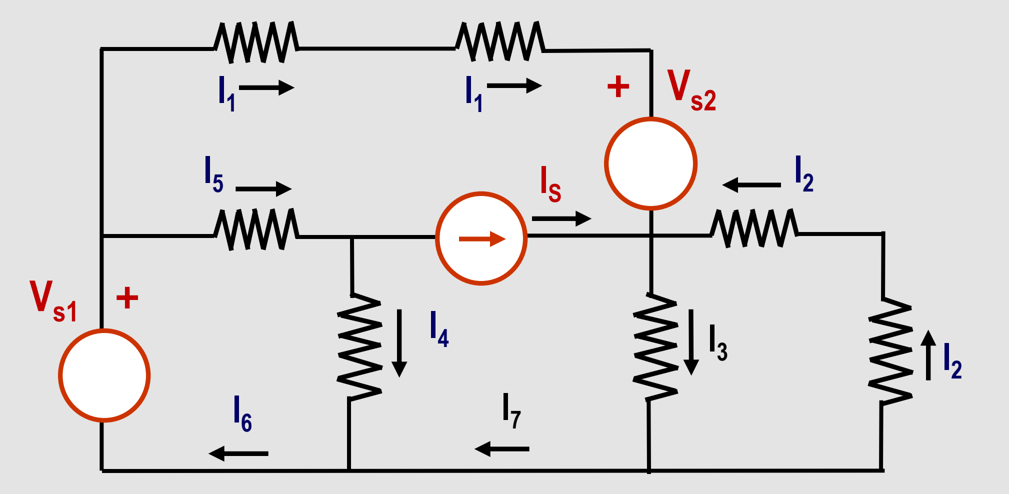
This cross-section line may be drawn arbitrarily passing through in any path

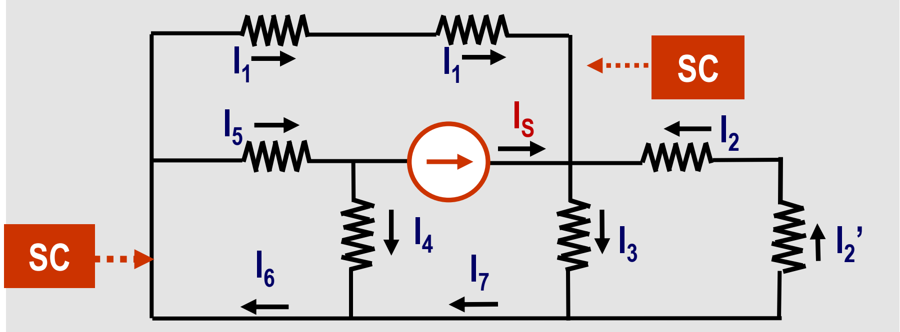
The above rule is actually nothing, but Kirchoff's Current Law (KCL)

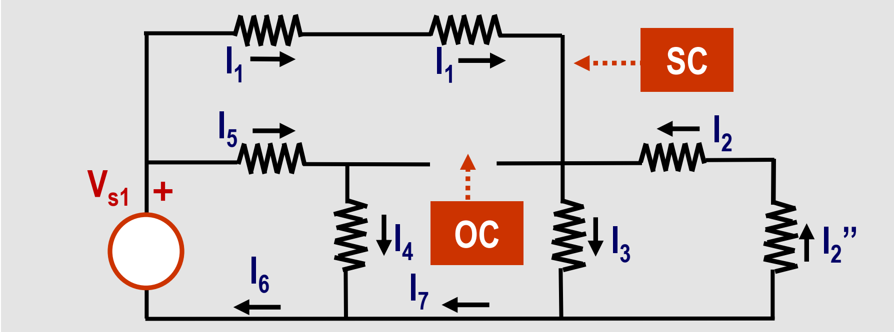
The Principle of Superposition

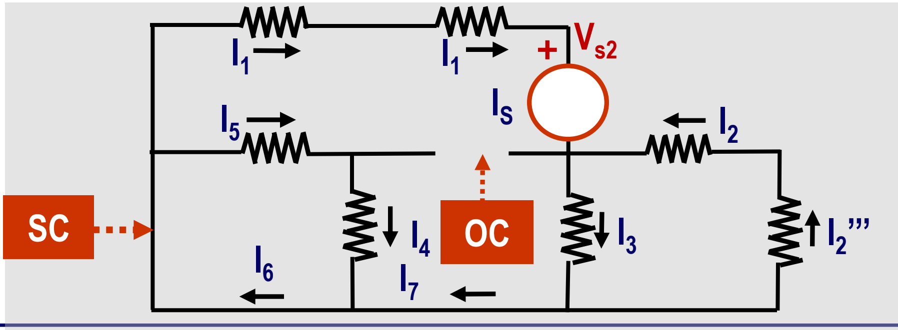
Method

- Kill all the sources except one,
- Solve the resulting circuit,
- Restore back the killed source,
- Kill another source,
- Repeate this procedure for all sources,
- Sum up all the solutions found





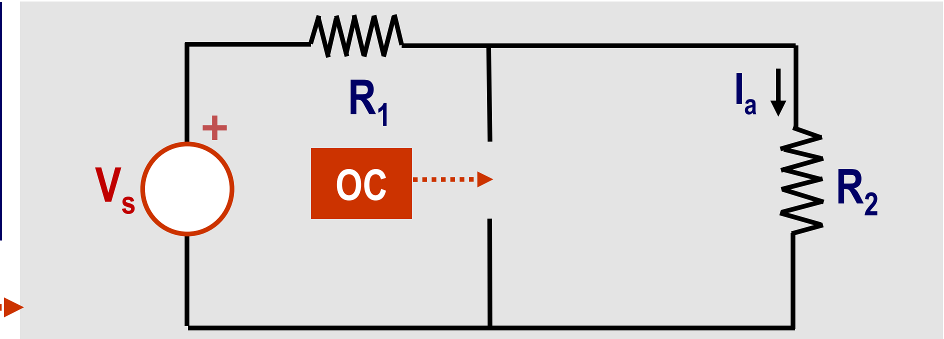




Example

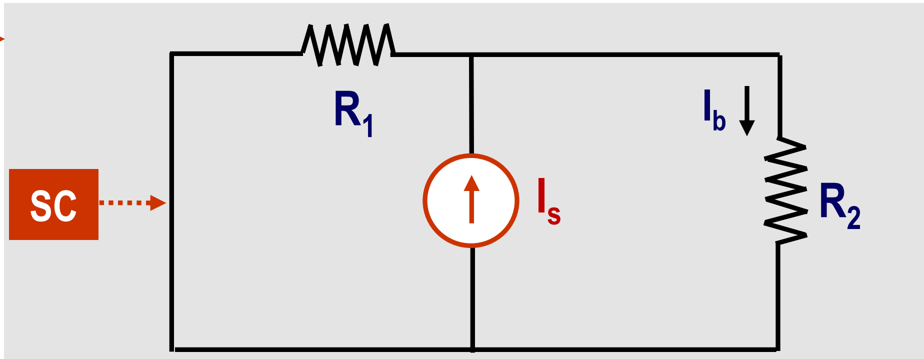
Find the current I_2 flowing in resistance R_2 in the following circuit by using the Principle of Superposition

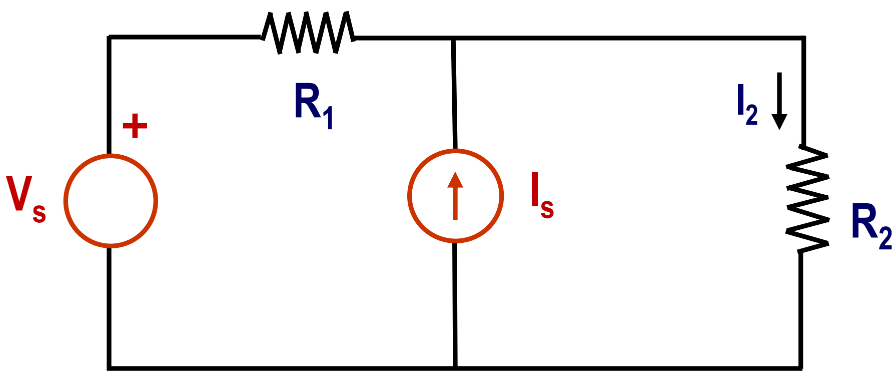
Kill the current source and solve the resulting cct



$$I_a = V_s / (R_1 + R_2)$$

Kill the voltage source and solve the resulting cct





$$I_b = I_S \times g_2 / (g_1 + g_2) \quad g_1 = 1/R_1, \; g_2 = 1/R_2$$

Sum up the resulting currents algebraically

$$I_2 = I_a + I_b$$

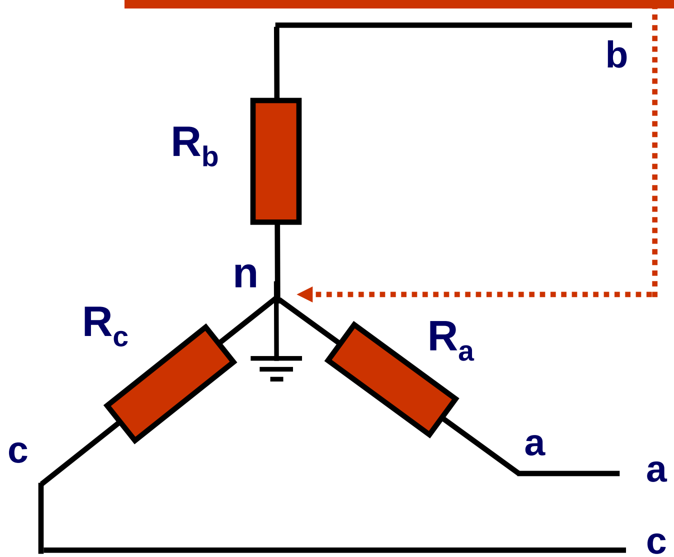
Star - Delta Conversion

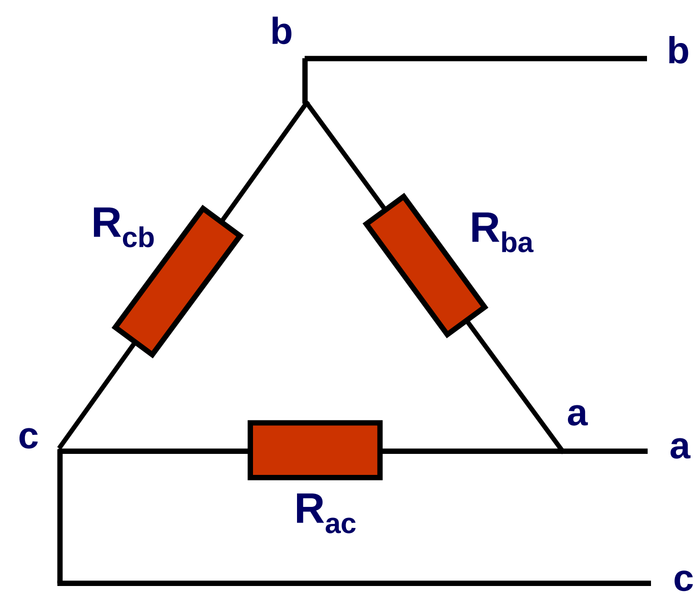
Formulation

A set of star connected resistances can be converted to a delta connection as shown on the RHS

$$R_{ba} = (R_a R_b + R_b R_c + R_c R_a) / R_c$$
$$R_{ac} = (R_a R_b + R_b R_c + R_c R_a) / R_b$$
$$R_{cb} = (R_a R_b + R_b R_c + R_c R_a) / R_a$$

Please note that the neutral node is eliminated by the conversion





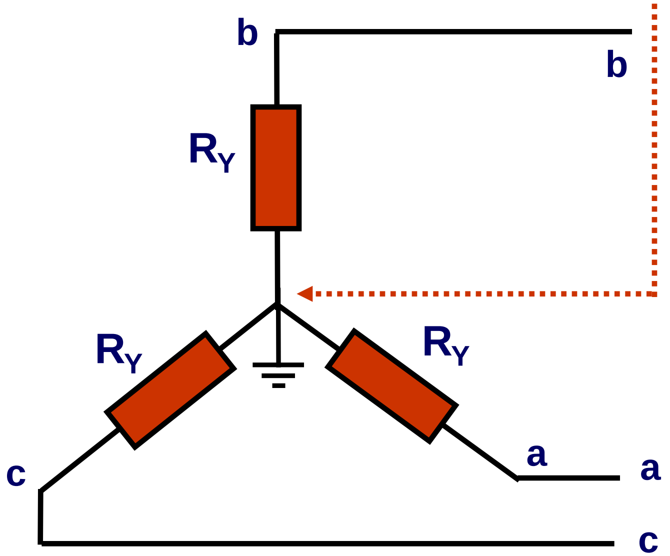
Star - Delta Conversion

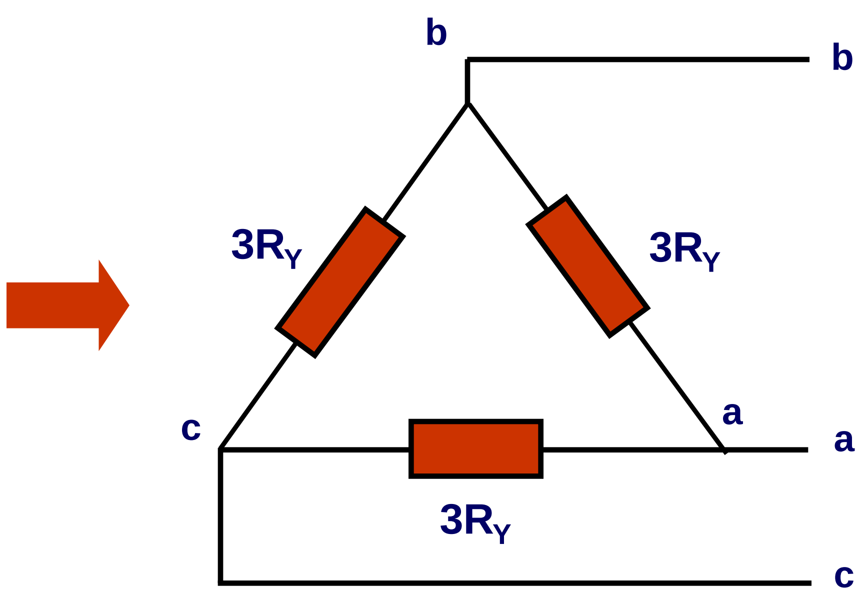
In case that the resistances are identical, the delta connection can further be simplified as shown on the RHS

Simplification

$$R_\Delta = (R_Y^2 + R_Y^2 + R_Y^2) / R_Y = 3 R_Y$$

Please note that the neutral node is eliminated by the conversion





Delta - Star Conversion

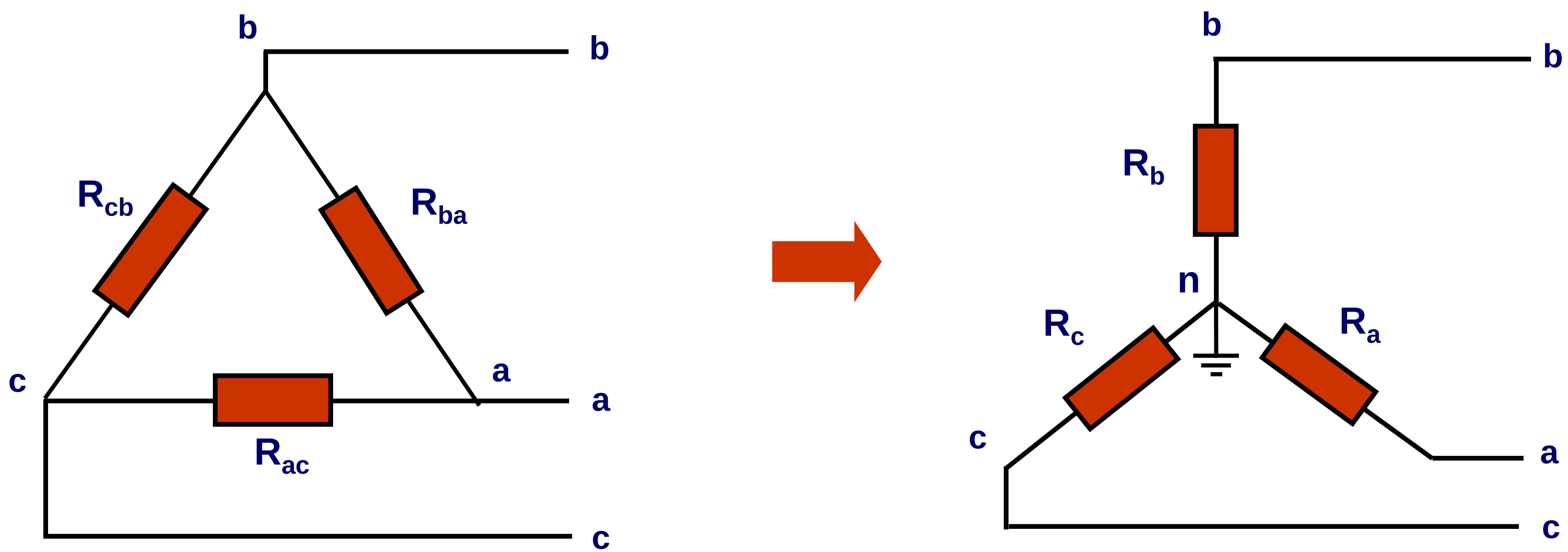
Formulation

A set of delta – connected resistances can be converted to a star connection as shown on the RHS

$$R_a = R_{ba} R_{ac} / (R_{ba} + R_{ac} + R_{cb})$$

$$R_b = R_{cb} R_{ba} / (R_{ba} + R_{ac} + R_{cb})$$

$$R_c = R_{ac} R_{cb} / (R_{ba} + R_{ac} + R_{cb})$$

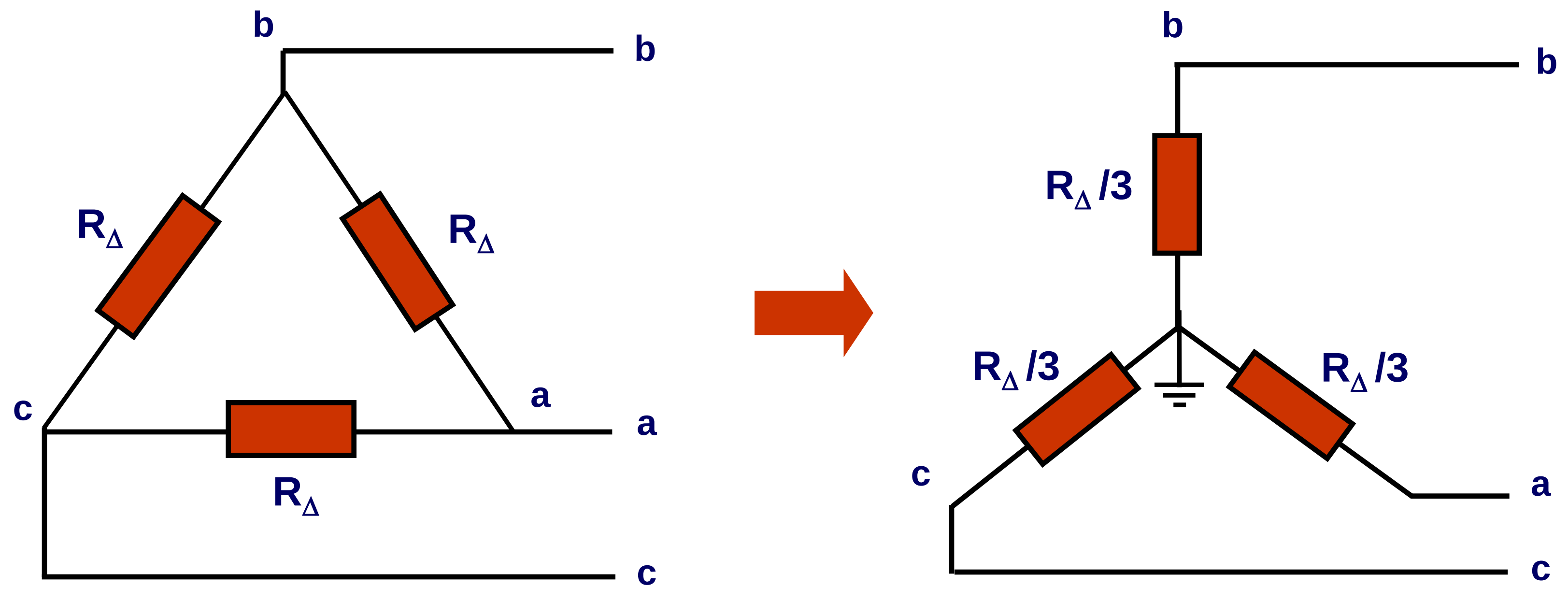


Delta - Star Conversion

In case that the resistances are identical, the equivalent star connection can further be simplified to the form shown on the RHS

Simplification

$$R_Y = R_\Delta^2 / (R_\Delta + R_\Delta + R_\Delta) = R_\Delta / 3$$



Circuit Analysis