3.2 Expectation

The expected value (or the *mean*) of a continuous random variable X with PDF $f_X(x)$ is defined as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

whenever the integral converges[1].

The following results can be show similarly as in the discrete case:

- Note that $E[X]$ is the center of gravity of the PDF.

- $E[g(X)] = \ldots$

- $\text{var}X = \ldots$

- If $Y = aX + b$, $E[Y] = \ldots$ $\text{var}Y = \ldots$

Ex: Compute expectations and variances for Uniform and Exponential random variables.

[1] If the integral is not finite, X is said to "not have a well-defined expectation".

3.3 Cumulative Distribution Functions

Definition 10 *The cumulative distribution function (CDF) of a random variable X is defined as*

$$F_X(x) = \mathrm{P}(X \leq x).$$

In particular,

$$F_X(x) = \begin{cases} \sum_{k \leq x} p_X(k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(\tau) d\tau, & \text{if } X \text{ is continuous} \end{cases}.$$

The CDF $F_X(x)$ accumulates probability "up to (and including)" the value x.

3.3.1 Properties of Cumulative Distribution Functions (CDFs)

(a) $0 \leq F_X(x) \leq 1$

(b) $F_X(-\infty) = \lim_{x \to -\infty} F_X(x) =$, $F_X(\infty) =$

(c) $\mathrm{P}(X > x) = 1 - F_X(x)$

(d) $F_X(x)$ is a monotonically nondecreasing function: if $x \leq y$, then $F_X(x) \leq F_X(y)$.
Proof:

(e) If X is discrete, CDF is a piecewise constant function of x.

(f) If X is continuous, CDF is a continuous function of x. (The definition of a continuous r.v. is based on this.)

(g) $\mathrm{P}(a < X \leq b) = F_X(b) - F_X(a)$.
Proof:

(h) $F_X(x)$ is continuous from the right. That is, for $\delta > 0$

$$\lim_{\delta \to 0} F_X(x+\delta) = F_X(x^+) = F_X(x).$$

Proof:

(i) $\mathrm{P}(X = x) = F_X(x) - F_X(x^-)$, where $F_X(x^-) = \lim_{\delta \to 0} F_X(x - \delta)$.
Proof:

(j) When X is discrete with integer values,

$$F_X(x) = \sum_{k \leq x} p_X(k),$$
$$p_X(x) = \mathrm{P}(X = x) = F_X(x) - F_X(x^-) = F_X(x) - F_X(x-1).$$

(k) If X is continuous,

$$F_X(x) = \int_{-\infty}^{x} f_X(\tau) d\tau,$$
$$f_X(x) = \frac{dF_X(x)}{dx}.$$

The second equality is valid for values of x where $F_X(x)$ is differentiable.

Ex: Let X be exponentially distributed with parameter λ. Derive and sketch $F_X(x)$, the CDF of X.

3.3.2 CDFs of Discrete Random Variables

Discrete random variables have piecewise constant CDFs.

Ex: Find and sketch the CDF of a geometric random variable with parameter p. Compare and contrast this with the CDF of an exponential random variable with rate λ, when p is selected as $1 - e^{\lambda\delta}$, as $\delta > 0$ becomes arbitrarily small.

Ex: You are allowed to take a certain exam n times, and let X_1, X_2, ..., X_n be the grades you get from each of these trials. Assume that the grades are independent and identically distributed. The maximum will be your final score. Find the CDF of your final score, in terms of the CDF of one exam grade. What happens to the CDF as n increases?

Ex: CDF of a r.v. which is neither continuous nor discrete