3.3.3 The Gaussian CDF

The random variable X is Gaussian, in other words, Normal, with parameters (μ, σ^2) if it has the PDF:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

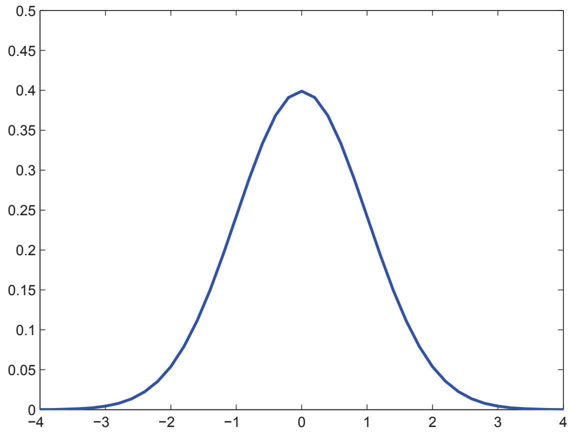

Figure 3.4: Gaussian (normal) PDF

X is said to be a Standard Normal if it's Normal (i.e. Gaussian) with mean 0 and variance 1. That is,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The CDF of the standard Gaussian is defined as follows:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Note that this is the area under the standard Gaussian curve, up to point x. We often use the function $\Phi(x)$ to make calculation involving general Gaussian random variables.

Normality is preserved by linear transformations: If X is Normal(μ, σ^2) and $Y = aX + b$, then Y is Normal$(a\mu + b, a^2\sigma^2)$ (We can prove this after we learn about Transforms of PDFs.) So, we can obtain any Gaussian by making a linear transformation on a standard Gaussian. That is, letting X be a standard Gaussian, if we let $Y = \sigma X + \mu$, then Y is Normal with mean μ and variance σ^2.

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

Then we have:

$$P(Y \leq y) = P((Y-\mu)/\sigma \leq (y-\mu)/\sigma) = P(X \leq (y-\mu)/\sigma) = \Phi((y-\mu)/\sigma)$$

Ex: (Adapted from Ex 3.7 from the textbook.) The annual snowfall at Elmadağ̃ is modeled as a normal random variable with a mean of $\mu = 150$ cm and a standard deviation of 50 cm. What is the probability that next year's snowfall will be at least 200 cm? (Note that from the standard normal table, $\Phi(1) = 0.8413$.)

Ex: Signal Detection (Adapted from Ex 3.7 from the textbook.) A binary message is transmitted as a signal S, which is either $+1$ or -1 with equal probability. The communication channel corrupts the transmission with additive Gaussian noise with mean $\mu = 0$ and variance σ^2. The receiver concludes that the signal $+1$ (or -1) was transmitted if the value received is not negative (or negative, respectively). Find the probability of