1.6 Independence

Definition: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$

- If $\mathrm{P}(A) > 0$, independence implies $\mathrm{P}(B|A) = \mathrm{P}(B)$.
- Symmetrically, if $\mathrm{P}(B) > 0$, independence implies $\mathrm{P}(A|B) = \mathrm{P}(A)$.
- Show that, if A and B are independent, so are A and B^c. (If A is independent of B, the occurrence (or non-occurrence) of B does not convey any information about A.)
- Show that, if A and B are disjoint, they are always *dependent*.

Ex: Consider two independent rolls of a tetrahedral die.

(a) Let $A_i = \{$the first outcome is $i\}$. Let $B_j = \{$the second outcome is $j\}$. "Independent rolls" implies A_i and B_j are independent for any i and j. Find $P(A_i, B_j)$.

(b) Let $A = \{$the max of the two rolls is $2\}$. Let $B = \{$the min of the two rolls is $2\}$. Are A and B independent?

(c) Note that independence can be counter-intuitive. For example, let $A_2 = \{$the first roll is $2\}$. Let $S_5 = \{$the sum of the two rolls is $5\}$. Show that A_2, S_5 are independent, although the sum of the two rolls and the first roll are dependent in general (try A_2, S_6 as a counterexample.)

1.6.1 Conditional Independence

Recall that conditional probabilities form a legitimate probability law. So, A and B are independent, conditional on C, if

$$\mathrm{P}(A \cap B|C) = \mathrm{P}(A|C)\mathrm{P}(B|C)$$

Show that this implies

$$\mathrm{P}(A|B \cap C) = \mathrm{P}(A|C)$$

(assuming $\mathrm{P}(B|C) \neq 0$, $\mathrm{P}(C) \neq 0$.)

Conditioning may affect independence.

Ex: Assume A and B are independent, but $A \cap B \cap C = \emptyset$. If we are told that C occurred, are A and B independent? (draw Venn Diagram exhibiting a counterexample.)

Ex: Two unfair coins, A and B.

$$P(H|\text{coinA}) = 0.9,\ P(\text{H}|\text{coinB}) = 0.1$$

Choose either coin with equal probability.

- Once we know it is coin A, are future tosses independent.
- If we don't know which coin it is, are future tosses independent?
- Compare

$$P(\text{5th toss is a T})$$

  and P(5th toss is a T|first 4 tosses are T).

Independence of a collection of events: Information on some of the events tells us nothing about the occurrence of the others.

- Events A_i, $i = 1, 2, \ldots, n$ are independent iff $P(\cap_{i \in S} A_i) = \pi_{i \in S} P(A_i)$ for any $S \subset \{1, 2, \ldots, n\}$
- Note that

$$P(A_5 \cup A_2 \cap (A_1 \cup A_4)^c | A_3 \cup A_6^c) = P(A_5 \cup A_2 \cap (A_1 \cup A_4)^c)$$

- Pairwise independence does not imply independence! (Checking $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all i and j is not sufficient for confirming independence.)
- For three events, checking $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ is not enough for confirming independence.

Ex: Consider two independent tosses of a fair coin. A = First toss is H.
B = Second toss is H.
C = First and second toss have the same outcome.
Are these events pairwise independent?

$$\begin{aligned} P(C) &= \\ P(C \cap A) &= \\ P(C \cap A \cap B) &= \\ P(C|B \cap A) &= \end{aligned}$$

Ex: Network Connectivity: In the electrical network in Fig. 1.2, each circuit element is "on" with probability p, independently of all others. What is the probability that there is a connection between points A and B?

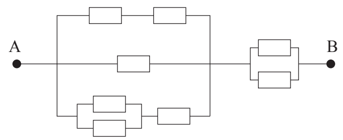

Figure 1.1: Electrical network with randomly operational elements.