1.4.2 Chain (Multiplication) Rule

Assuming that all of the conditioning events have positive probability, the following expression holds

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\dots P(A_n|\bigcap_{i=1}^{n-1} A_i).$$

Ex: There are two balls in an urn numbered with 0 and 1. A ball is drawn. If it is 0, the ball is simply put back. If it is 1, another ball numbered with 1 is put in the urn along with the drawn ball. The same operation is performed once more. A ball is drawn in the third time. What is the probability that the drawn balls are all labeled 1?

1.4.3 Total Probability Theorem

- This is the "divide and conquer" idea. Very useful in modelling and solving problems.
- Partition set B into $A_1, A_2, \dots, A_n$. The A_i's should be disjoint inside B. That is, $A_i \cap A_j \cap B = \emptyset$ for all i, j.

- One way of computing $\mathrm{P}(B)$:

$$\mathrm{P}(B) = \mathrm{P}(A_1)P(B|A_1) + \mathrm{P}(A_2)\mathrm{P}(B|A_2) + \dots \mathrm{P}(A_n)\mathrm{P}(B|A_n)$$

- An often used partition is A and A^c, where A is any event in the sample space, not disjoint with B.

Ex: The "Monty Hall Problem" (Example 1.12 in textbook.) There is a prize behind one of three identical doors. You are told to pick a door. The game show host then opens one of the remaining doors with no prize behind it. At this point, you have the option to switch to the unopened door, or stick to your original choice. What is the better strategy- to stick or to switch? (Hint: Examine each strategy separately. In each, let B be the event of winning, and A the event that the initially chosen door has the prize behind it.) Show that, when you adopt a randomized strategy (you decide whether to switch or not by tossing a fair coin) the probability of winning is 1/2.

1.4.4 Bayes's Rule

This is a rule for combining "evidence".

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B)} \text{ Note how } A \text{ and } B \text{ changed places} \end{aligned}$$

Ex: Criminal X and Criminal Y are both 20 percent likely to commit a certain crime, and they are both 50 percent likely to be near the site of the crime at a given time. As a result of the investigation, it is revealed that Criminal X was near the site at the time of the crime, but Y was not. What are the posterior probabilities of committing the crime for X and Y?

Bayes's rule is often applied to events A_i that form a partition of the given event B.

$$\begin{aligned} P(A_i|B) &= \frac{P(A_i \cap B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{P(B)} \\ &= \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)} \end{aligned}$$

Where, going from the second line to the third, we applied the Total Probability Theorem.

Ex: Let B be the event that the sum of the numbers obtained on two tosses of a die is seven. Given that B happened, find the probability that the first toss resulted in a 3.

1.5 Modelling using conditional probability

Ex: If an aircraft is present in a certain area, a radar detects it and generates and alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm with probability 0.10. We assume that an aircraft is present with probability 0.05.
Event A: Airplane is flying above
Event B: Something registers on the radar screen

(a) $P(A \cap B) =$

(b) $P(B) =$

(c) $P(A|B) =$? (Discuss how to improve this probability.)

Ex: The "false positive puzzle" (Example 1.18 in textbook.): A test for a certain disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person is not sick, the test results are negative with probability 0.95. A person randomly drawn from the population has the disease with probability 0.01. Given that a person is tested positive, what is the probability that the person is actually sick?