Chapter 2

Discrete Random Variables

2.1 Preliminaries

Definition 4 *A random variable is a mapping (a function) from the sample space into real numbers.*

- We can define an arbitrary number of different random variables on the same sample space.

Ex: Toss a fair 6-sided die. Let the random variable X take on the value 1 if the outcome is 6, and 0 otherwise. Let the random variable Y be equal to the outcome of the die. Illustrate the mappings from the sample space associated with X and Y. (Note that $\{X = 1\} = \{\text{outcome is } 6\} = A$, and $\{X = 0\} = A^c$.)

Definition 5 *A discrete random variable takes a discrete set of values. The Probability Mass Function (PMF) of a discrete random variable is defined as*

$$p_X(x) = \mathrm{P}(X = x)$$

Ex: Find and plot the PMFs of X and Y defined in the previous example.

- A discrete random variable is completely characterized by its PMF.

Ex: Let M be the maximum of the two rolls of a fair die. Find $p_M(m)$ for all m. (Think of the sample space description and the sets of outcomes where M takes on the value m.)

2.2 Some Discrete Random Variables

2.2.1 The Bernoulli Random Variable

In the rest of this course, we shall define the Bernoulli random variable with parameter p as the following:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

In shorthand we say $X \sim$Ber(p).

Ex: Express and sketch the PMF of a Bernoulli(p) random variable.

Despite its simplicity, the Bernoulli r.v. is very important since it can model generic probabilistic situations with just two outcomes (often referred to as binary r.v.).

Examples:

- Indicator function: Consider the random variable X defined previously. $X(w) = 1$ if outcome $w \in A$, and $X(w) = 0$ otherwise. So, X indicates whether the outcome is in set A or A^c. X, a Bernoulli random variable, is sometimes called the "indicator function" of the

  event A. This is sometimes denoted as $X(w) = I_A(w)$.

- Consider n tosses of a coin. Let $X_i = 1$ if the i^{th} roll comes up H, and $X_i = 0$ if it comes up T. Each of the X_i's are *independent* Bernoulli random variables. The X_i's, $i = 1, 2, \ldots$ are a sequence of independent "Bernoulli Trials".

- Let Z be the total number of successes in n independent Bernoulli trials. Express Z in terms of n independent Bernoulli random variables.

2.2.2 The Geometric Random Variable

Consider a sequence of independent Bernoulli trials where the probability of success in each trial is p (We will later call this a "Bernoulli Process".) Let Y be the number of trials up to and including the first success. Y is a *Geometric* random variable with parameter p.

P($Y = k$) = for $k =$

Sketch $p_Y(k)$ for all k.

Check that this is a legitimate PMF.

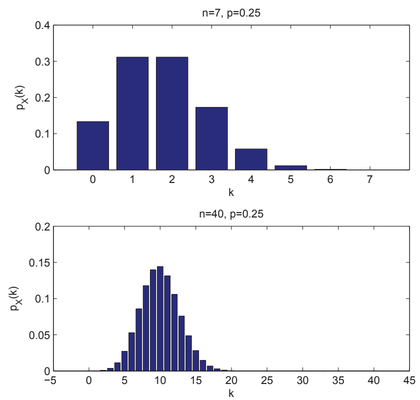
Ex: Let Z be the number of trials up to (but not including) the first success. Find and sketch $p_Z(z)$.

2.2.3 The Binomial Random Variable

Consider n independent Bernoulli Trials each with probability of success p, and let B be the number of successes in the n trials. B is Binomial with parameters (n, p).

$\mathrm{P}(B = k) =$ for $k =$

Ex: Let R be the number of Heads in n independent tosses of a coin with bias p.



2.2.4 The Poisson Random Variable

A Poisson random variable X with parameter λ has the PMF

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

Ex: Show that $\sum_k p_X(k) = 1$ (Hint: use the Taylor series expansion of e^λ.

- The Binomial is a good approximation for the Poisson with $\lambda = np$ when n is very large and p is small, for small values of k. That is, if $k \ll n$

$$\frac{\lambda^k e^{-\lambda}}{k!} \approx \frac{n!}{k!(n-k)!} p^k (1-p)^{(n-k)}$$

2.2.5 The Discrete Uniform R.V.

The discrete uniform random variable takes consecutive integer values within a finite range with equal probability. That is, X is Discrete Uniform in $[a, b]$, $b > a$ if and only if

$$p_X(k) = 1/(b - a + 1) \text{ for } k = a, a+1, a+2, \ldots, b$$

Ex: A four-sided die is rolled. Let X be equal to the outcome, Y be equal to the outcome divided by three, and Z be equal to the square of the outcome.

(Note that Y and Z both take four equally likely values, however they do not have the discrete uniform distribution.)

2.3 Functions of Random Variables

$$Y = f(X)$$

Ex: Let X be the temperature in Celsius, and Y be the temperature in Fahrenheit. Clearly, Y can be obtained if you know X.

$$Y = 1.8X + 32$$

Ex: $\mathrm{P}(Y \geq 14) = \mathrm{P}(X \geq ?)$

Ex: A uniform r.v. X whose range is the integers in $[-2, 2]$. It is passed through a transformation $Y = |X|$.

To obtain $p_Y(y)$ for any y, we add the probabilities of the values x that results in $g(x) = y$:

$$p_Y(y) = \sum_{x:g(x)=y} p_X(x).$$

Ex: A uniform r.v. whose range is the integers in $[-3, 3]$. It is passed through a transformation $Y = u(X)$ where $u(\cdot)$ is the discrete unit step function.

2.4 Expectation, Mean, and Variance

We are sometimes interested in a summary of certain properties of a random variable.

Ex: Instead of comparing your grade with each of the other grades in class, as a first approximation you could compare it with the class average.

Ex: A fair die is thrown in a casino. If 1 or 2 shows, the casino will pay you a net amount of $30,000$ TL (so they will give you your money back plus 30,000), if $3, 4, 5$ or 6 shows you they will take the money you put down. Up to how much would you pay to play this game?

Ex: Alternatively, suppose they give you a total of $30,000$ if you win (regardless of how much you put down), and nothing if you lose. How much would you pay to play this game?

(answer: the value of the first game (the break-even point) is 15,000, and for the second game, it is 10,000. In the second game, you expect to get 30,000 with probability 1/3, so you expect to get 10,000 on average.)