2.4 Expectation and Variance

We are sometimes interested in a summary of certain properties of a random variable.

Ex: Instead of comparing your grade with each of the other grades in class, as a first approximation you could compare it with the class average.

Ex: A fair die is thrown in a casino. If 1 or 2 shows, the casino will pay you a net amount of $30,000$ TL (so they will give you your money back plus 30,000), if $3, 4, 5$ or 6 shows you they will take the money you put down. Up to how much would you pay to play this game?

Ex: Alternatively, suppose they give you a total of $30,000$ if you win (regardless of how much you put down), and nothing if you lose. How much would you pay to play this game?

(Answer: the value of the first game (the break-even point) is 15,000, and for the second game, it is 10,000. In the second game, you expect to get 30,000 with probability 1/3, so you expect to get 10,000 on average.)

Definition 6 *The* ***expected value*** *or* ***mean*** *of a discrete r.v.* X *is defined as*

$$E[X] = \sum_x xP(X = x) = \sum_x xp_X(x).$$

The intuition for the definition is a weighted some of the values the r.v. takes, where the weights are the probability masses of these values.

The mean of X is a representative value, which lies somewhere in the middle of its range. The definition above tells us that the mean corresponds to the *center of gravity* of the PMF.

Ex: Let X be your net earnings in the (first) Casino problem above, where you put down $12,000$ TL to play the game. Find $E(X)$.

Answer: $E(X) = 2000$ (You expect to make money, and the Casino expects to lose money. A more realistic Casino would charge you something strictly more than 15,000, so that they can expect to make a profit.)

2.4.1 Variance, Moments, and the Expected Value Rule

A very important quantity that provides a measure of the spread of X around its mean is variance.

$$var(X) = E[(X - E[X])^2] \tag{2.1}$$

The variance is always nonnegative. One way to calculate $var(X)$ is to use the PMF of $(X - E[X])^2$.

Ex: Find the variance of the random variables X with the following PMFs.

(a) $p_X(15) = p_X(20) = p_X(25) = 1/3$.

(b) $p_X(15) = p_X(25) = 1/2$.

The standard deviation of X is also a measure of the spread of X around its mean: $\sigma_X = \sqrt{var(X)}$. It is usually simpler to interpret since it has the same units as X.

Another way to evaluate $var(X)$ is by using the following result.

Theorem 2 *Let X be a r.v. with PMF $p_X(x)$ and $g(X)$ be a function of X. Then,*

$$E[g(X)] = \sum_x g(x)p_X(x).$$

Proof: Exercise.

Note: Unless $g(x)$ is a linear function, $E[g(X)]$ is in general not equal to $g(E[X])$.

Ex: When I listen to Radyo ODTU in the morning, I drive at a speed of 50 km per hour, and otherwise I drive at 70 km per hour. Suppose I listen to Radyo ODTU with probability 0.3 on any given day. What is the average duration of my 5 km trip to work?
Answer: 4.8 minutes.

Notes: The trip duration T is a nonlinear function $T = D/V$ of the speed V. In fact it is a convex function, which means $E[g(X)] > g(E[X])$. So it would be **wrong** to calculate the expected speed, which is 0.3*50+0.7*70=64 km/hour, and find the expected duration as 5/64*60=4.68 min.