2.4.2 Properties of Expectation and Variance

Expectation is always linear: $E(aX + b) = aE(X) + b$, which follows from the definition (note that the definition is a linear sum.)

Evaluating variance in terms of moments is sometimes more convenient.

$$var(X) = E[X^2] - (E[X])^2 \tag{2.2}$$

Proof:

Variance is NOT linear: $\text{var}(aX + b) = a^2\text{var}(X)$.

Proof:

Consequently,

- adding a constant to a random variable does not change its variance,
- scaling a random variable by a scales the variance by a^2,
- the variance of a constant is 0 (and conversely, a random variable with

zero variance is a deterministic constant.)

Ex: As exercise, derive the mean and variance of the

- Bernoulli(p) random variable

- Discrete Uniform[a,b] random variable.

- Poisson(λ) random variable.

2.5 Joint PMFs of multiple random variables

Often, we need to be able to think about more than one random variable defined on the same probability space. They may or may not contain information about each other. Consider:

- two signals received as a result of two radar measurements
- the current workload at each of a group of network routers
- your grades received from three consecutive exams

Let X and Y be random variables defined on the same probability space. Their joint PMF is defined as the following.

$$p_{X,Y}(x, y) = P(X = x, Y = y)$$

More precise notations for $\mathrm{P}(X = x, Y = y)$: $\mathrm{P}(X = x \text{ and } Y = y)$, $\mathrm{P}(\{X = x\} \cap \{Y = y\})$, $\mathrm{P}(\{X = x, Y = y\})$.

$$\mathrm{P}((X, Y) \in A) =$$

The term **marginal PMF** is used for $p_X(x)$ and $p_Y(y)$ to distinguish them from the joint PMF. Can one find marginal PMFs from the joint PMF?

Note that the event $\{X = x\}$ is the union of the disjoint sets $\{X = x, Y = y\}$ as y ranges over all the different values of Y. Then,

$$\begin{aligned} p_X(x) &= \mathrm{P}(X = x) \\ &= \mathrm{P}(\{X = x\}) = \mathrm{P}(\bigcup_y \{X = x, Y = y\}) \\ &= \sum_y \mathrm{P}(\{X = x, Y = y\}) = \sum_y \mathrm{P}(X = x, Y = y) \\ &= \sum_y p_{X,Y}(x, y). \end{aligned}$$

Similarly, $p_Y(y) = \sum_x p_{X,Y}(x, y)$.

The tabular method can be utilized to obtain the marginal PMFs from the joint PMF.

Ex: Two r.v.s X and Y have the joint PMF given in the 2-D table.

	x=1	2	3
y=1	0	1/10	2/10
2	1/10	1/15	1/30
3	2/10	2/10	1/10

Ex: For the joint PMF in the previous example, please compute the following:

1. $\mathrm{P}(X < Y) =$

2. $p_X(x) =$

Ex: Consider the joint PMF of random variables X and Y which take positive integer values:

$$p_{X,Y}(x,y) = \begin{cases} c & 1 < x+y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

1. $c =?$

2. Find the marginals.

2.6 Functions of Multiple Random Variables

Let $Z = g(X,Y)$.

$$p_Z(z) = \sum_{\{(x,y)|g(x,y)=z\}} p_{X,Y}(x,y)$$

The expected value rule naturally extends to functions of more than one random variable:

$$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$$

Special case when g is linear: $g(X,Y) = aX + bY + c$

$$E[aX + bY + c] =$$

Ex: Expectation of the Binomial r.v.

Ex: The hat problem: The hats of n people are shuffled and randomly redistributed to them. What is the expected number of people getting their own hat? (Alternatively, consider a string of length n randomly formed by shuffling the string 123..n without any regard to the integers. Note that the probability of integer i occuring in the ith location is $1/n$ for each i, by symmetry. Now, compute the expected number of integers staying in their original locations.)

Ex: Multi-sensor laser communication: On/off signaling can be used to transmit bits in laser communication. In an on period of 1μsec, the number of photons detected at a sensor is a Poisson with parameter $\lambda = 100$. In a high-quality system, five such sensors are used to enhance communication. Find the expected value of the total number of photons detected in the system in 1μsec.