Ex: Consider an optical communications receiver that uses a photodetector that counts the number of photons received within a constant time unit. The sender conveys information to the receiver by transmitting or not transmitting photons. There is shot noise at the receiver, and consequently even if nothing is transmitted during that time unit, there may be a positive count of photons. If the sender transmits (which happens with probability 1/2), the number of photons counted (including the noise) is Poisson with parameter $a + n$. If nothing is transmitted, the number of photons counted by the detector is again Poisson with parameter n. Given that the detector counted k photons, what is the probability that a signal was sent? Examine the behavior of this probability with a, n and k.

2.7.1 Conditioning one random variable on another

$$p_{X|Y}(x|y) = \mathrm{P}(X = x|Y = y)$$

Show that

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

The function $p_{X|Y}(x|y)$ has the same shape as $p_{X,Y}(x, y)$ (a slice through the joint pmf at a fixed value of y), and because of the normalization (division by $p_Y(y)$), it is a legitimate PMF.

Ex: The joint PMF of two r.v.s X and Y that share the same range of values $\{0, 1, 2, 3\}$ is given by

$$p_{X,Y}(x, y) = \begin{cases} 1/7 & 1 < x + y \leq 3 \\ 0 & \text{otherwise} \end{cases}.$$

Find $p_{X|Y}(x|y)$ and $p_{Y|X}(y|x)$.

One can obtain the following sequential expressions directly from the definition:

$$\begin{aligned} p_{X,Y}(x,y) &= p_X(x)p_{Y|X}(y|x) \\ &= p_Y(y)p_{X|Y}(x|y). \end{aligned}$$

Ex: A die is tossed and the number on the face is denoted by X. A fair coin is tossed X times and the total number of heads is recorded as Y.

(a) Find $p_{Y|X}(y|x)$.

(b) Find $p_Y(y)$.

2.7.2 Conditional Expectation

Let X and Y be random variables defined in the same probability space, and let A be an event such that $\mathrm{P}(A) > 0$.

$E(X|A) =$

$E(g(X)|A) =$

$E(X|Y = y) =$

Furthermore, let A_i, $i = 1, \dots, n$ be a disjoint partition of the sample space. Then,

$E(X) = \sum_i E(X|A_i)P(A_i)$

$E(X|B) = \sum_i E(X|B \cap A_i)P(A_i|B)$

$E(X) = \sum_y p_Y(y)E(X|Y = y)$

The above three are statements of the "Total Expectation Theorem".

Ex: Data flows entering a router are low rate with probability 0.7, and high rate with probability 0.3. Low rate sessions have a mean rate of 10 Kbps, and high rate ones have a rate of 200 Kbps. What is the mean rate of flow entering the router?

Ex: Find the mean and variance of the Geometric random variable (with parameter p) using the Total Expectation Theorem. (Hint: condition on the events $\{X = 1\}$ and $\{X > 1\}$.

Ex: X and Y have the following joint distribution:

$$p_{XY}(x,y) = \begin{cases} 1/27 & x \in \{4,5,6\}, y \in \{4,5,6\} \\ 2/27 & x \in \{1,2,3\}, y \in \{1,2,3\} \end{cases}$$

Find $E(X)$ using the total expectation theorem.

Ex: Consider three rolls of a fair die. Let X be the total number of 6's, and Y be the total number of 1's. Find the joint PMF of X and Y, $E(X|Y)$ and $E(X)$.

Reading assignment: Example 2.18: The two envelopes paradox, and Problem 2.34: The spider and the fly problem.