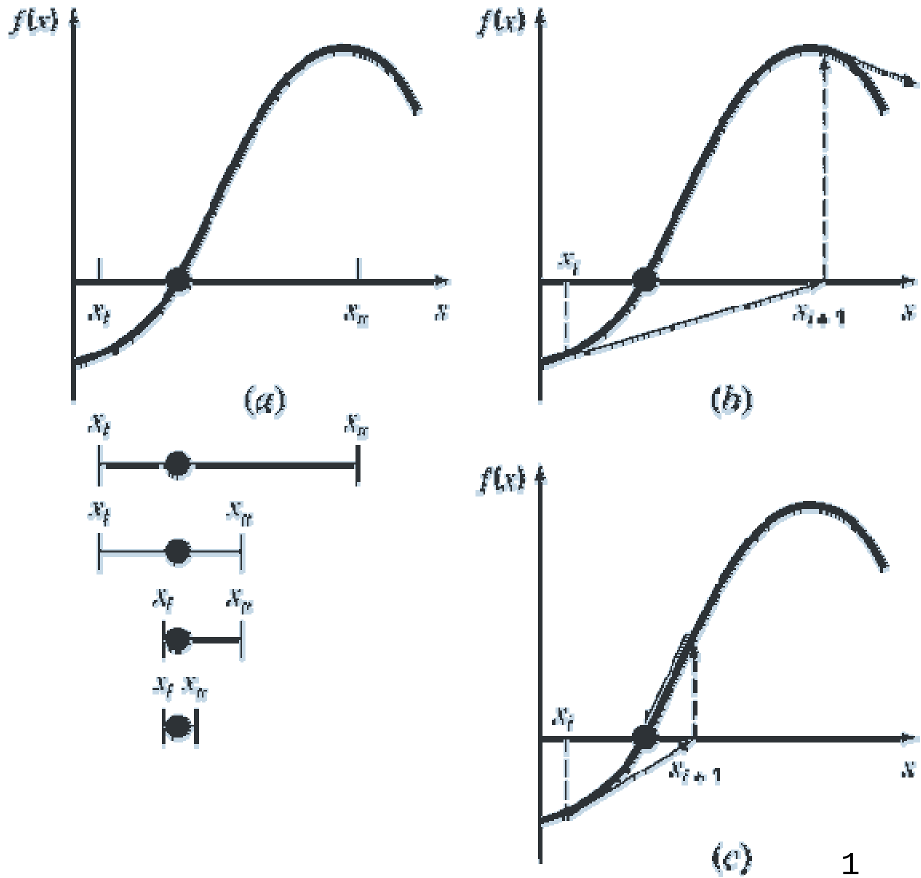
Open Methods

* Require a single starting point.
* Convergence to a solution is not guarantied.



Fixed Point Iteration

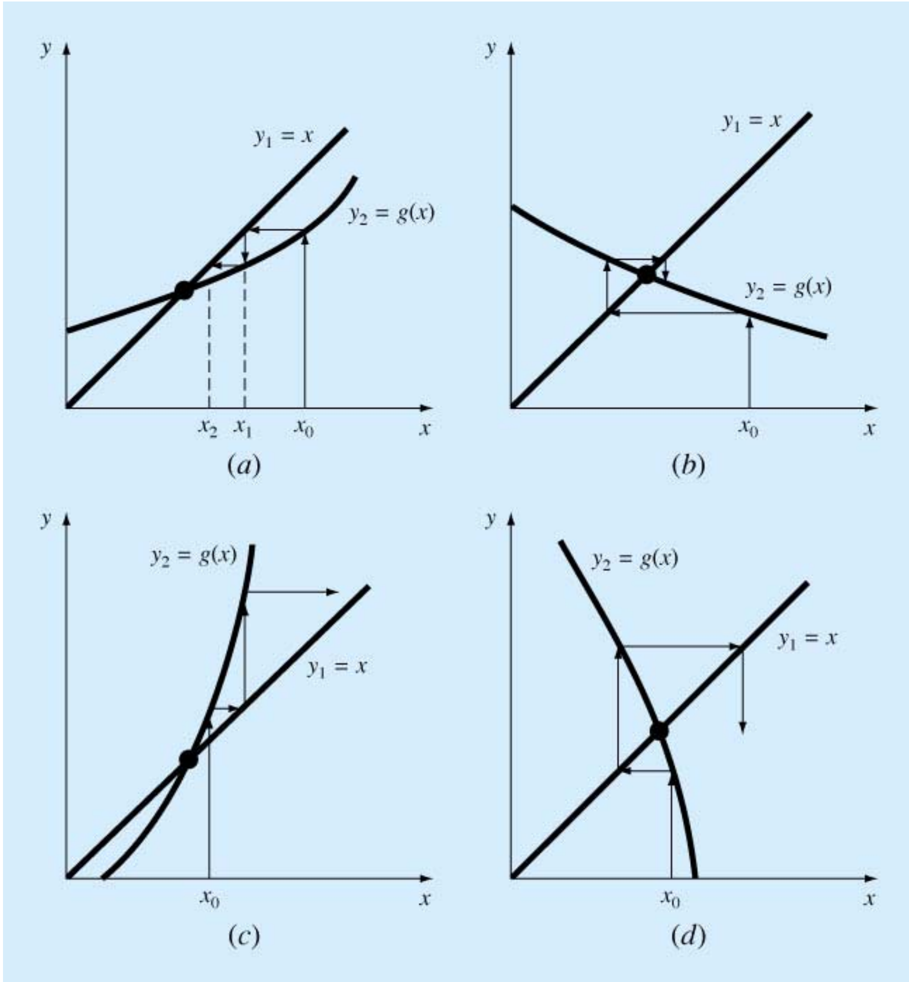
A function f(x) can be converted to a form:

$x=g(x)$

A guess on x (x_i) can be refined as:

$x_{i+1}=g(x_i)$

Converges if $|g'(x)|<1$ in the region of interest.



(a) Monotone convergence (b) Spiral convergence
(c) Monotone divergence (d) Spiral divergence

Example:

$$f(x) = e^x - 2 - x$$

$x_0 = -2.4$

or

$$x = e^x - 2$$

$$x_{i+1} = e^{x_i} - 2$$

Recursive Formula

	x	f(x)	$\%\varepsilon_a$
1	-1.909282047	0.057468786	-25.7017
2	-1.85181326	0.008765573	-3.10338
3	-1.843047687	0.001381824	-0.4756
4	-1.841665863	0.000218941	-0.07503
5	-1.841446922	3.47176E-05	-0.01189
6	-1.841412204	5.50588E-06	-0.00189
7	-1.841406698	8.73199E-07	-0.0003
8	-1.841405825	1.38484E-07	-4.7E-05
9	-1.841405687	2.19628E-08	-7.5E-06
10	-1.841405665	3.48318E-09	-1.2E-06

Ex: $f(x)=x^3-x-1$

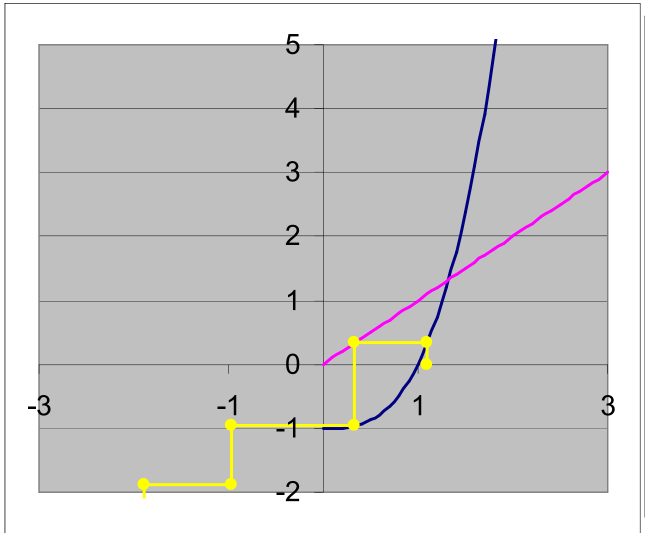
1: $x_{i+1}=x_i^3-1$

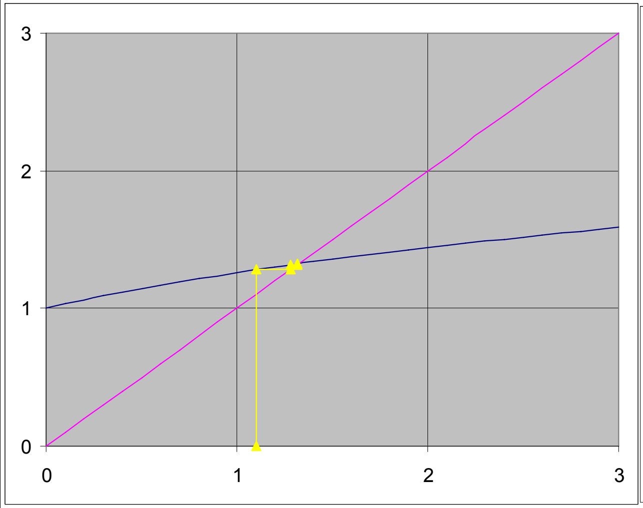
2: $x_{i+1}=\sqrt[3]{x_i+1}$

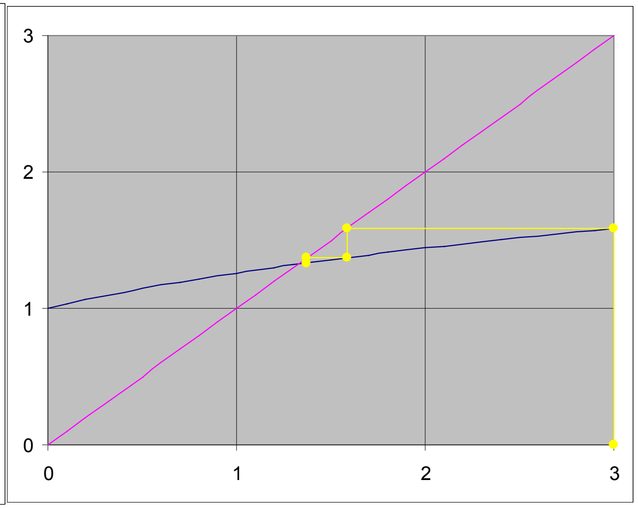
	x_i	ε_a
0	1.1	
1	0.331	232.3263
2	-0.96374	134.3455
3	-1.8951	49.14604
4	-7.80611	75.72281
5	-476.668	98.36236
6	-1.1E+08	99.99956
7	-1.3E+24	100
8	-2.1E+72	100
9	-9E+216	100

	x_i	ε_a
0	3	
1	1.587401	88.98816
2	1.372844	15.62864
3	1.333797	2.92752
4	1.32644	0.554624
5	1.325045	0.105297
6	1.32478	0.019999
7	1.32473	0.003799
8	1.32472	0.000722
9	1.324718	0.000137

	x_i	ε_a
0	1.1	
1	1.280579	14.10137
2	1.31628	2.712275
3	1.323113	0.516434
4	1.324413	0.09814
5	1.32466	0.018643
6	1.324707	0.003541
7	1.324716	0.000673
8	1.324718	0.000128
9	1.324718	2.43E-05







$$f(x) = x - x^{\frac{1}{3}} - 2 = 0$$

rewrite as:

$$x_{i+1} = g_1(x_i) = x_i^{1/3} + 2$$

or

$$x_{i+1} = g_2(x_i) = (x_i - 2)^3$$

or

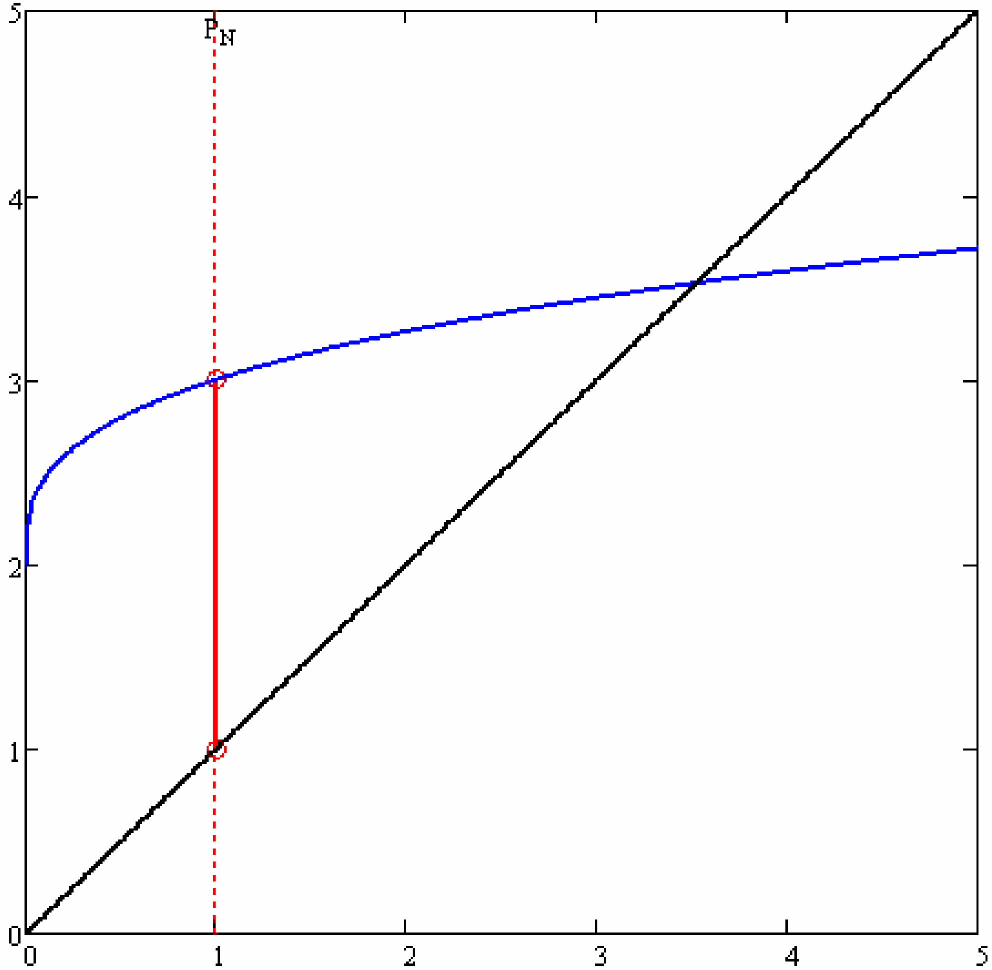
$$x_{i+1} = g_3(x_i) = \frac{2x_i^{2/3}}{x_i^{2/3} - 1}$$

$x_0=3$

i	g_1	g_2	g_3
0	3	3	3
1	3.44224957	1	3.85170813
2	3.50989745	-1	3.37252145
3	3.51972430	-27	3.60141144
4	3.52114127	-24389	3.48199250
5	3.52134537	-1.4511E+13	3.54165817
6	3.52137476	-3.0554E+39	3.51117472
7	3.52137899	-2.8523E+118	3.52657511
8	3.52137960	#NUM!	3.51875018
9	3.52137969	#NUM!	3.52271455
10	3.52137970	#NUM!	3.52070312

Fixed-Point Iteration

$n = $ 0 $P_n = $ 1



$x_0 := 0.5$

© ERES

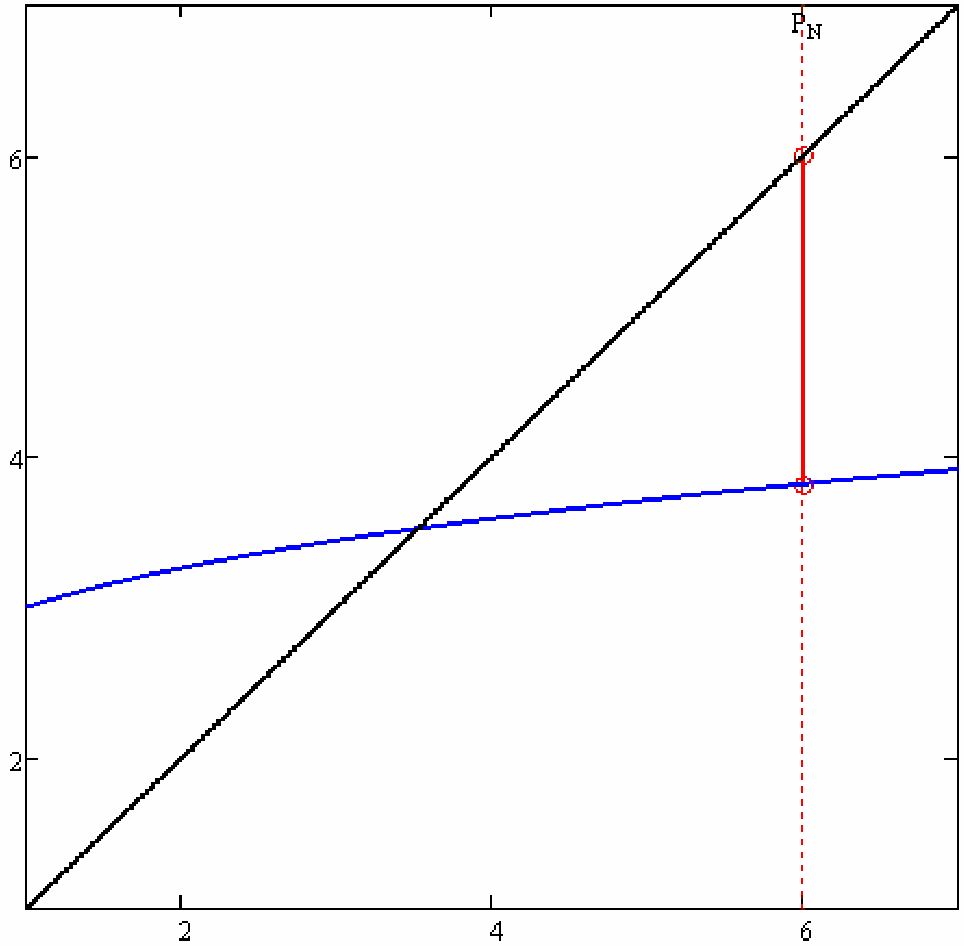
$$f(x) = x - x^{1/3} - 2$$

$$g(x) = x^{1/3} + 2$$

Fixed-Point Iteration

$n =$ 0

$P_n =$ 6



$f(x) = x - x^{1/3} - 2$

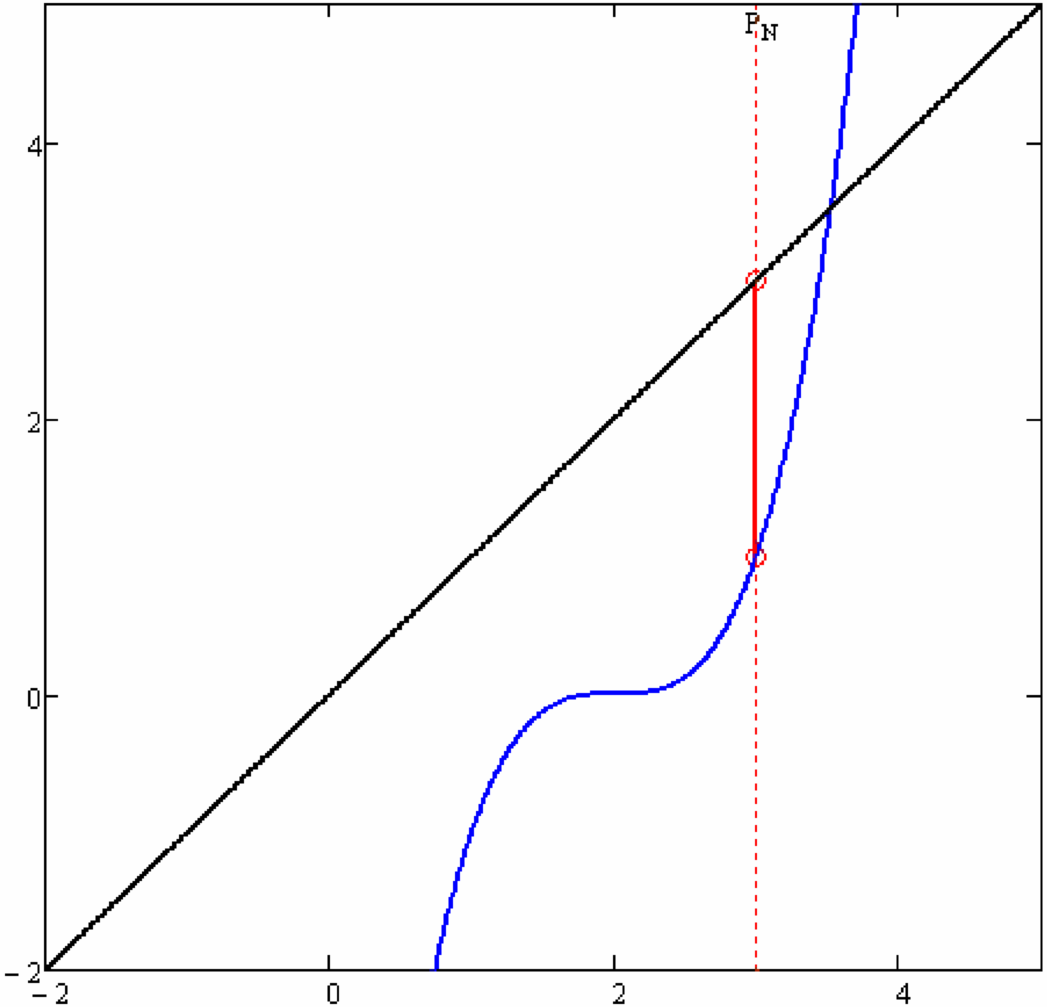
$g(x) = x^{1/3} + 2$

$x_0 := 6$

Fixed-Point Iteration

n = 0

P_n = 3



$$f(x) = x - (x-2)^3$$

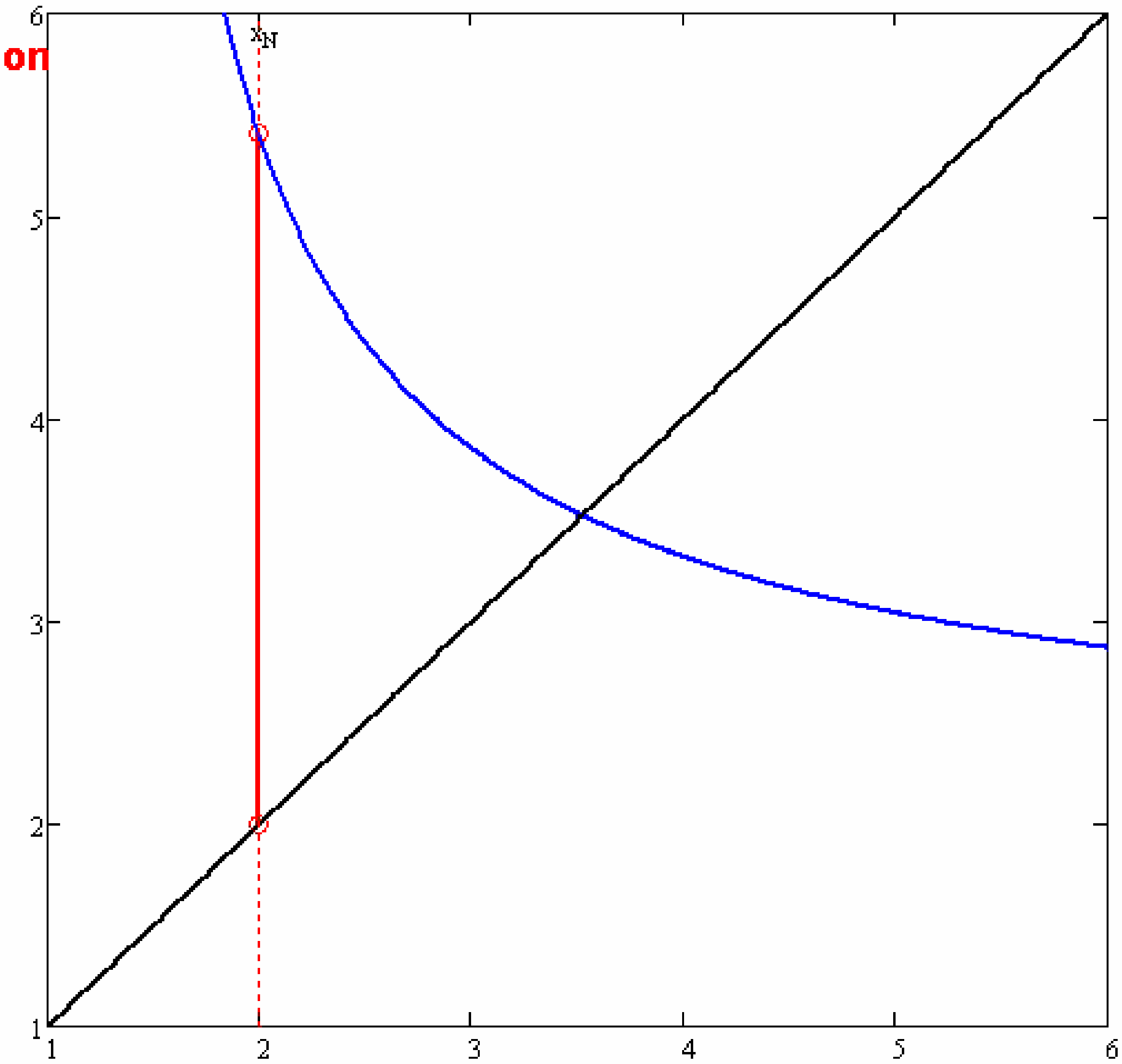
$$g(x) = (x-2)^3$$

$x_0 := 3$

Fixed-Point Iteration

$n =$ 0

$x_n =$ 2



$x_0 := 2$

$$g(x) = 2\frac{x^{2/3}}{\left(x^{2/3}-1\right)}$$

```
Function FixedPoint(x0, Eps, Imax)
xr = x0
Iter = 0
Do
   xrold = xr
   xr = g (xrold)
   Iter = Iter + 1
   If xr <> 0 Then
   ea = Abs((xr - xrold) / xr) * 100
   End If
Loop Until (ea < Eps) Or (Iter > Imax)
FixedPoint=xr
End Function
```

Convergence of fixed point Iteration

$x_{i+1}=g(x_i)$

Let xr be the true solution:

$x_r=g(x_r)$

Subtracting the two equations:

$x_r-x_{i+1} = g(x_r) - g(x_i)$

Derivative mean value theorem states that there exists a value ξ within [a,b] such that:

$$g'(\xi) = \frac{g(b)-g(a)}{b-a}$$

If $a=x_i$ and $b=x_r$ then:

$$g(x_r)-g(x_i)=(x_r-x_i)g'(\xi)$$
$$x_r-x_{i+1}=(x_r-x_i)g'(\xi)$$
$$E_{t,i+1}=E_{t,i}g'(\xi)$$

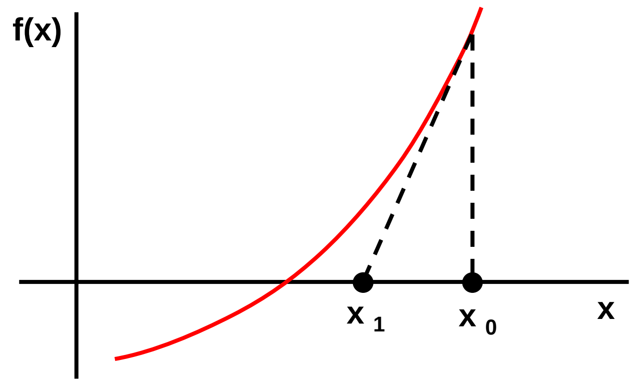
Converges if $|g'(x)|<1$ in the region of interest

Newton-Raphson Method

Geometrically: Use the slope information

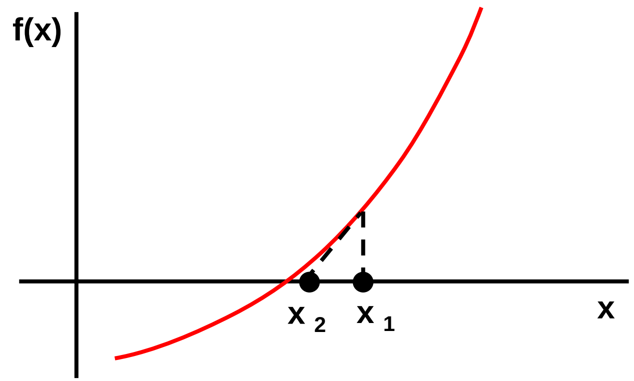
$$Slope = f'(x) \approx \frac{f(x_i) - 0}{x_{i+1} - x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x)}$$



Use Taylor Series expansion of function f(x):

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots\ldots\ldots$$



Neglect higher order terms and let $f(x_{i+1})=0$:

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$which\ can\ be\ solved\ for\ x_{i+1}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

New guess will be away from the old when $f'(x_i) \approx 0$

Example

$$f(x) = x - x^{\frac{1}{3}} - 2 = 0$$

$$f'(x) = 1 - \frac{1}{3} x^{-2/3}$$

Recursive Formula:

$$x_{i+1} = x_i - \frac{x_i - x_i^{1/3} - 2}{1 - \frac{1}{3} x_i^{-2/3}}$$

i	x	ε_a	f(x)	f'(x)
0	3	-	-0.44224957	0.839750048
1	3.52664429	14.93329776	0.004506792	0.856129758
2	3.52138015	-0.149490981	3.77141E-07	0.855986412
3	3.52137971	-1.25119E-05	0.0000E+00	0.8559864
4	3.52137971	-8.82786E-14	0.0000E+00	0.8559864
5	3.52137971	1.26112E-14	0.0000E+00	0.8559864

Convergence of Newton Raphson Method:

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(\xi)}{2!}(x_{i+1} - x_i)^2$$

approximate:

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i) \qquad [1]$$

Let $x_{i+1}=x_r$ be the true value of the root. Then:

$$0 = f(x_i) + f'(x_i)(x_r - x_i) + \frac{f''(\xi)}{2!}(x_r - x_i)^2 \qquad [2]$$

Subtract [1] from [2]:

$$0 = f'(x_i)(x_r - x_{i+1}) + \frac{f''(\xi)}{2!}(x_r - x_i)^2 \qquad [3]$$

$E_{t,i+1}= x_r-x_{i+1}$= True error at (i+1)st step

$$0 = f'(x_i)E_{t,i+1} + \frac{f''(\xi)}{2!}E_{t,i}^2$$

Assume at the convergence both x_i and ξ can be approximated by x_r

$$E_{t,i+1} = \frac{-f''(x_r)}{2f'(x_r)}E_{t,i}^2 \qquad \text{quadratic convergence}$$

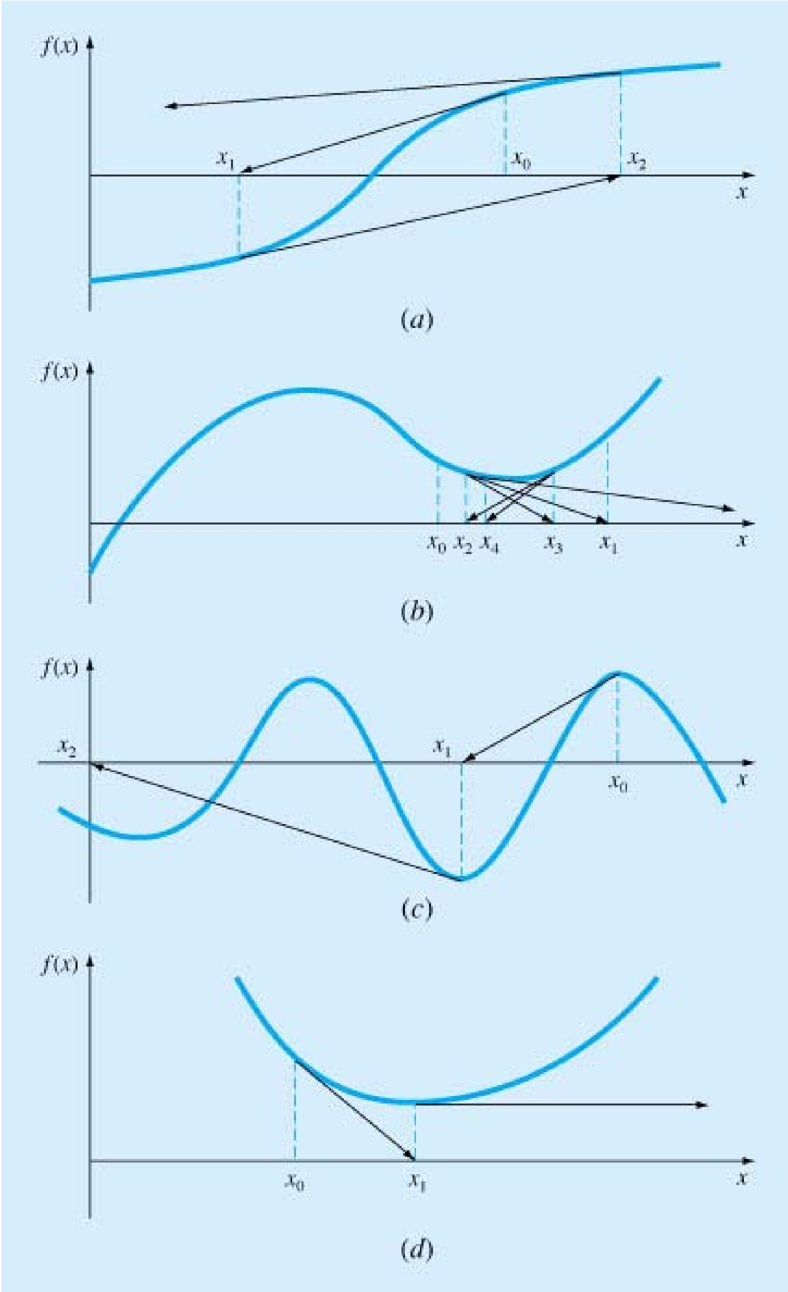
Pitfalls

a) Slow convergence or divergence (depending on x_0)

b) Oscillation

c) Convergence to a different root or divergence

d) Shooting out



-Need to know the derivative of the function.

-May diverge around inflection points.

-May give oscillations around local minimums or maximums.

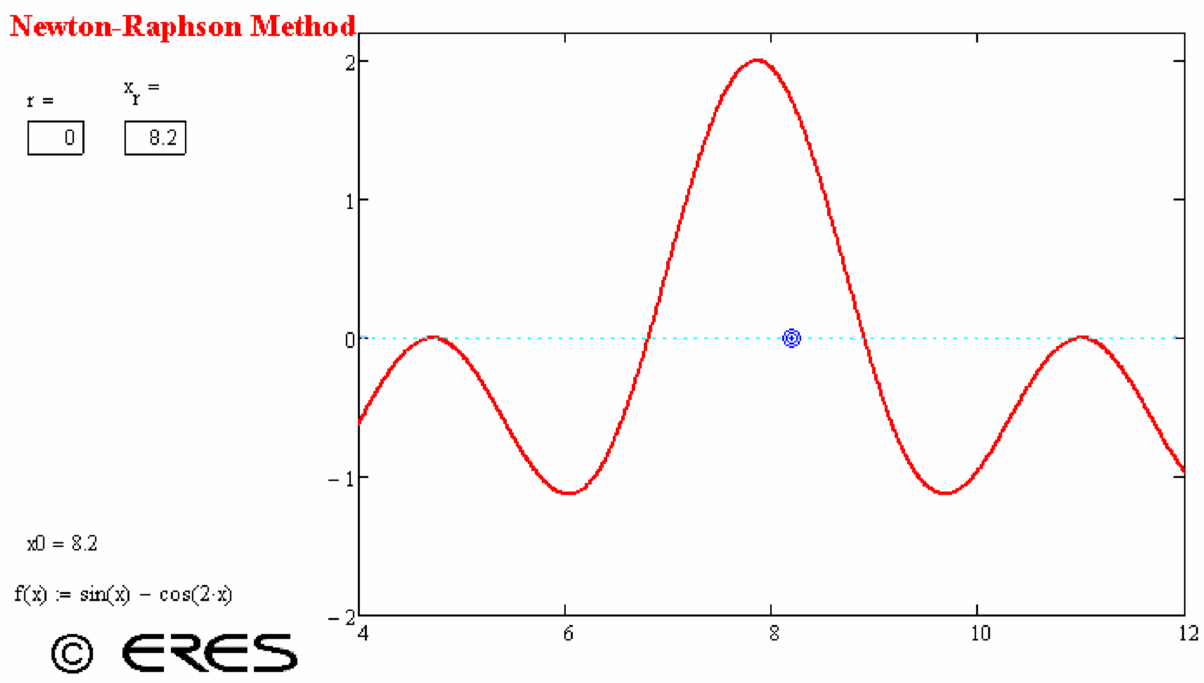
- Zero or near zero slope is a problem, because f ' is at the denominator.

- Convergence may be slow if the initial guess is poor.

Example :

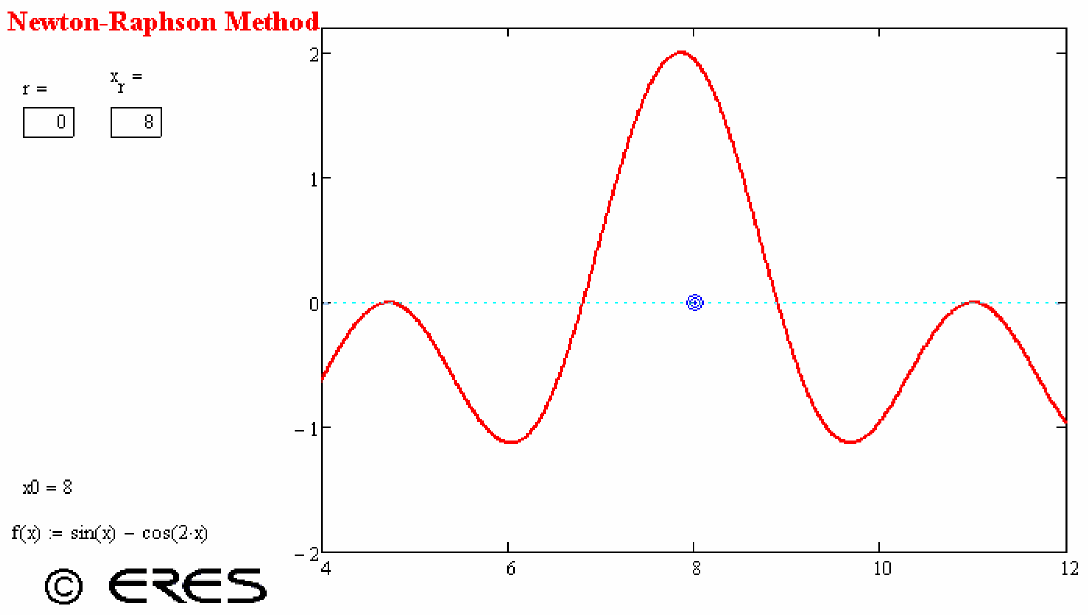
$$f(x) = \sin(x) - \cos(2x)$$

$$f'(x) = \cos(x) + 2\sin(2x)$$



x_0=8

x_0=8.2

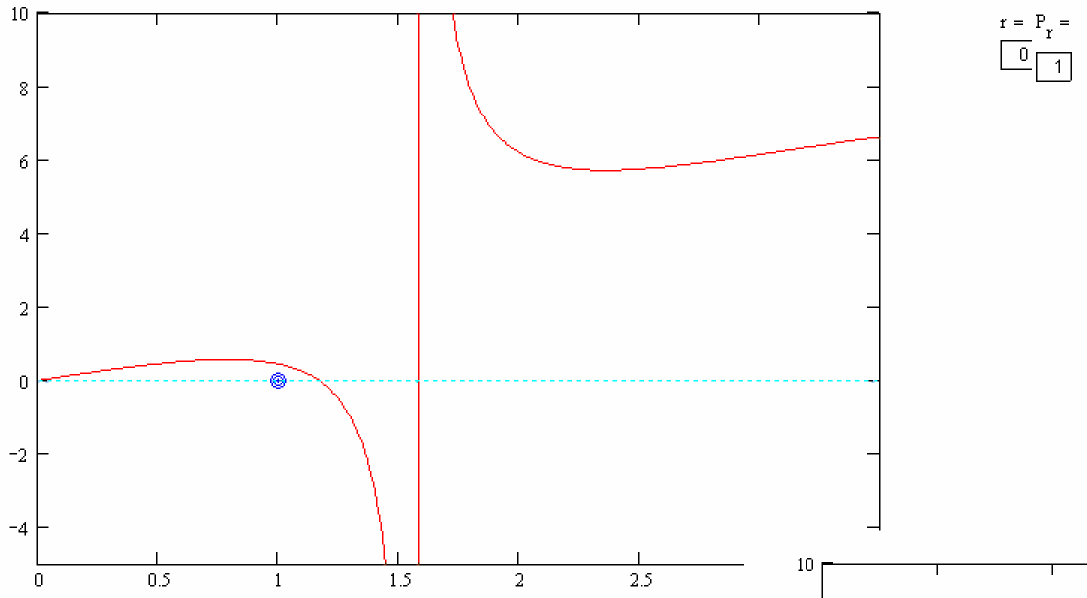


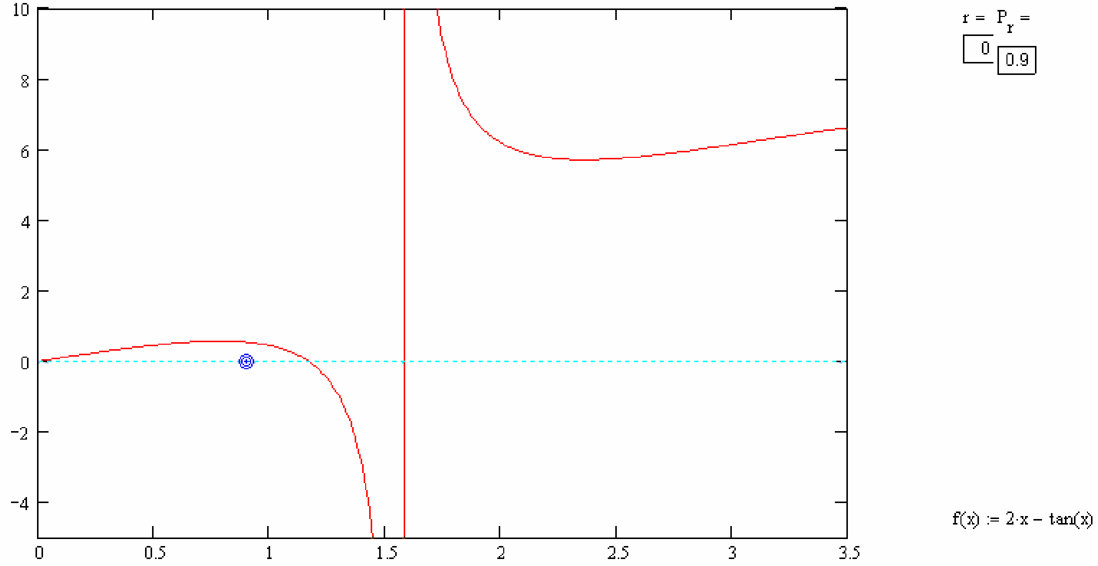
Example f(x)=2x-tan(x)

X_0=1 or 0.9

$$f'(x) = 2 - \frac{1}{\cos^2 x}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$





```
Function NewtonRaphson(x0, Eps, Imax)
xr = x0
Iter = 0
F0 = F(x0)
DF0 = DevF (x0)
Do
    xrold = xr
    xr = xrold - F0 / DF0
    F0 = F (xr)
    DF0 = DevF (xr)
    Iter = Iter + 1
    If xr <> 0 Then
    Ea = Abs((xr - xrold) / xr) * 100
    End If
Loop Until (Ea < Eps) Or (Iter > Imax)
NewtonRaphson = xr
End Function
```

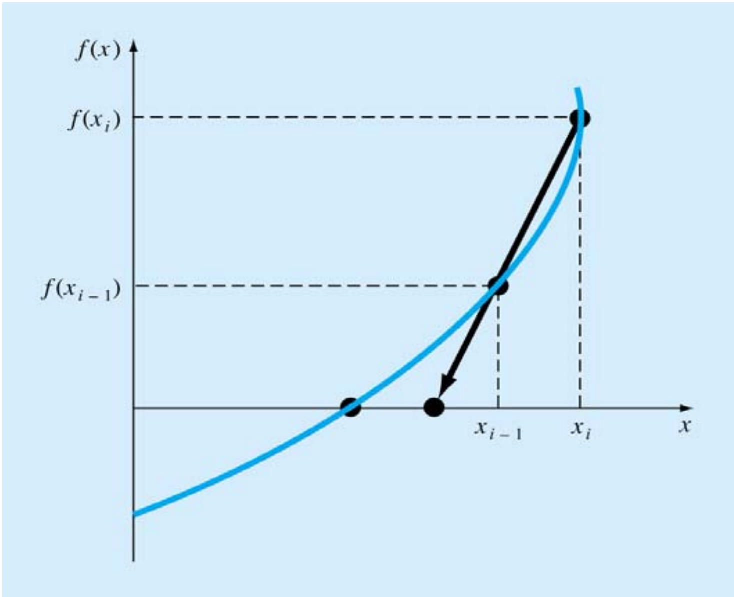
Function F(x) and its derivative DevF(x) must be supplied as function routines

Secant Method

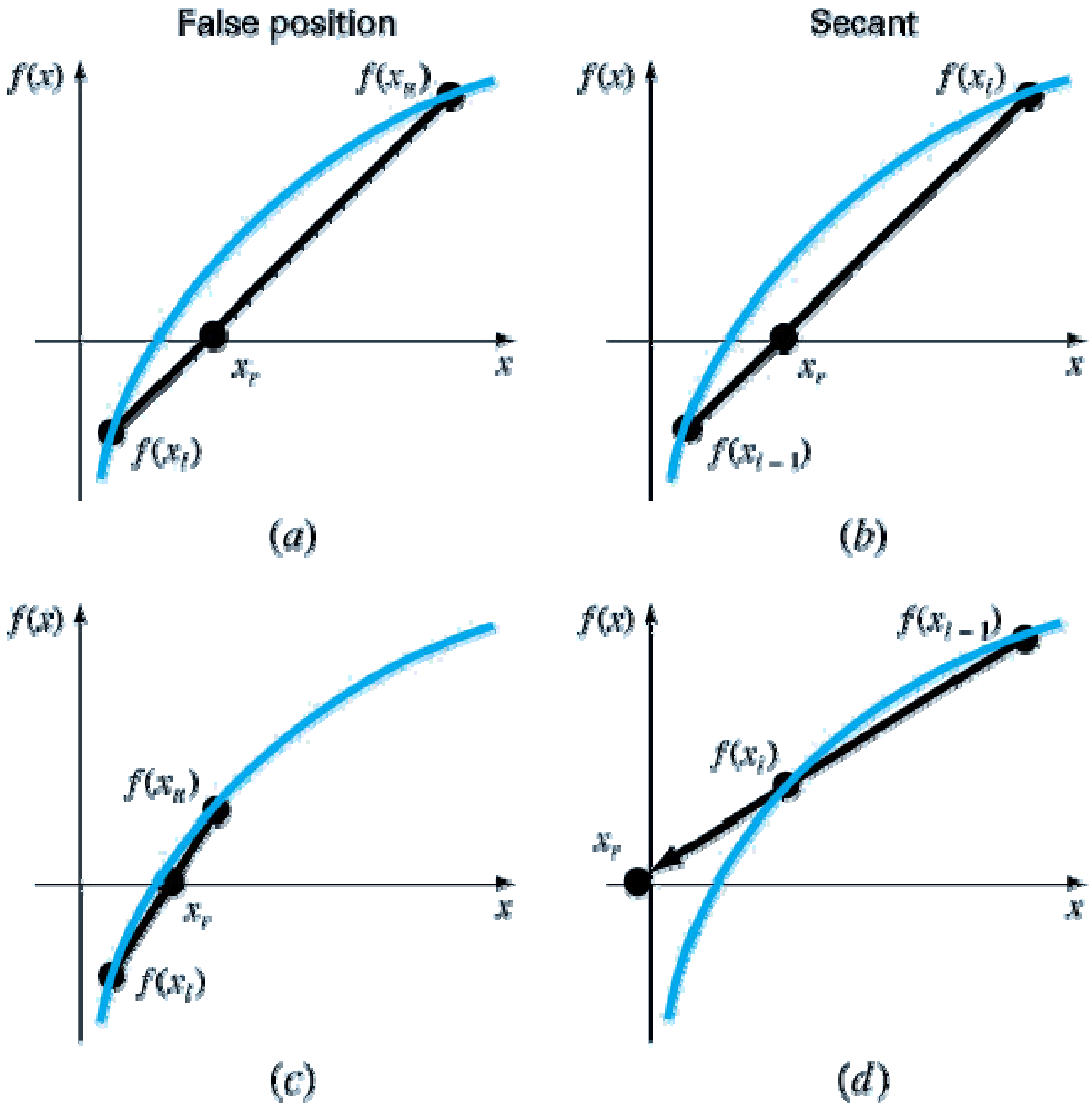
In case where the derivative of a function can not be obtained, approximate the slope by two points:

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$



Important: Although the formula is the same as the false position method, the logic of the two methods is different.



First iterations of both methods are the same.

Second iterations are different in terms of how the previous estimates are replaced with the newly calculated root.

False-position Method drops one of previous estimates so that the remaining ones bracket the root.

Secant Method always drops the oldest estimate.

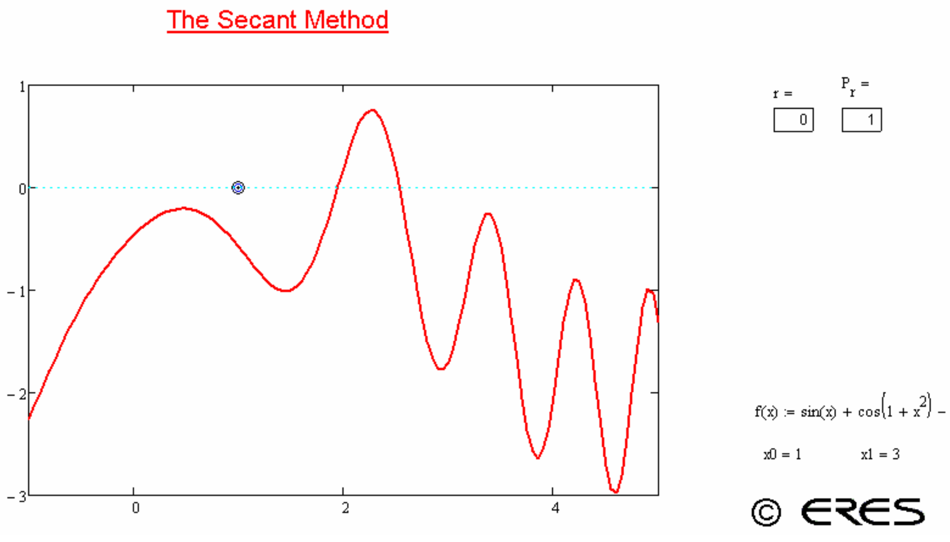
Example (Problem 6.6 page 158):

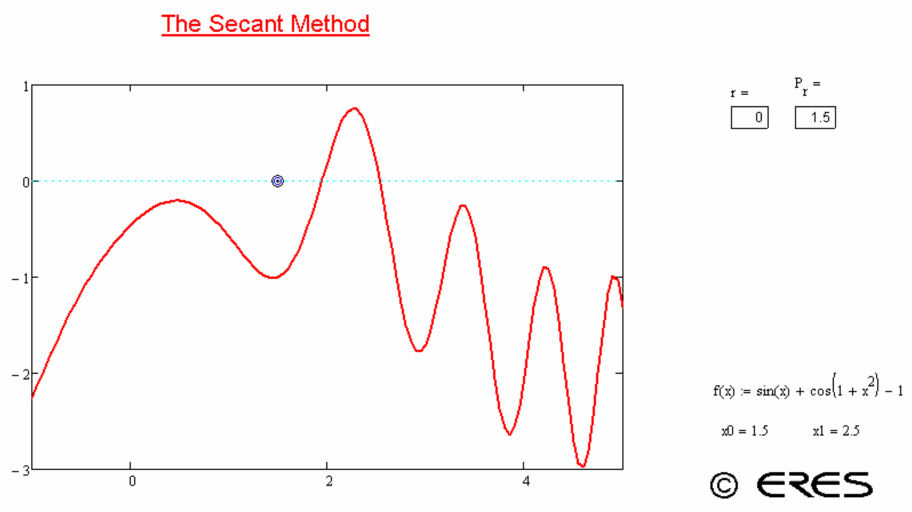
$$f(x) = \sin(x) + \cos(1 + x^2) - 1$$

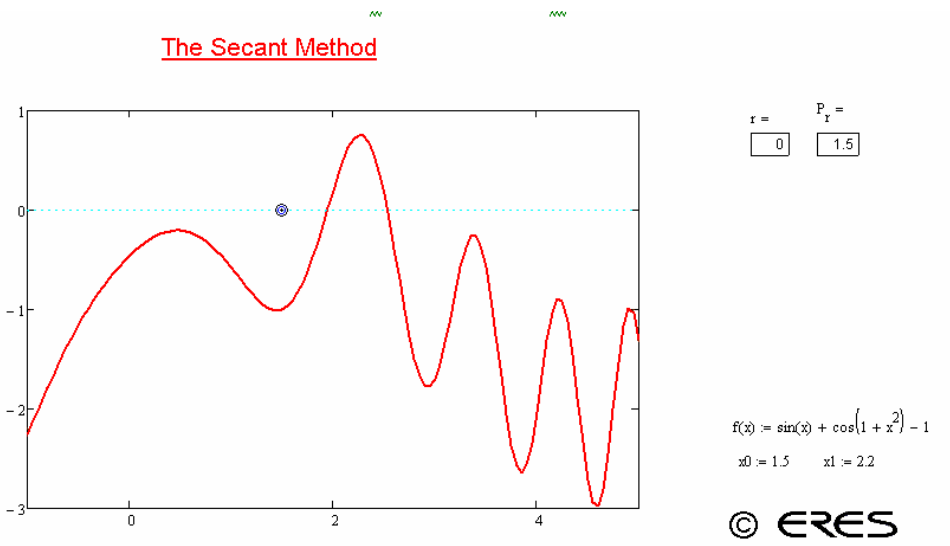
$a)$ $x_0 = 1$ $x_1 = 3$

b) $x_0 = 1.5$ $x_1 = 2.5$

c) $x_0 = 1.5$ $x_1 = 2.2$

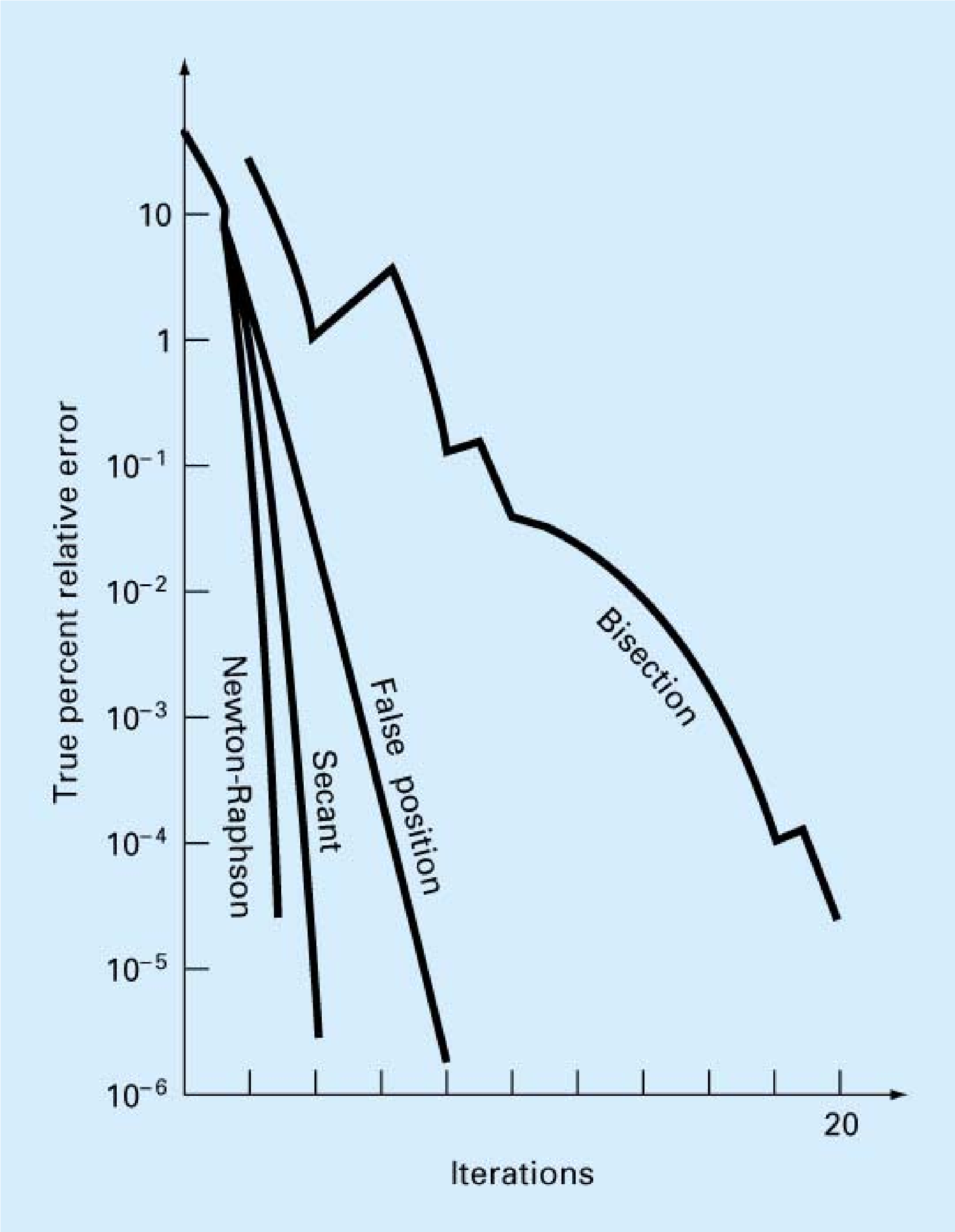






```
Function Secant(x0, x1, Eps, Imax)
Iter = 0
F0 = Fun(x0)
F1 = Fun(x1)
Do
    x2 = (F1 * x0 - F0 * x1) / (F1 - F0)
    x0 = x1
    F0 = F1
    x1 = x2
    F1 = Fun(x1)
    Iter = Iter + 1
    If x1 <> 0 Then
    Ea = Abs((x1 - x0) / x1) * 100
    End If
Loop Until (Ea < Eps) Or (Iter > Imax)
Secant = x1
End Function
```

(compare this algorith with the false position)


True percent relative error
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
Newton-Raphson
Secant
False position
Bisection
20
Iterations

F(x)=x-0.2*sin(x)-0.5

Bisection

x_u	0.5	$f(x_u)$	-0.096
x_l	1	$f(x_l)$	0.3317

	x_r	$f(x_r)$	(x_u-x_l)	ε_a
0	0.7500000000	0.11367225	0.25	
1	0.6250000000	0.00798055	0.125	20
2	0.5625000000	-0.04416053	0.0625	11.1111111
3	0.5937500000	-0.01814463	0.03125	5.26315789
4	0.6093750000	-0.00509601	0.015625	2.56410256
5	0.6171875000	0.00143873	0.0078125	1.26582278
6	0.6132812500	-0.00182952	0.00390625	0.63694268
7	0.6152343750	-0.00019561	0.001953125	0.31746032
8	0.6162109375	0.00062151	0.000976563	0.15847861
9	0.6157226563	0.00021293	0.000488281	0.07930214
10	0.6154785156	8.6566E-06	0.000244141	0.0396668

False Position

	x_r	$f(x_r)$	ε_a
0	0.6121224812	-0.0027986900	
1	0.6153677259	-0.0000840405	0.527366733
2	0.6154651511	-0.0000025255	0.015829525
3	0.6154680788	-0.0000000759	0.000475685
4	0.6154681668	-0.0000000023	1.4295E-05
5	0.6154681694	-0.0000000001	4.29588E-07
6	0.6154681695	0.0000000000	1.29098E-08
7	0.6154681695	0.0000000000	3.87958E-10
8			
9			
10			

Fixed Position

	x_r	$f(x_r)$	ε_a
0	0.5	-0.0958851077	
1	0.595885108	-0.0163631994	16.0912073
2	0.612248307	-0.0026934591	2.67264102
3	0.614941766	-0.0004404253	0.43800231
4	0.615382192	-0.0000719373	0.07156939
5	0.615454129	-0.0000117478	0.01168849
6	0.615465877	-0.0000019184	0.00190877
7	0.615467795	-0.0000003133	0.0003117
8	0.615468108	-0.0000000512	5.0902E-05
9	0.61546816	-0.0000000084	8.3122E-06
10	0.615468168	-0.0000000014	1.3574E-06

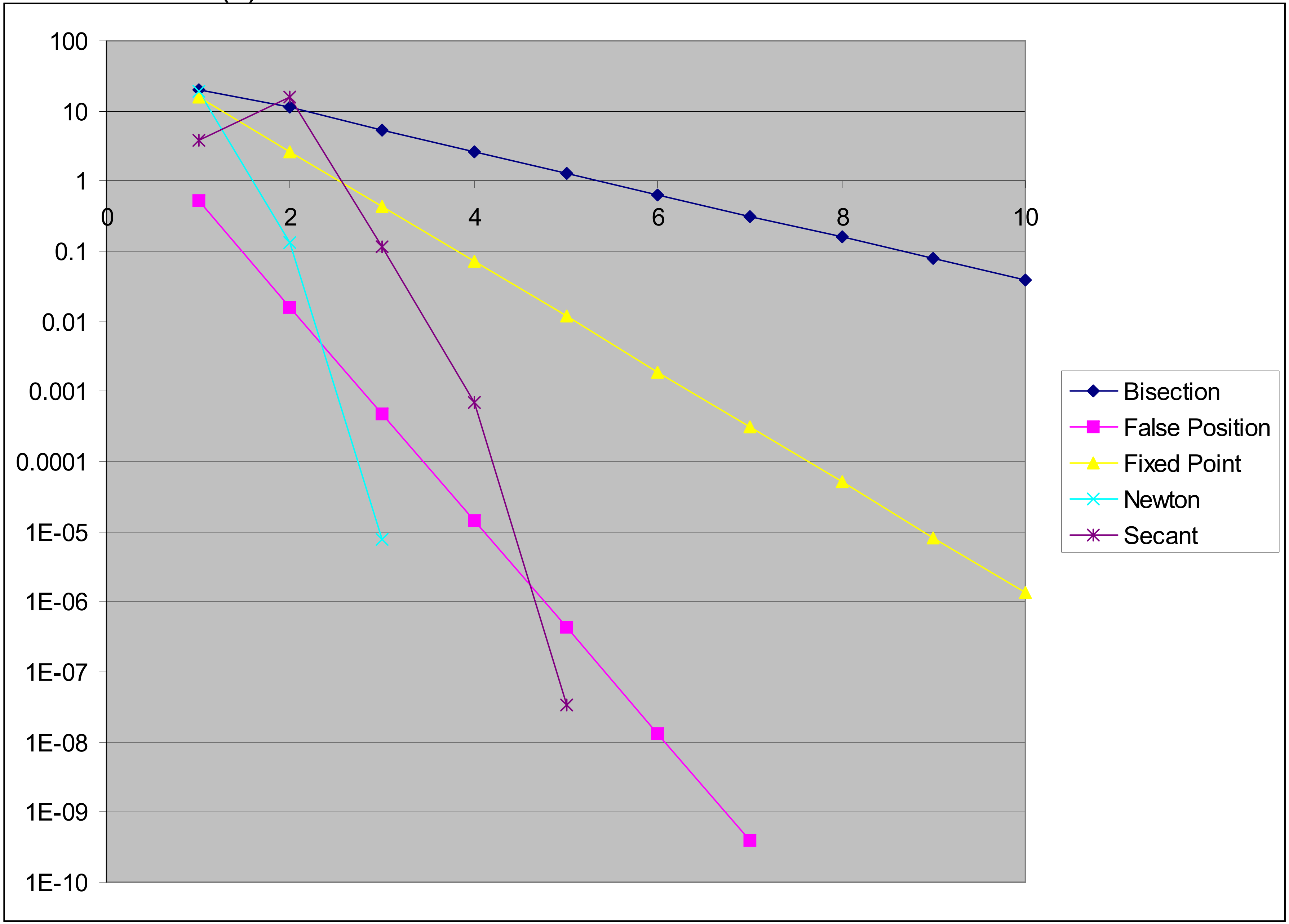
Newton

	x_r	$f(x_r)$	$f'(x_r)$	ε_a
0	0.5000000000	-0.09588511	0.824483488	
1	0.6162971838	0.00069368	0.836795132	18.8703091
2	0.6154682169	3.9705E-08	0.836699357	0.13468882
3	0.6154681695	0	0.836699351	7.7103E-06
4	0.6154681695	0	0.836699351	0
5				
6				
7				
8				
9				
10				

Secant

	x_r	$f(x_r)$	ε_a
0	0.5000000000	-0.0958851077	
1	0.5200000000	-0.0793760276	3.8461538462
2	0.6161604485	0.0005792571	15.6063974432
3	0.6154637889	-0.0000036653	0.1131926258
4	0.6154681693	-0.0000000002	0.0007117227
5	0.6154681695	0.0000000000	0.0000000340
6	0.6154681695	0.0000000000	0.0000000000
7			
8			
9			
10			

Root of f(x)=x-0.2sinx-0.5



$$f(x) = e^{x} - 2 - x = 0$$

Bisection

x_u	-2.4	$f(x_u)$	0.4907
x_l	-1.6	$f(x_l)$	-0.198

	x_r	$f(x_r)$	(x_u-x_l)	ε_a
0	-2.0000000000	0.1353352832	0.4	
1	-1.8000000000	-0.0347011118	0.2	11.1111111
2	-1.9000000000	0.0495686192	0.1	5.26315789
3	-1.8500000000	0.0072371663	0.05	2.7027027
4	-1.8250000000	-0.0137823559	0.025	1.36986301
5	-1.8375000000	-0.0032850336	0.0125	0.68027211
6	-1.8437500000	0.0019729760	0.00625	0.33898305
7	-1.8406250000	-0.0006568038	0.003125	0.16977929
8	-1.8421875000	0.0006578927	0.0015625	0.08481764
9	-1.8414062500	0.0000004961	0.00078125	0.04242681
10	-1.8410156250	-0.0003281660	0.000390625	0.02121791

False Position

	x_r	$f(x_r)$	ε_a
0	-1.8300781850	-0.0095207887	
1	-1.8416939157	0.0002425462	0.630709078
2	-1.8413983618	-0.0000061411	0.016050512
3	-1.8414058453	0.0000001555	0.000406399
4	-1.8414056558	-0.0000000039	1.02913E-05
5	-1.8414056606	0.0000000001	2.60609E-07
6	-1.8414056604	0.0000000000	6.59947E-09
7	-1.8414056604	0.0000000000	1.67118E-10
8			
9			
10			

Fixed Position

	x_r	$f(x_r)$	ε_a
0	-1.6	-0.1981034820	
1	-1.798103482	-0.0362838400	11.0173571
2	-1.834387322	-0.0059013546	1.9779814
3	-1.840288677	-0.0009397376	0.32067548
4	-1.841228414	-0.0001491335	0.05103862
5	-1.841377548	-0.0000236542	0.00809902
6	-1.841401202	-0.0000037515	0.00128457
7	-1.841404953	-0.0000005950	0.00020373
8	-1.841405548	-0.0000000944	3.231E-05
9	-1.841405643	-0.0000000150	5.1242E-06
10	-1.841405658	-0.0000000024	8.1268E-07

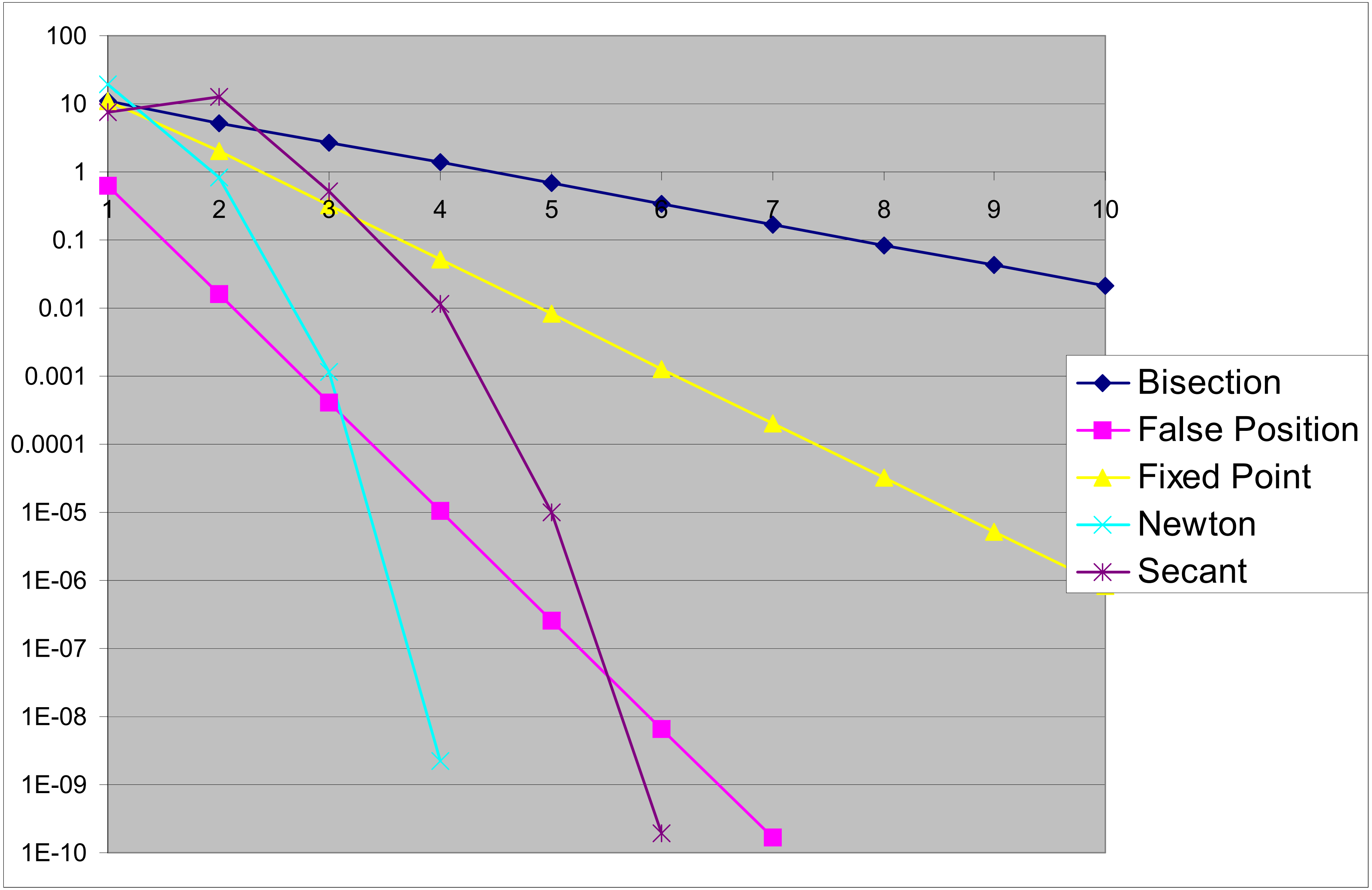
Newton

	x_r	$f(x_r)$	$f'(x_r)$	ε_a
0	-1.5000000000	-0.2768698399	-0.77686984	
1	-1.8563915416	0.0126269249	-0.843764617	19.1980804
2	-1.8414265565	0.0000175821	-0.841408974	0.81268433
3	-1.8414056605	0.0000000000	-0.84140566	0.00113479
4	-1.8414056604	0.0000000000	-0.84140566	2.2347E-09
5	-1.8414056604	0.0000000000	-0.84140566	0
6				
7				
8				
9				
10				

Secant

	x_r	$f(x_r)$	ε_a
0	-1.5000000000	-0.2768698399	
1	-1.6200000000	-0.1821013009	7.4074074074
2	-1.8505844994	0.0077297875	12.4600902840
3	-1.8411952618	-0.0001770271	0.5099533865
4	-1.8414054791	-0.0000001526	0.0114161336
5	-1.8414056604	0.0000000000	0.0000098464
6	-1.8414056604	0.0000000000	0.0000000002
7			
8			
9			
10			

$$f(x) = e^x - 2 - x = 0$$



100
10
1
0.1
0.01
0.001
0.0001
1E-05
1E-06
1E-07
1E-08
1E-09
1E-10
1
2
3
4
5
6
7
8
9
10
Bisection
False Position
Fixed Point
Newton
Secant

Multiple Roots

■ At even multiple roots, bracketing methods can not be used at all.

■ Open methods still work but $f'(x)$ also goes to zero at a multiple root. Possibility of division by zero for Secant and Newton-Raphson Methods. $f(x)$ will reach zero faster than $f'(x)$, therefore use a zero-check for $f(x)$ and stop properly. They converge slowly (linear instead of quadratic convergence).

■ Some modifications can be made for speed up.

i) If you know the multiplicity of the root NR can be modified as

$$x_{i+1} = x_i - m\frac{f(x_i)}{f'(x_i)}$$

m=2 for a double root, m=3 for a triple root, etc.

ii) Another alternative is to define a new function $u(x)=f(x)/f'(x)$ and use it in the formulation of Newton-Raphson Method.

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)} = \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$$

$f''(x)$ must be known and evaluated

(Similar modifications can be made for the Secant Method. See the book for details.)