

Linear Algebraic Equations

Linear Equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots + a_{2n}x_n = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + \dots + a_{3n}x_n = c_3$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + a_{m4}x_4 + \dots + a_{mn}x_n = c_m$$

m equations in n unknowns

Where a_{ij} are constant coefficients , c_j are constants.

Brief Review of Matrices

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

[m*n] matrix
If m=n square matrix

If m=1, [1*n] row vector and if n=1 [m*1] column vector

$$C = [c_1 \quad c_2 \quad \dots \quad c_n]$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

-
- Matrix is **symmetric** if: $a_{ij}=a_{ji}$
 - Matrix is **diagonal** if $a_{ij}=0$ for $i \neq j$ for all i
 - Matrix is **identity Matrix**, I , if $a_{ij}=0$ for $i \neq j$ for all i and $a_{ii}=1$
 - **Upper** and **Lower Triangular** Matrices
 - **Banded** matrix
 - **Transpose** of a matrix

Matrix Operations

- $[A]=[B]$ if both matrices have the same m rows, n columns and $a_{ij}=b_{ij}$ for all i,j
- Addition and Subtraction:

$$[C]=[A]+[B], \quad c_{ij}=a_{ij}+b_{ij}$$

$$[C]=[B]+[A] = [A]+[B] \text{ Commutative}$$

$$([A]+[B])+[C]= [A]+([B]+[C]) \text{ Associative}$$

Multiplication by a constant:

$$k[A] = G \quad g_{ij} = k a_{ij}$$

Multiplication of Matrices:

$$[C] = [A] \quad [B]$$

$$m * l \quad m * n \quad n * l$$

$$c_{ij} = \sum a_{ik} b_{kj}$$

$$(AB)C = A(BC) \text{ (Associative)}$$

$$A(B+C) = AB + AC \text{ (Distributive)}$$

$$AB \neq BA \text{ (is not commutative)}$$

- Trace of a Matrix: $tr[A] = \sum_{i=1}^n a_{ii}$
 - Augmentation:
-

Matrix A augmented by an identity Matrix:

$$[A, I] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{bmatrix}$$

$$AX = B$$

Matrix **A** augmented by **B**:

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Inverse of a Matrix A^{-1}

A^{-1} (Inverse of Matrix A) is such that $A^{-1}A = AA^{-1} = I$

If $AX = B$ Multiply both sides by A^{-1}

$$A^{-1}AX = A^{-1}B$$

Since

$$A^{-1}A = I$$

$$X = A^{-1}B$$

DETERMINANT (Defined for square matrices only)

Determinant of Matrix A

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2x2

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

3x3

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Gauss Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots + a_{2n}x_n = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + \dots + a_{3n}x_n = c_3$$

....

....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + a_{n4}x_4 + \dots + a_{nn}x_n = c_n$$

n equations in n unknowns

When n is small: (Graphical Method)

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad 3x_1 + 2x_2 = 18$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad -x_1 + 2x_2 = 2$$

$$x_2 = \frac{3}{2}x_1 + 9$$

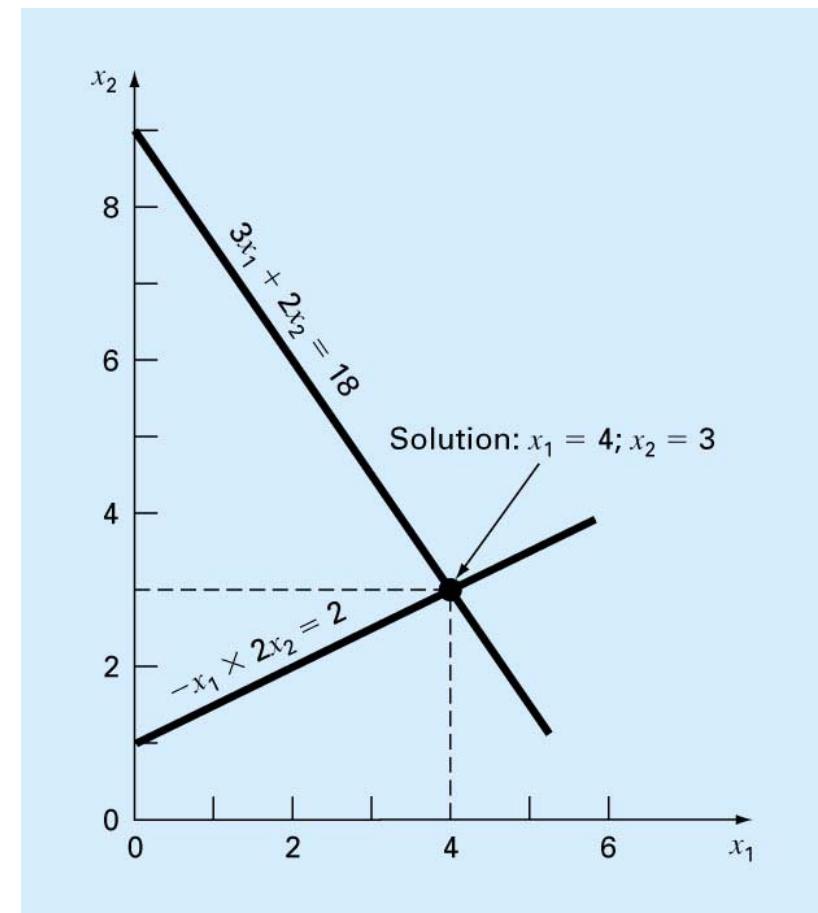
$$x_2 = \frac{1}{2}x_1 + 1$$

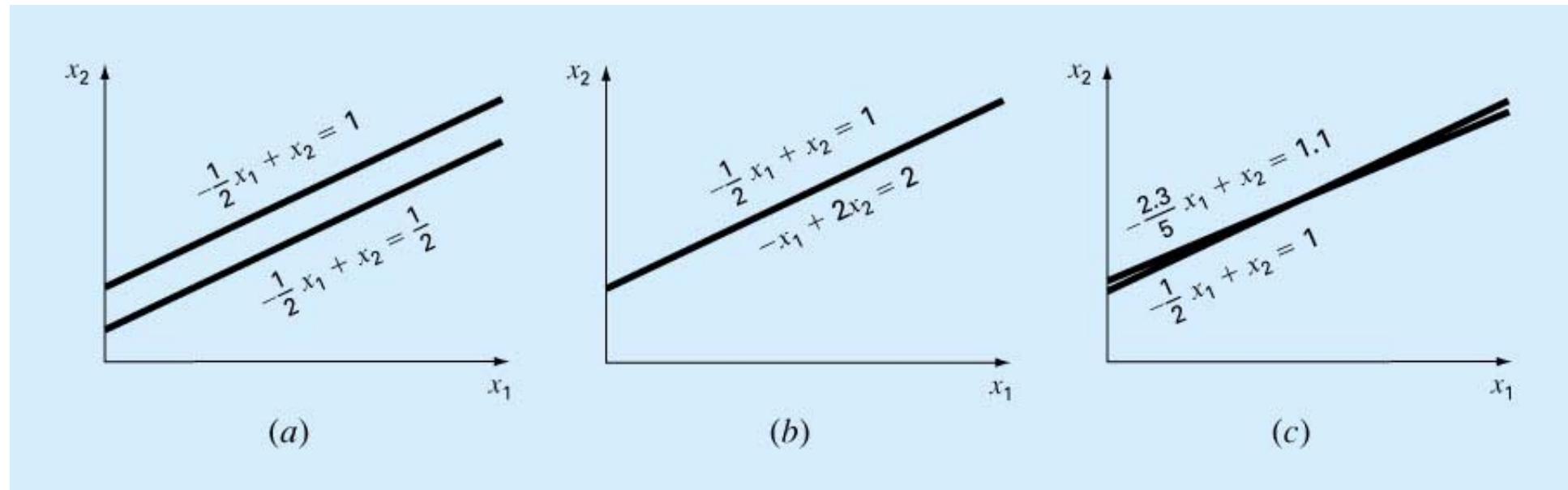
Solve for x_2

$$x_2 = -\frac{a_{11}}{a_{12}}x_1 + b_1$$

$$x_2 = -\frac{a_{21}}{a_{22}}x_1 + b_2$$

$x_2 = \text{slope} * x_1 + x_1 \text{ intercept}$





No Solution

Infinite Solutions

Ill Conditioned

When n is small

Cramer's Rule

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_1 & a_{22} & a_{23} \\ b_1 & a_{32} & a_{33} \end{vmatrix}}{D}$$

D is the determinant of
the coefficient Matrix A

When n is small:

Elimination of the unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$a_{21}a_{11}x_1 + a_{21}a_{12}x_2 = a_{21}b_1$$

$$a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2$$

$$a_{22}a_{11}x_2 - a_{21}a_{12}x_2 = a_{11}b_2 - a_{21}b_1$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{22}a_{11} - a_{21}a_{12}}$$

Naive Gauss Elimination

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots + a_{2n}x_n = c_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + \dots + a_{3n}x_n = c_3$$

....

....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + a_{n4}x_4 + \dots + a_{nn}x_n = c_n$$

Step 1: eliminate x_1 from the second upto n th equation

Multiply the first equation by: $\frac{a_{21}}{a_{11}}$

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \frac{a_{21}}{a_{11}}a_{13}x_3 + \frac{a_{21}}{a_{11}}a_{14}x_4 + \dots + \frac{a_{21}}{a_{11}}a_{1n}x_n = c_1 \frac{a_{21}}{a_{11}}$$

Subtract it from the second equation: the result is:

$$\left(a_{22} - \frac{a_{21}}{a_{11}}a_{12} \right)x_2 + \left(a_{23} - \frac{a_{21}}{a_{11}}a_{13} \right)x_3 + \dots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n} \right)x_n = c_2 - \frac{a_{21}}{a_{11}}c_1$$

or

$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_n$$

Repeat the same procedure to eliminate x_1 from the other equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots + a_{1n}x_n = c_1$$

$$\dots \quad a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 + \dots + a'_{2n}x_n = c'_2$$

$$\underline{\dots \quad a'_{32}x_2 + a'_{33}x_3 + a'_{34}x_4 + \dots + a'_{3n}x_n = c'_3}$$

....

....

$$\dots \quad a'_{n2}x_2 + a'_{n3}x_3 + a'_{n4}x_4 + \dots + a'_{nn}x_n = c'_n$$

First equation is the **pivot equation** and a_{11} is the **pivot coefficient**

Repeat the above procedure to eliminate x_2 from the third upto nth equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots + a_{1n}x_n = c_1$$

$$\dots \quad a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 + \dots + a'_{2n}x_n = c'_2$$

$$\dots \quad + a''_{33}x_3 + a''_{34}x_4 + \dots + a''_{3n}x_n = c''_3$$

....

....

$$\dots \quad + a''_{n3}x_3 + a''_{n4}x_4 + \dots + a''_{nn}x_n = c''_n$$

Repeat the procedure (n-1) times till x_{n-1} is eliminated from the nth equation.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots + a_{1n}x_n = c_1$$

$$\dots \quad a'_{22}x_2 + a'_{23}x_3 + a'_{24}x_4 + \dots + a'_{2n}x_n = c'_2$$

$$\dots + a''_{33}x_3 + a''_{34}x_4 + \dots + a''_{3n}x_n = c''_3$$

....

....

$$\dots + a^{(n-1)}_{nn}x_n = c^{(n-1)}$$

Upper triangular system is formed.

Determine x_i by back substitution

$$x_n = \frac{c_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{c_i^{(i-1)} - \sum_{j=i+1}^n a_{ij}^{(i-1)}x_j}{a_{ii}^{(i-1)}}$$

Forward elimination

(a)

```
DO k = 1, n - 1
  DO i = k + 1, n
    factor = ai,k / ak,k
    DO j = k + 1 to n
      ai,j = ai,j - factor · ak,j
    END DO
    bi = bi - factor · bk
  END DO
END DO
```

Back Sustitution

(b)

```
xn = bn / an,n
DO i = n - 1, 1, -1
  sum = 0
  DO j = i + 1, n
    sum = sum + ai,j · xj
  END DO
  xi = (bi - sum) / ai,i
END DO
```

Example (Pr.9.10):

$$2x_1 + x_2 - x_3 = 1$$

$$5x_1 + 2x_2 + 2x_3 = -4$$

$$3x_1 + x_2 + x_3 = 5$$

Forward Elimination

$$2x_1 + x_2 - x_3 = 1$$

$$-0.5x_2 + 4.5x_3 = -6.5$$

$$-0.5x_2 + 2.5x_3 = 3.5$$

$$2x_1 + x_2 - x_3 = 1$$

$$-0.5x_2 + 4.5x_3 = -6.5$$

$$-2x_3 = 10$$

Back Substitution

$$x_3 = -\frac{10}{2} = -5$$

$$x_2 = \left(\begin{array}{c} 1 \\ -0.5 \end{array} \right) [-6.5 - 4.5x_3] = -32$$

$$x_1 = \left(\begin{array}{c} 1 \\ 2 \end{array} \right) [1 - (-1)x_3 - 1x_2] = 14$$

Check your result !!!

Example

Augmented Matrix

$$\left[\begin{array}{cccc} 3 & -1 & 2 & 12 \\ 1 & 2 & 3 & 11 \\ 2 & -2 & -1 & 2 \end{array} \right]$$

We shall use 3 digits

$$\text{Row2}-\frac{1}{3}\text{Row1} \dots \left[\begin{array}{cccc} 3 & -1 & 2 & 12 \\ 0 & 2.333 & 2.334 & 7.004 \\ 0 & -1.334 & -2.332 & -5.992 \end{array} \right]$$

$$\text{Row3}-(-1.334/2.333)\text{Row1} \dots \left[\begin{array}{cccc} 3 & -1 & 2 & 12 \\ 0 & 2.333 & 2.334 & 7.004 \\ 0 & 0 & -1 & -1.993 \end{array} \right]$$

$$X_3 = 1.993, \quad x_2 = (7.004 - 2.334 \cdot 1.993) / 2.333 = 1.008,$$

$$X_1 = (12 - 2 \cdot 1.993 + 1 \cdot 1.008) / 3 = 3.007 \text{ (actual result is 2,1,3)}$$

Operation Count

Elimination:

Outer Loop k	Middle Loop i	FLOPS
1	2,n	$[n-1][1+n]$
2	3,n	$[n-2][n]$
..	..	
k	k+1,n	$\frac{[n-k][1n+2-k]}{k}$
n-1	n,n	[1][3]

$$\text{Number of Mult\&Div} = \frac{n^3}{3} + O(n^2)$$

Back Substitution: $O(n^2)$

(a)

```

DO k = 1, n - 1
DO i = k + 1, n
factor = ai,k / ak,k
DO j = k + 1 to n
ai,j = ai,j - factor * ak,j
END DO

```

(b)

```

bi = bi - factor * bk
END DO
END DO
xn = bn / an,n
DO i = n - 1, 1, -1
sum = 0
DO j = i + 1, n
sum = sum + ai,j * xj
END DO

```

Therefore, the number of operations is proportional to $n^3/3$

Pitfalls:

1. Division by zero:

If the pivot element is zero i.e. $a_{11}=0$

2. Round off errors (especially when n is large)

3. Ill conditioned systems (when the determinant $D \neq 0$ but, $D \sim 0$)

Improvements:

1. Use more significant figures (time?)
2. Pivoting

Partial pivoting: Determine the largest coefficient in the column below the pivot element and switch rows (equations).

Complete pivoting: Determine the largest element (in rows and columns and switch the rows (equations) and columns (x's) Too costly!!!

3. Scaling (normalization- make the highest coefficient equal to unity)

Example (Partial pivoting):

We shall use 4 digit accuracy.

No Pivoting

$$0.0003x_1 + 1.566x_2 = 1.569$$

$$0.3454x_1 - 2.436x_2 = 1.018$$

With Pivoting

$$0.3454x_1 - 2.436x_2 = 1.018$$

$$0.0003x_1 + 1.566x_2 = 1.569$$

Multiply the first eq. By

$$0.3454/0.0003 = 1151$$

Multiply the first eq. By

$$0.0003/0.3454 = 0.0009$$

$$0.0003x_1 + 1.566x_2 = 1.569$$

$$-1804x_2 = -1805$$

$$0.3454x_1 - 2.436x_2 = 1.018$$

$$+1.5682x_2 = 1.5681$$

$$X_2 = 1.001$$

$$X_1 = (1.569 - 1.566 * 1.001) / .0003 = 3.333$$

$$X_2 = 0.9999$$

$$X_1 = (1.018 + 2.436 * 0.9999) / 0.3454 = 9.9993$$

Example (Scaling): We shall use 3 significant digits

$$\begin{array}{rcl} x_1 + 50000x_2 & = & 50000 \\ x_1 + x_2 & = & 2 \\ \hline \end{array}$$

If we scale the equations

$$\begin{array}{rcl} 0.00002x_1 + x_2 & = & 1 \\ x_1 + x_2 & = & 2 \\ \hline \end{array}$$

$$\begin{array}{rcl} x_1 + 50000x_2 & = & 50000 \\ -50000x_2 & = & -50000 \end{array}$$

Pivot

$$x_1 + x_2 = 2$$

$$x_2 = 1 \text{ and } x_1 = 0$$

$$0.00002x_1 + x_2 = 1$$

$$x_1 + x_2 = 2$$

Pivot, but do not scale

$$x_2 = 1$$

$$x_1 + x_2 = 2$$

$$x_1 + 50000x_2 = 50000$$

$$x_2 = 1 \text{ and } x_1 = 1$$

$$x_1 + x_2 = 2$$

$$50000x_2 = 50000$$

Example (Partial Pivoting)

$$2x_2 + 2x_4 = 0$$

$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$

$$4x_1 - 3x_2 + x_4 = -7 \quad \rightarrow$$

$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

Eliminate

Pivot change					
$\left[\begin{array}{ccccc} 0 & 2 & 0 & 2 & 0 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 6 & 1 & -6 & -5 & 6 \end{array} \right]$			
$\left[\begin{array}{ccccc} 2 & 2 & 3 & 2 & -2 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 2 & 2 & 3 & 2 & -2 \end{array} \right]$			
$\left[\begin{array}{ccccc} 4 & -3 & 0 & 1 & -7 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 4 & -3 & 0 & 1 & -7 \end{array} \right]$			
$\left[\begin{array}{ccccc} 6 & 1 & -6 & -5 & 6 \end{array} \right]$		$\left[\begin{array}{ccccc} 0 & 2 & 0 & 2 & 0 \end{array} \right]$			

\rightarrow

Pivot change					
$\left[\begin{array}{ccccc} 6 & 1 & -6 & -5 & 6 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 6 & 1 & -6 & -5 & 6 \end{array} \right]$			
$\left[\begin{array}{ccccc} 0 & 1.6667 & 5 & 3.6667 & -4 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 0 & -3.6667 & 4 & 4.3333 & -11 \end{array} \right]$			
$\left[\begin{array}{ccccc} 0 & -3.6667 & 4 & 4.3333 & -11 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 0 & 1.6667 & 5 & 3.6667 & -4 \end{array} \right]$			
$\left[\begin{array}{ccccc} 0 & 2 & 0 & 2 & 0 \end{array} \right]$		$\left[\begin{array}{ccccc} 0 & 2 & 0 & 2 & 0 \end{array} \right]$			

\rightarrow

Eliminate	Eliminate				
$\left[\begin{array}{ccccc} 6 & 1 & -6 & -5 & 6 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 6 & 1 & -6 & -5 & 6 \end{array} \right]$			
$\left[\begin{array}{ccccc} 0 & -3.6667 & 4 & 4.3333 & -11 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 0 & -3.6667 & 4 & 4.3333 & -11 \end{array} \right]$			
$\left[\begin{array}{ccccc} 0 & 0 & 6.8181 & 5.6364 & -9 \end{array} \right]$	\rightarrow	$\left[\begin{array}{ccccc} 0 & 0 & 6.8181 & 5.6364 & -9 \end{array} \right]$			
$\left[\begin{array}{ccccc} 0 & 0 & 2.1818 & 4.3636 & -6 \end{array} \right]$		$\left[\begin{array}{ccccc} 0 & 0 & 2.1818 & 4.3636 & -6 \end{array} \right]$			
		$\left[\begin{array}{ccccc} 0 & 0 & 2.5600 & -3.12 & 0 \end{array} \right]$			

Use back substitution to obtain:

$$x_4 = -1.2188; \quad x_3 = -0.3125; \quad x_2 = 1.2188; \quad x_1 = -.5313$$

Pseudo Code

```
SUB Gauss (a, b, n, x, tol, er)
  DIMENSION s (n)
  er = 0
  DO i = 1, n
    si = ABS(ai,1)
  -----
  DO j = 2, n
    IF ABS(ai,j)>si THEN si = ABS(ai,j)
  END DO
  END DO
  CALL Eliminate(a, s, n, b, tol, er)
  IF er ≠ -1 THEN
    CALL Substitute(a, n, b, x)
  END IF
END Gauss

SUB Eliminate (a, s, n, b, tol, er)
  DO k = 1, n - 1
    CALL Pivot (a, b, s, n, k)
    IF ABS (ak,k/sk) < tol THEN
      er = -1
      EXIT DO
    END IF
    DO i = k + 1, n
      factor = ai,k/ak,k
      DO j = k + 1, n
        ai,j = ai,j - factor*ak,j
      END DO
      bi = bi - factor * bk
    END DO
  END DO
  IF ABS(ak,k/sk) < tol THEN er = -1
END Eliminate

SUB Pivot (a, b, s, n, k)
  p = k
  big = ABS(ak,k/sk)
  DO ii = k + 1, n
    dummy = ABS(aii,k/sii)
    IF dummy > big THEN
      big = dummy
      p = ii
    END IF
  END DO
  IF p ≠ k THEN
    DO jj = k, n
      dummy = ap,jj
      ap,jj = ak,jj
      ak,jj = dummy
    END DO
    dummy = bp
    bp = bk
    bk = dummy
    dummy = sp
    sp = sk
    sk = dummy
  END IF
END pivot

SUB Substitute (a, n, b, x)
  xn = bn/an,n
  DO i = n - 1, 1, -1
    sum = 0
    DO j = i + 1, n
      sum = sum + ai,j * xj
    END DO
    xi = (bi - sum) / ai,i
  END DO
```

Comments on pseudocode

- Equations are not scaled but scaled values of the elements are used to determine whether pivoting is required.
 - Diagonal term is monitored to determine near-zero occurrence
-

s_i: is the value of the maximum element on *i*th row.

a: Coefficient matrix, **b**: column vector of a set of equations **ax=b**

n: number of equations (or unknowns) to be solved.

tol, a small value defined by the user. If the diagonal element <tol the matrix is considered as singular and **er** is set to -1. **er** is to be returned by the program.

Gauss Jordan

Augmented Matrix

Interchange row 1&4 and normalize

$$2x_2 + 2x_4 = 0$$

$$2x_1 + 2x_2 + 3x_3 + 2x_4 = -2$$

$$\underline{4x_1 - 3x_2 + x_4 = -7}$$

$$6x_1 + x_2 - 6x_3 - 5x_4 = 6$$

$\begin{bmatrix} 0 & 2 & 0 & 2 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$	\rightarrow	$\begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix}$
---	---------------	---

Eliminate

$$\rightarrow \begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 4.3333 & -11 \\ 0 & -3.6667 & 4 & 3.6667 & -4 \\ 0 & 2 & 0 & 2 & 0 \end{bmatrix}$$

Interchange Row 2&3, normalize & eliminate all elements in the second column.

$$\rightarrow \begin{bmatrix} 1 & 0 & -0.8182 & -0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 6.8182 & 5.6364 & -9 \\ 0 & 0 & 2.1818 & 4.3636 & -6 \end{bmatrix}$$

Normalize 3rd row and eliminate

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0.0400 & -0.58 \\ 0 & 1 & 0 & -2.800 & 1.56 \\ 0 & 0 & 1 & 0.8267 & -1.32 \\ 0 & 0 & 0 & 2.5600 & -3.12 \end{bmatrix}$$

Normalize 4th row and eliminate

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -0.5313 \\ 0 & 1 & 0 & 0 & 1.2188 \\ 0 & 0 & 1 & 0 & -0.3125 \\ 0 & 0 & 0 & 1 & -1.2188 \end{bmatrix}$$

Requires more work
 $(n^{3/2}$, compared to $n^{3/3}$
 In Gauss elimination)

Gauss-Jordan Method

Eliminate the unknown from all but one of the equations.

Pivot and Normalize

$$2x_1 + x_2 - x_3 = 1$$

$$5x_1 + 2x_2 + 2x_3 = -4$$

$$3x_1 + x_2 + x_3 = 5$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 \\ 5 & 2 & 2 & -4 \\ 3 & 1 & 1 & 5 \end{array} \right] \xrightarrow{\text{Pivot and Normalize}} \left[\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 5/3 \\ 1 & 2/5 & 2/5 & -4/5 \\ 1 & 1/2 & -1/2 & 1/2 \end{array} \right]$$

Eliminate

$$\left[\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 5/3 \\ 0 & 1/15 & 1/15 & -37/15 \\ 0 & 1/6 & -5/6 & -7/6 \end{array} \right] \xrightarrow{\text{Row Operations}} \left[\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 5/3 \\ 0 & 1 & 1 & -37 \\ 0 & 1 & -5 & -7 \end{array} \right] \xrightarrow{\text{Row Operations}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 14 \\ 0 & 1 & 1 & -37 \\ 0 & 0 & -6 & 30 \end{array} \right]$$

$$\xrightarrow{\text{Row Operations}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 14 \\ 0 & 1 & 1 & -37 \\ 0 & 0 & 1 & -5 \end{array} \right] \xrightarrow{\text{Row Operations}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & -32 \\ 0 & 0 & 1 & -5 \end{array} \right]$$

Right Column is the result. No back substitution

Inverse of a matrix Using Gauss-Jordan:

AA⁻¹=I Augment A with I Change Row 1&2 and normalize

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\quad} \begin{array}{ccccccc|ccccc} 1 & 0 & 0.3333 & 0 & 0.3333 & 0 & 0 \\ 1 & -1 & 1.6667 & 1 & -0.3333 & 0 & 0 \\ 0 & 0 & 1.6667 & 0 & -0.3333 & 1 & 0 \end{array}$$

Normalize and eliminate

$$\xrightarrow{\quad} \begin{array}{ccccccc|ccccc} 1 & 0 & 0.3333 & 0 & 0.3333 & 0 & 0 \\ 0 & 1 & -1.6667 & -1 & 0.3333 & 0 & 0 \\ 0 & 0 & 1.6667 & 0 & -0.3333 & 1 & 0 \end{array} \xrightarrow{\quad}$$

Normalize and eliminate

$$\xrightarrow{\quad} \begin{array}{ccccccc|ccccc} 1 & 0 & 0 & 0 & 0.4 & -0.2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -0.2 & 0.6 & 0 \end{array} \quad A^{-1} = \begin{bmatrix} 0 & 0.4 & -0.2 \\ -1 & 0 & 1 \\ 0 & -0.2 & 0.6 \end{bmatrix}$$

One can also use LU decomposition explained in the book (pg 273).

Iterative Methods

Guess a solution and improve your solution

The Equations must be
diagonally dominant

Consider:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Solve the first eq. For x_1 , second for x_2
and the third for x_3 :

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Iterative Method 1: Gauss-Seidel

-
1. Assume $x_i=0$ ($i=0, 1, 2, n$)
 2. Solve for x_1 from the first eq.
 3. Solve for x_2 from the second eq. (use x_1 obtained in step 2)
 4. Solve for x_3 from the 3rd eq. (use x_1 and x_2 values)
 5. Solve for x_1 from the 1st eq. (use the latest values available)
 6. Repeat until:

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

$$\left| \mathcal{E}_{a,i} \right| = \left| \frac{x_i^j - x_i^{j-1}}{x_i^j} \right| * 100\% < \mathcal{E}_s$$

$$8x_1 + x_2 - x_3 = 8$$

$$x_1 - 7x_2 + 2x_3 = -4$$

$$2x_1 + x_2 + 9x_3 = 12$$

$$x_1 = 1 - 0.125x_2 + 0.125x_3$$

$$x_2 = 0.571 + 0.143x_1 + 0.286x_3$$

$$x_3 = 1.333 - 0.222x_1 + 0.111x_2$$

	0	1	2	3	4	5	6	7	8
x_1	0.000000	1.000000	1.039718	0.996848	1.000565	0.999930	1.000010	0.999999	1.000000
x_2	0.000000	0.714000	1.014759	0.996559	1.000390	0.999942	1.000008	0.999999	1.000000
x_3	0.000000	1.031746	0.989544	1.001082	0.999831	1.000022	0.999997	1.000000	1.000000

$\varepsilon_s < 10^{-6}$

PseudoCode for GaussSeidel

```
SUBROUTINE Gseid (a,b,n,x,imax,es,lambda)
DO i = 1,n
    dummy = ai,i
    DO j = 1,n
        ai,j = ai,j/dummy
    END DO
    bi = bi/dummy
END DO
DO i = 1, n
    sum = bi
    DO j = 1, n
        IF i≠j THEN sum = sum - ai,j*xj
    END DO
    xi=sum
END DO
iter=1
DO
    sentinel = 1
    DO i = 1,n
        old = xi
        sum = bi
        DO j = 1,n
            IF i≠j THEN sum = sum - ai,j*xj
        END DO
        xi = lambda*sum +(1.-lambda)*old
        IF sentinel = 1 AND xi ≠ 0. THEN
            ea = ABS((xi - old)/xi)*100.
            IF ea > es THEN sentinel = 0
        END IF
    END DO
    iter = iter + 1
    IF sentinel = 1 OR (iter ≥ imax) EXIT
END DO
END Gseid
```

‘Start Iteration

Iter=1

Do While Iter<MaxIterations

Sentinel=1

For i= 0 to n-1

Old=x(i)

x(i)= BC(i)

For j= 0 to n-1

If i<>j Then x(i)=x(j)-AC(i,j)*x(j)

Next j

x(i)=Lambda*x(i)+(1-Lambda)*Old

Relaxation:

$$x_i^{new} = \lambda x_i^{new} + (1 - \lambda) x_i^{old}$$

$$0 < \lambda < 2$$

$\lambda=1$ Original Gauss-Seidel

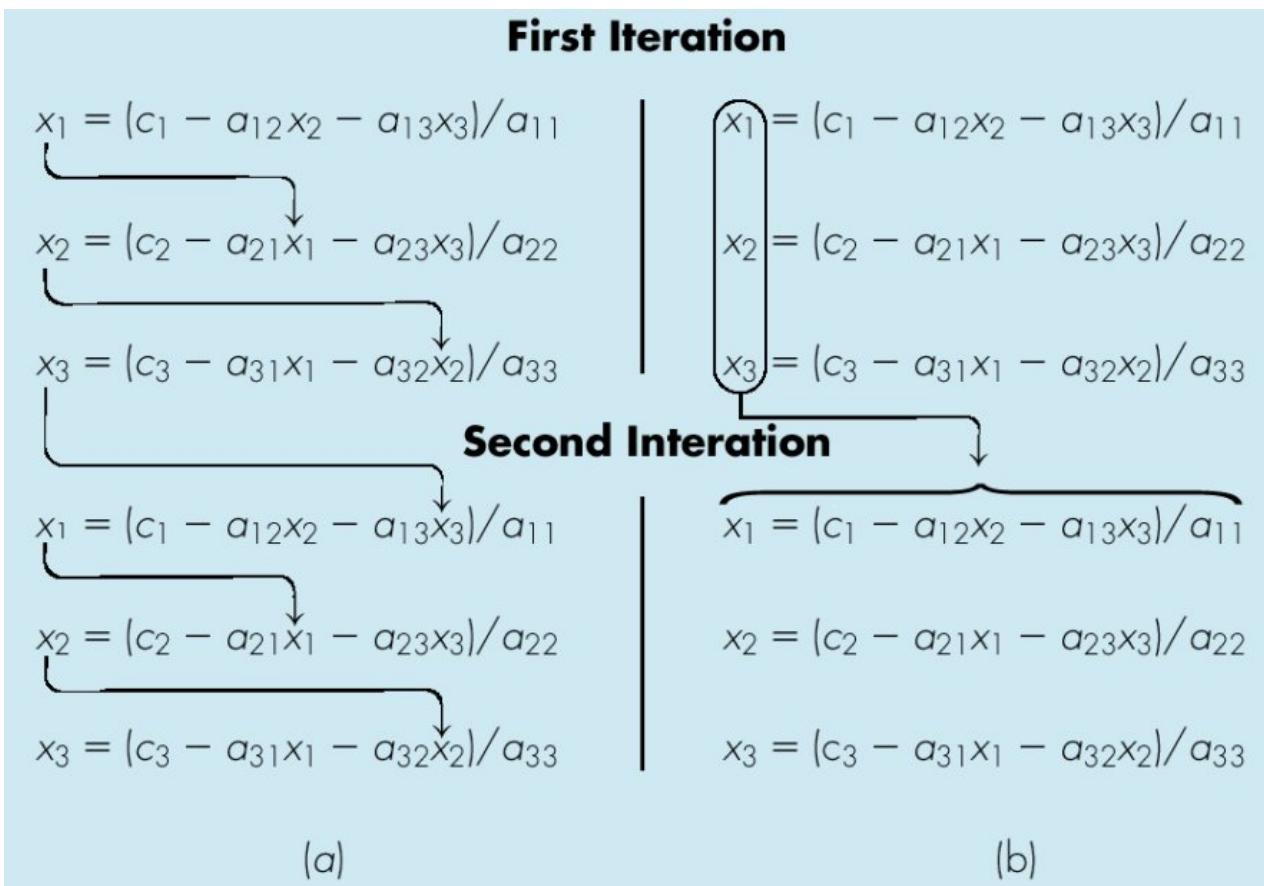
$\lambda < 1$ underrelaxation

$\lambda > 1$ overrelaxation

Iterative Method 2: Jacobi

Compute a set of new x values before using them in the equations.

Gauss-Seidel



Example:

$$\begin{array}{l}
 8x_1 + x_2 - x_3 = 8 \\
 -x_1 - 7x_2 + 2x_3 = -4 \\
 2x_1 + x_2 + 9x_3 = 12
 \end{array}
 \quad
 \begin{array}{l}
 x_1 = 1 - 0.125x_2 + 0.125x_3 \\
 x_2 = 0.571 + 0.143x_1 + 0.286x_3 \\
 x_3 = 1.333 - 0.222x_1 + 0.111x_2
 \end{array}$$

	0	1	2	3	4	5	6	7	8	9	10	11
x_1	0.0000	1.0000	1.0953	0.9940	0.9926	1.0010	1.0005	0.9999	1.0000	1.0000	1.0000	1.0000
x_2	0.0000	0.5710	1.0952	1.0272	0.9901	0.9985	1.0009	1.0001	0.9999	1.0000	1.0000	1.0000
x_3	0.0000	1.3330	1.0476	0.9683	0.9983	1.0027	0.9999	0.9998	1.0000	1.0000	1.0000	1.0000

Jacobi

	0	1	2	3	4	5	6
x_1	0.0000	1.0000	1.0397	0.9968	1.0006	0.9999	1.0000
x_2	0.0000	0.7140	1.0148	0.9966	1.0004	0.9999	1.0000
x_3	0.0000	1.0317	0.9895	1.0011	0.9998	1.0000	1.0000

Gauss-Seidel

$\varepsilon_s < 10^{-4}$

Matrix Norms and Matrix Condition Number

Norm is the largest sum
of the columns

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$$

Norm is largest sum of
the rows

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

“Uniform Matrix Norm” or
“Row-Sum Norm”

Condition Number of
Matrix A (>1)

$$Cond[A] = \|A\| \|A^{-1}\|$$

System of Nonlinear equations

N nonlinear equations in n unknowns

$$f_1(x_1, x_2, x_3, x_4, \dots, x_n) = 0$$

$$f_2(x_1, x_2, x_3, x_4, \dots, x_n) = 0$$

....

$$f_n(x_1, x_2, x_3, x_4, \dots, x_n) = 0$$

Sometimes you can use a fixed point iteration or an iteration similar to Gauss Seidel Iteration for linear equations.

Newton Raphson method for n variables

Newton Raphson Method for one variable:

$$f_{1,i+1} = f_{1,i} + \frac{\partial f_{1,i}}{\partial x_1} (x_{1,i+1} - x_{1,i}) + \frac{\partial f_{1,i}}{\partial x_2} (x_{2,i+1} - x_{2,i}) + \dots + \frac{\partial f_{1,i}}{\partial x_n} (x_{n,i+1} - x_{n,i}) + HOTerms \approx 0$$

$$f_{2,i+1} = f_{2,i} + \frac{\partial f_{2,i}}{\partial x_1} (x_{1,i+1} - x_{1,i}) + \frac{\partial f_{2,i}}{\partial x_2} (x_{2,i+1} - x_{2,i}) + \dots + \frac{\partial f_{2,i}}{\partial x_n} (x_{n,i+1} - x_{n,i}) + HOTerms \approx 0$$

....

$$f_{n,i+1} = f_{n,i} + \frac{\partial f_{n,i}}{\partial x_1} (x_{1,i+1} - x_{1,i}) + \frac{\partial f_{n,i}}{\partial x_2} (x_{2,i+1} - x_{2,i}) + \dots + \frac{\partial f_{n,i}}{\partial x_n} (x_{n,i+1} - x_{n,i}) + HOTerms \approx 0$$

$$\frac{\partial f_{1,i}}{\partial x_1} x_{1,i+1} + \frac{\partial f_{1,i}}{\partial x_2} x_{2,i+1} + \dots + \frac{\partial f_{1,i}}{\partial x_n} x_{n,i+1} = -f_{1,i} + \frac{\partial f_{1,i}}{\partial x_1} x_{1,i} + \frac{\partial f_{1,i}}{\partial x_2} x_{2,i} + \dots + \frac{\partial f_{1,i}}{\partial x_n} x_{n,i}$$

$$\frac{\partial f_{2,i}}{\partial x_1} x_{1,i+1} + \frac{\partial f_{2,i}}{\partial x_2} x_{2,i+1} + \dots + \frac{\partial f_{2,i}}{\partial x_n} x_{n,i+1} = -f_{2,i} + \frac{\partial f_{2,i}}{\partial x_1} x_{1,i} + \frac{\partial f_{2,i}}{\partial x_2} x_{2,i} + \dots + \frac{\partial f_{2,i}}{\partial x_n} x_{n,i}$$

....

$$\frac{\partial f_{n,i}}{\partial x_1} x_{1,i+1} + \frac{\partial f_{n,i}}{\partial x_2} x_{2,i+1} + \dots + \frac{\partial f_{n,i}}{\partial x_n} x_{n,i+1} = -f_{n,i} + \frac{\partial f_{n,i}}{\partial x_1} x_{1,i} + \frac{\partial f_{n,i}}{\partial x_2} x_{2,i} + \dots + \frac{\partial f_{n,i}}{\partial x_n} x_{n,i}$$

$$J_i x_{i+1} = B_i$$

$$J_i = \begin{bmatrix} \frac{\partial f_1}{\partial x_{1,i}} & \frac{\partial f_1}{\partial x_{2,i}} & \dots & \frac{\partial f_1}{\partial x_{n,i}} \\ \frac{\partial f_2}{\partial x_{1,i}} & \frac{\partial f_2}{\partial x_{2,i}} & \dots & \frac{\partial f_2}{\partial x_{n,i}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_{1,i}} & \frac{\partial f_n}{\partial x_{2,i}} & \dots & \frac{\partial f_n}{\partial x_{n,i}} \end{bmatrix}$$

$$B_i = \begin{bmatrix} -f_{1,i} + \frac{\partial f_1}{\partial x_{1,i}} x_{1,i} + \frac{\partial f_1}{\partial x_{2,i}} x_{2,i} + \dots + \frac{\partial f_1}{\partial x_{n,i}} x_{n,i} \\ -f_{2,i} + \frac{\partial f_2}{\partial x_{1,i}} x_{1,i} + \frac{\partial f_2}{\partial x_{2,i}} x_{2,i} + \dots + \frac{\partial f_2}{\partial x_{n,i}} x_{n,i} \\ \dots \\ -f_{n,i} + \frac{\partial f_n}{\partial x_{1,i}} x_{1,i} + \frac{\partial f_n}{\partial x_{2,i}} x_{2,i} + \dots + \frac{\partial f_n}{\partial x_{n,i}} x_{n,i} \end{bmatrix}$$

Solve for x_{i+1} using any one of the methods discussed and repeat till there is convergence

Or, if we let $u_{i+1} = x_{i+1} - x_i$ then

$$J_i u_{i+1} = C_i$$

Where

$$C_i = \begin{bmatrix} -f_{1,i} \\ -f_{2,i} \\ \dots \\ -f_{n,i} \end{bmatrix}$$

Solve for u_{i+1} , $x_{i+1} = u_{i+1} + x_i$

Or

Let: $\delta x_{k,i} = x_{k,i+1} - x_{k,i}$

Then: $A_i \delta_i = B_i$

$$A_i = J_i = \begin{bmatrix} \frac{\partial f_1}{\partial x_{1,i}} & \frac{\partial f_1}{\partial x_{2,i}} & \dots & \frac{\partial f_1}{\partial x_{n,i}} \\ \frac{\partial f_2}{\partial x_{1,i}} & \frac{\partial f_2}{\partial x_{2,i}} & \ddots & \frac{\partial f_2}{\partial x_{n,i}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_{1,i}} & \frac{\partial f_n}{\partial x_{2,i}} & \ddots & \frac{\partial f_n}{\partial x_{n,i}} \end{bmatrix} \quad B_i = \begin{bmatrix} -f_{1,i} \\ -f_{2,i} \\ \dots \\ -f_{2,i} \end{bmatrix} \quad \delta_i = \begin{bmatrix} \delta_{1,i} \\ \delta_{2,i} \\ \dots \\ \delta_{2,i} \end{bmatrix}$$

$$x_{k,i+1} = x_{k,i} + \delta x_{k,i}$$