Chapter 3

General Random Variables

3.1 Continuous Random Variables

Definition 9 *A random variable X is continuous if there is a nonnegative function f_X, called the probability density function (PDF) such that*

$$P(X \in B) = \int_{x \in B} f_X(x)dx$$

for every subset B of the real line.

The probability that the value of X falls within an interval is

$$\mathrm{P}(a \leq X \leq b) = \int_a^b f_X(x)dx,$$

which can be interpreted as the area under the graph of the PDF.

3.1.1 Properties of PDF

If $f_X(x)$ is a PDF , the following hold.

1. Nonnegativity: $f_X(x) \geq 0$
2. Normalization property:
3. (for small δ) $\mathrm{P}(x < X \leq x + \delta) = \int_x^{x+\delta} f_X(a)da \approx$

By the last item, $f_X(x)$ can be viewed as the "probability mass per unit length near x". Although it is used to evaluate probabilities of some events, $f_X(x)$ is not itself an event's probability. It tells us the relative concentration of probability around the point x.

Ex: ($f_X(x)$ may be larger than 1)

$$f_X(x) = \begin{cases} cx^2 & , 0 \leq x \leq 1 \\ 0 & , o.w. \end{cases}$$

1. Find c.
2. Find $P(|X|^2 \leq 0.5)$.

Ex: (A PDF can take arbitrarily large values) Sketch the following PDF.

$$f_X(x) = \begin{cases} c/(\sqrt{x}) & , |x| \leq 2 \\ 0 & , o.w. \end{cases}$$

3.1.2 Some Continuous Random Variables and Their PDFs

Continuous Uniform R.V.

We sometimes have information only about the interval of a random variable and nothing else. A PDF used very commonly in such a case is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & , o.w. \end{cases}.$$

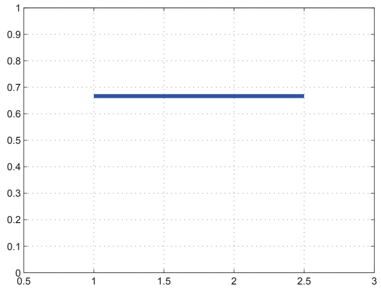

Figure 3.1: Uniform PDF

Gaussian R.V.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

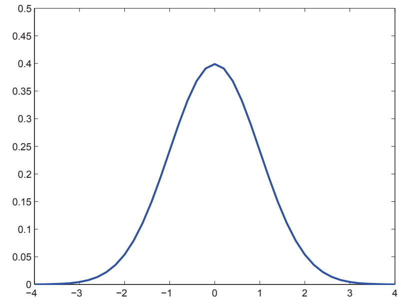

Figure 3.2: Gaussian (normal) PDF

Exponential R.V.

An exponential r.v. has the following PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0, & \text{o.w.} \end{cases},$$

where λ is a positive parameter.

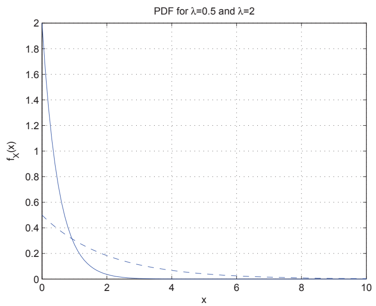

Figure 3.3: Exponential PDF