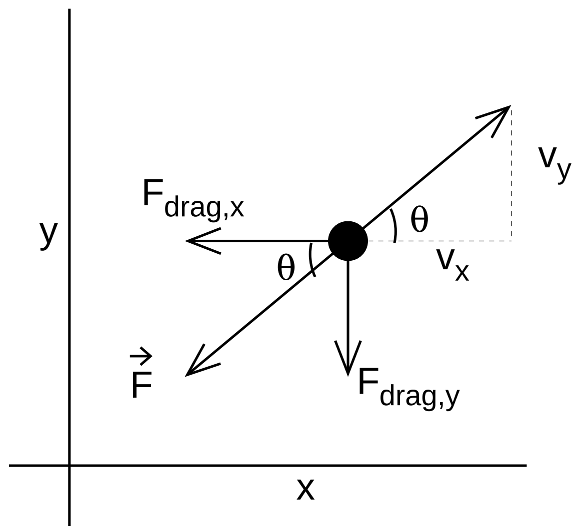
Lecture IV : Realistic motion in multiple dimentsions — cannon and baseball

I. CANNON BALL (CONTINUED)

In the last lab session, you wrote a program that calculates the range of a cannonball launched with a given angle and initial velocity in the absence of air drag. In this lecture and in your homework, we are going to look at how air drag effects the range and the trajectory.

A. Constant air drag

For the velocities that are relevant for a cannon shell, we shall use the drag force that goes like v^2 again. The additional complication here is that the drag force is proportional to the square of the norm of the velocity but its direction is opposite to the velocity at every instant of the motion. We thus have to break it down to its components as you see in the following figure.



The components of the drag force are

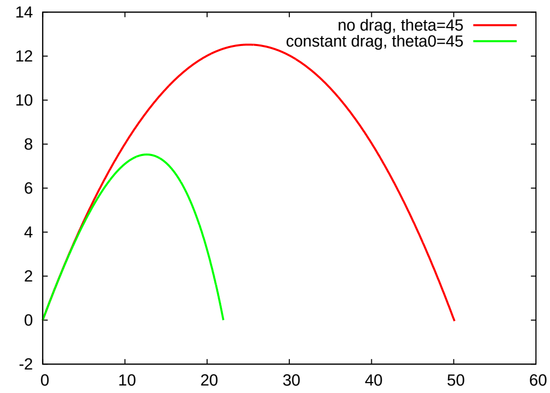
$$\begin{aligned} F_{drag,x} &= -B_2 v^2 \cos\theta = -B_2 v^2 \frac{v_x}{v} = -B_2 v v_x \\ F_{drag,y} &= -B_2 v^2 \sin\theta = -B_2 v^2 \frac{v_y}{v} = -B_2 v v_y \end{aligned} \tag{1}$$

Adding this to the set of discretized equations for motion without drag, we have

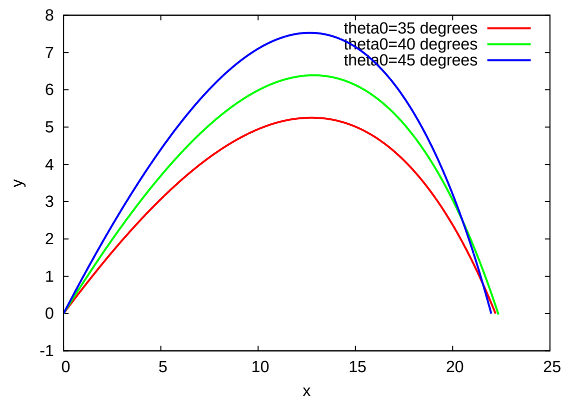
$$\begin{aligned} x_{n+1} &= x_n + v_{x,n}\Delta t \\ v_{x,n+1} &= v_{x,n} - \frac{B_2 v v_{x,n}}{m}\Delta t \\ y_{n+1} &= y_n + v_{y,n}\Delta t \\ v_{y,n+1} &= v_{y,n} - g\Delta t - \frac{B_2 v v_{y,n}}{m}\Delta t \end{aligned} \tag{2}$$

In your homework, you will make a few changes to the code without the drag and obtain the trajectory for a set of angles. You will see that there are many notable differences between the trajectories in the resence and absence of drag.

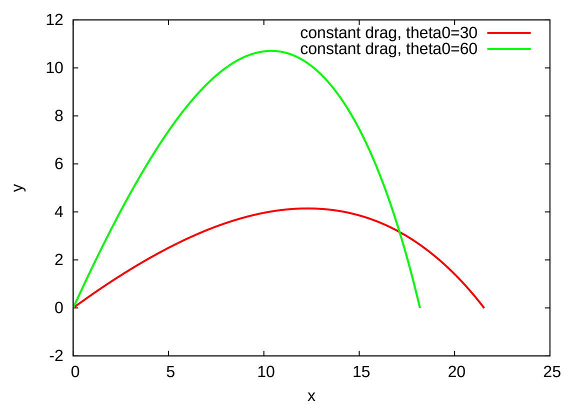
1. The range has fallen to about half of its original value.



2. The trajectories are no longer parabolic and a launch angle of 45^o no longer gives the maximum range.



3. Launch angles whose sums are 90^o no longer give identical trajectories.



B. Effect of altitude

As is well-known, the density of the atmosphere drops with increasing altitude. Among the many factors that effect the density profile of the atmosphere with respect to altitude are the effect of gravity, collisions between air molecules, conductivity of air molecules and the decreasing temperature gradient (Why does it decrease?). Most of the physics associated with trajectories happen in the troposphere, which is the lowest 10-20 km. of the atmosphere. It is also the densest layer and contains about 75% of the mass of the entire atmosphere. Assuming (quite correctly) that air is a poor conductor of heat and using the adiabatic approximation, we find the density profile to be

$$\rho(y) = \rho_0 \left(1 - \frac{ay}{T_0}\right)^{\alpha} \tag{3}$$

where α and a are constants that are derived from experimental data and ρ_0 and T_0 are respectively the density and the temperature at the sea level.

For regular trajectories, such as that of a soccerball, we don't expect the variations in the density profile to affect the motion very much. However, cannonballs typically travel up to altitudes of several kilometers. In that case, we expect there to be a substantial change from the constant friction case.

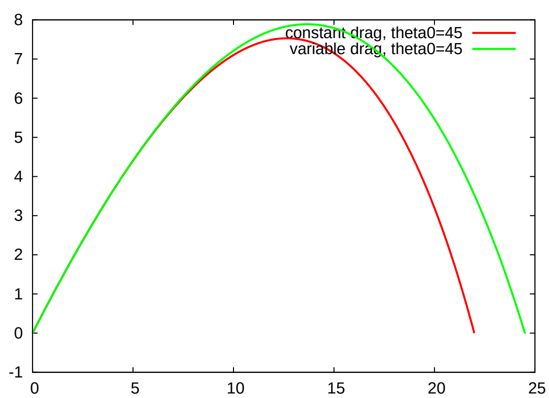
Even though this adds an analytic complexity to the problem, which is very hard to tackle even with computer programmes that deal with analytical expressions, it requires almost no work to solve this problem analytically. All we have to do is add a term to modify the drag force in Eq. 2 to contain a height dependent B_2, namely

$$B_2 = B_2(y) = \frac{1}{2} C A \rho(y) = \frac{1}{2} C A \rho_0 \left(1 - \frac{ay}{T_0}\right)^{\alpha} \tag{4}$$

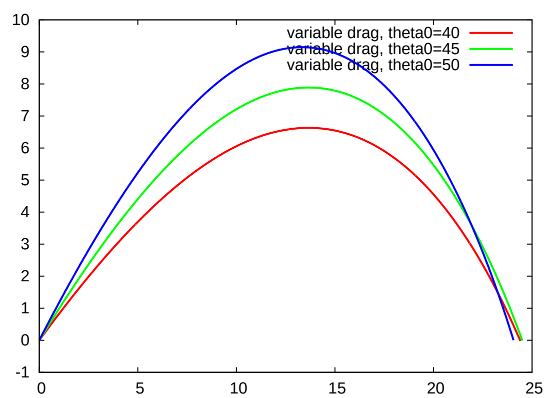
Because of the great altitudes that a cannonball reaches when launched, this variation in the atmospheric density makes a large difference in the trajectories. Employing the following emprically determined constants

Symbol	Value
`T0`	300 K
`a`	6.5×10^{-3} K/m
`alpha`	2.5
`m`	10 kg

and making the necessary modification in the program `cannon_constant_drag.m` while saving it as `cannon_variable_drag.m`, we obtain the following results :



Comparison to the constant drag case

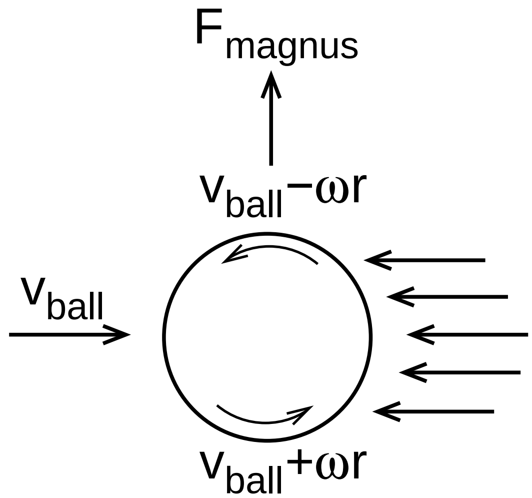


Different angles for variable drag

On the left hand side, we see a graph that compares the ranges for the variable and the constant drag cases. The range for the variable drag is several kilometers longer as expected because the average drag is less in comparison to the constant drag case. On the right hand side, we see the ranges corresponding to 40^o, 45^o and 50^o. Although 45^o is probably still not the exact angle for maximum range, it is closer to being the maximum than it was in the constant drag case.

II. BASEBALL : MOTION IN THREE DIMENSIONS AND THE EFFECT OF SPIN

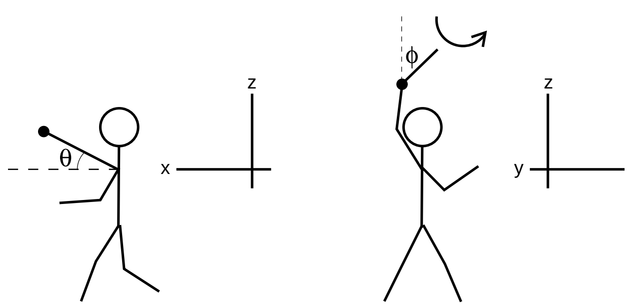
When a baseball is thrown, in addition to the gravity and air drag, it experiences an additional force perpendicular to its trajectory called *the Magnus force*. This force arises due to the spin of the baseball and the direction of the stiches with respect to the direction of travel. The direction of the Magnus force is perpendicular to the velocity of the ball pulling it along the side spinning against the velocity.



Many scientists have tried to give a complete explanation to the exact origin of the Magnus with varying success. Some explanations follow :

- Adair : When a person throws a ball spinning with a rate of 1800 rpm and the ball travels with a mean velocity of 110 kmph, faster-spinning side travels with a velocity of about 130 kmph and the slower side travels with about 95 kmph. Thus there will be a large difference between the two sides, resulting in an upward force.
- Magnus : "A spinning ball induces in the air around it a kind of whirlpool of air in addition to the motion of air past the ball as the ball flies through the air." So there's less flow of air on the slower air flow past the lower side of the ball and faster air flow on the upper side of the ball. According to Bernoulli's theorem, faster flow corresponds to lower pressure resulting in an upward force.
- Briggs : The faster-traveling side causes more turbulance behind the ball, making it easier for the ball to slip in air. Thus, there is a force in the upward direction.

Let's first orient ourselves by discussing the motion of the baseball pitcher. There are two angles associated with the throw. The first angle θ is the usual elevation angle, which determines the initial components of the velocity of the ball. The second is the axis of rotation, ϕ. Both angles are demonstrated in the figure above. While θ changes throughout the motion, ϕ and ω are assumed to stay the same.



In view of the above axes, the vectoral angular velocity, $\vec{\omega}$ is $\vec{\omega} = \omega(0, \sin\phi, \cos\phi)$.

The direction of the Magnus force is perpendicular to both the axis of rotation and the direction of the velocity. We may thus find its direction by taking the cross-product between these two vectors. Its magnitude is proportional to both the linear and angular velocity. The constant of proportionality is an experimentally determined value, which we'll call S_0 here. Thus, the Magnus force is

$$\vec{F}_M = S_0\,\vec{\omega} \times \vec{v} \tag{5}$$

Let's now evaluate the cross product in Eq. 5. We use the usual determinantal method for this.

$$\vec{\omega} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & \omega \sin\phi & \omega\cos\phi \\ v_x & v_y & v_z \end{vmatrix} = \omega[(v_z \sin\phi - v_y \cos\phi)\hat{\imath} + v_x \cos\phi \hat{\jmath} - v_x \sin\phi \hat{k}] \tag{6}$$

In addition to the Magnus force that acts laterally, we still have the drag force that acts on the ball in a direction that is opposite the direction of velocity. While working with the cannonball, we assumed that the drag coefficient is independent of velocity. This is a fairly good assumption considering that the cannonball is massive enough to be unaffected by changes in the surrounding air flow when its velocity increases. In the case of the baseball however, the drag coefficient is no longer a constant and depends on the velocity. The additional complication arises here because the baseball is small enough and it goes up to high enough speeds for the drag coefficient to exhibit velocity-dependence. At lower speeds, the air flow around the ball is *laminar*, that is to say, smooth. At lower speeds, the flow is *turbulent*. At lower speeds, laminar flow creates a large pressure difference between the front and back surfaces of the ball causing a large drag. Turbulent flow on the other hand decreases this difference and decreases the drag on the ball. Thus, the drag coefficient is now velocity-dependent. The transition between the laminar and turbulent regimes is reflected in the plot below.

To summarize, we need to include three types of forces in our equations of motion :

1. The usual drag force
2. The lateral Magnus force
3. The vertical gravity

The equations of motion may thus be written as follows

$$\frac{dx}{dt} = v_x \qquad \frac{dv_x}{dt} = -F(v)\, v\, v_x + \frac{S_0}{m}\,\omega\,(v_z\, \sin\phi - v_y\, \cos\phi) \tag{7}$$

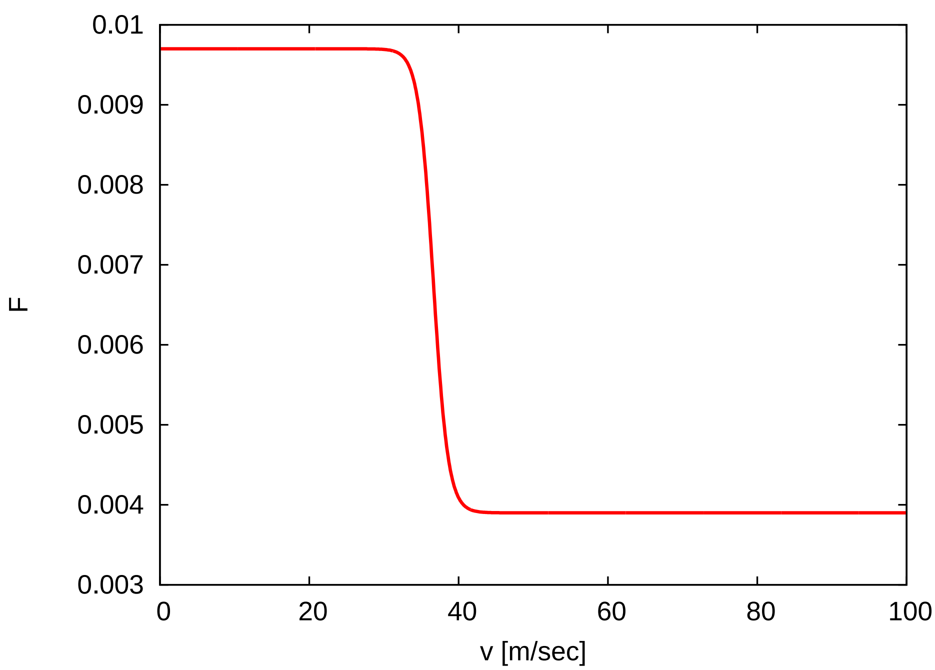
$$\tag{8}$$

$$\frac{dy}{dt} = v_y \qquad \frac{dv_y}{dt} = -F(v)\, v\, v_y + \frac{S_0}{m}\,\omega\, v_x\, \cos\phi \tag{9}$$

$$\frac{dz}{dt} = v_z \qquad \frac{dv_z}{dt} = -g - F(v)\, v\, v_z - \frac{S_0}{m}\,\omega\, v_x\, \sin\phi \tag{10}$$

$$\tag{11}$$

In the above equations, we have introduced a different drag coefficient, $F(v)$, than the one we have been using so far. Beware that it also includes the mass in it. As described in the textbook by Giordano, through wind tunnel experiments, $F(v)$ has been measured experimentally for a baseball and a functional form has been fitted as in the following graph and formula.



$$F(v) = 0.0039 + \frac{0.0058}{1 + \exp[(v - v_d)/\Delta]} \tag{12}$$

where $v_d = 35$ m/sec and $\Delta = 5$ m/sec are the parameters of the fit. If we let $F(v) = m\,B_2$, we go back to the previous case with the cannon ball where the drag coefficient is a constant.