

Lecture VI : Solving partial differential equations, waves in strings

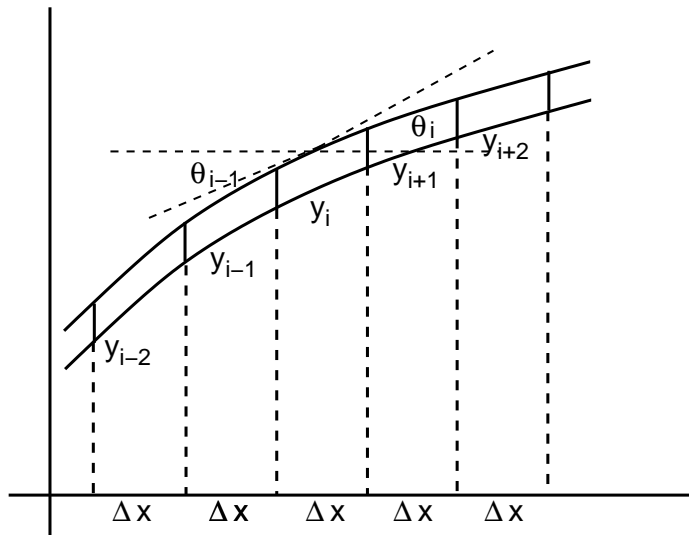
I. THE WAVE EQUATION

Waves occur in various media ranging from vacuum (electromagnetic waves) to water to strings of musical instruments. The equation describing a traveling wave, that is a wave who carries momentum, is given by a second order partial differential equation as follows (since we are going to be dealing with waves in strings, we are going to deal with waves in one-dimension)

$$\frac{\partial^2 y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 y(x,t)}{\partial x^2}$$

where c is the velocity of waves in the medium in question, x is the horizontal coordinate and $y(x,t)$ is the vertical displacement as a function of the horizontal coordinate and time. This equation describes a rather unrealistic string which does not show any resistance to bending. We will now prove this equation, which will simultaneously facilitate casting it into our `Octave` code.

As we normally do in computer simulations, the first step in dealing with a continuum equation like the wave equation is to *emph* discretize it, which is to say dividing it into small elements. Let's divide the x coordinate up into pieces of equal length, Δx , labeled with subscript i , $x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots$. The corresponding y coordinates are labeled as $y_i = y(x_i)$. Let's focus on a particular segment i and concentrate on its vertical motion. For an ideal string, the only vertical force on any such segment is the vertical component of the tension, T as can be seen in the figure below.



On the particular segment we are interested in, labeled by i , this is given by

$$F_{i,y} = T \sin \theta_i - T \sin \theta_{i-1} \approx T \left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i+1}} \right) = \frac{T}{\Delta x} (y_{i+1} + y_{i-1} - 2y_i)$$

The approximate equation is written for small discrete segments.

By Newton's second law, the vertical force on this segment should also be related to the vertical component of its acceleration. We thus have

$$\begin{aligned} m_i \frac{\partial^2 y_i}{\partial t^2} &= T \frac{y_{i+1} + y_{i-1} - 2y_i}{\Delta x} \Rightarrow \\ \rho \Delta x \frac{\partial^2 y_i}{\partial t^2} &= T \frac{y_{i+1} + y_{i-1} - 2y_i}{\Delta x} \Rightarrow \\ \frac{\partial^2 y_i}{\partial t^2} &= \frac{T}{\rho} \frac{y_{i+1} + y_{i-1} - 2y_i}{\Delta x^2} \end{aligned}$$

The above equation is reminiscent of the wave equation up above. If we set $c = \sqrt{T/\rho}$ and prove that the fraction on the right hand side is equal to the second derivative with respect to the horizontal coordinate then we'll have proven the wave equation. We can do this by remembering the finite difference (discrete) form of the first derivative with respect to x . The difference this time is going to be that instead of the off-centered definition of the derivative we have given previously, we'll give a centered definition making use of imaginary points between successive x_i 's.

$$\begin{aligned}\frac{\partial y}{\partial x} &\approx \frac{y_{i+1} - y_i}{\Delta x} && \text{(off-centered difference)} \\ \frac{\partial y}{\partial x} &\approx \frac{y_{i+1/2} - y_{i-1/2}}{\Delta x} && \text{(centered difference)}\end{aligned}$$

Applying the centered difference twice yields

$$\frac{\partial}{\partial x} \left(\frac{\partial y}{\partial x} \right) \approx \frac{y_{i+1} + y_{i-1} - 2y_i}{\Delta x^2}$$

which is identical to the expression in the equation above. We have thus proven the wave equation.

II. THE SIMULATION

Let's now extend this discretization process to the time derivative by means of choosing an appropriate time step, Δt . The criteria for choosing Δx and Δt will be discussed later. Discretizing the wave equation both in time and in space yields the following

$$y(i, n+1) + y(i, n-1) - 2y(i, n) = \left(\frac{c \Delta t}{\Delta x} \right)^2 (y(i+1, n) + y(i-1, n) - 2y(i, n))$$

where we have designated the spatial steps with the discrete index i and the temporal steps with n . In our simulation we are interested in finding the profile of the string at the next time step given the profile at previous time steps. We may thus isolate $y(i, n+1)$ at the same time giving a special name r to the combination of the constants involved. Thus, we have

$$y(i, n+1) = r^2[y(i+1, n) + y(i-1, n)] + 2(1-r^2)y(i, n) - y(i, n-1)$$

Now, we are ready to cast this equation into computer program. Given an initial profile, we can now calculate the profile at all future steps.

A. Choice of discrete steps

In contrast to the earlier examples, where we were only concerned with the choice of a spatial step, this time we are faced with two such choices that we need to make, namely Δx and Δt . Through simple considerations, we see that choosing each of them independently from the other may at best cost us a major loss of accuracy. It turns out that the optimum stability is achieved for the simulation when r is chosen to be 1.

- If r is chosen to be larger than 1, the simulation is unstable and the iterations diverge. This is because information can travel along the string with a maximum speed of c . Choosing r larger than 1 is equivalent to choosing $\frac{\Delta x}{\Delta t}$ larger than c , which is unphysical.
- Choosing r on the other hand does yield a stable solution, however, we lose the benefit of *cancellation of errors* that we have if r is chosen to be 1.

B. Boundary conditions

From elementary courses, we are familiar with the fact that any function having the form

$$y(x, t) = f(x + ct) - g(x - ct)$$

will be a solution to the wave equation. What yields the particular solution for a given system is the boundary conditions. There are several boundary conditions we may enforce :

1. Fixed boundary conditions : Enforce the segments at the ends to have the same displacement as they did in the previous time step.
2. Free boundary conditions : Let the segments at the ends evolve according to the equations of motion. We need to be careful in this case because the segments at the ends have a single neighbor in contrast with the segments in the inner part of the string.
3. Driven boundary conditions : Drive one end of the string with a time-dependent modulation while keeping the other fixed.
4. Strings of different mass : You can have strings of different density, i.e. different velocities attached at a given point, probably in the middle.

If coded up correctly, your code should display the reflection characteristics associated with the particular boundary conditions.