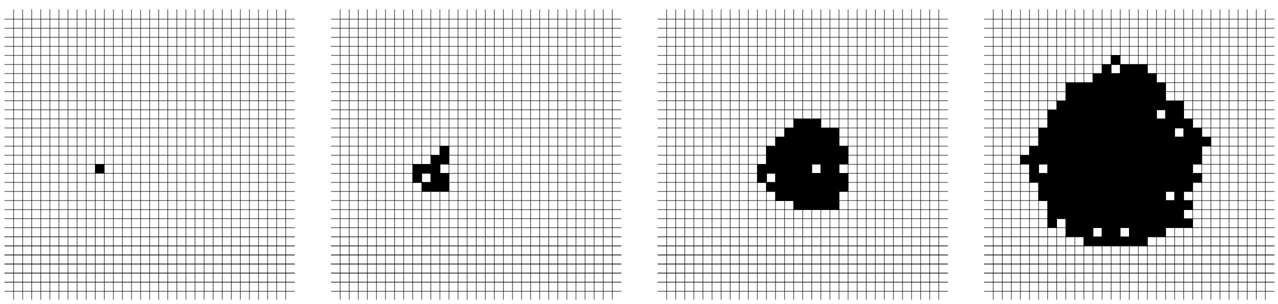
Lecture VIII : Fractals and fractal dimension

I. RANDOM WALK AND CLUSTER GROWTH

One of the most commonly seen examples of random walk and random systems in nature is *cluster formation*. The term *cluster* often refers to a collection of particles having, to a greater or lesser extent, an irregular shape. These particles could be atoms (Fe clusters), molecules (H_2O clusters) and even clusters (C_{60} clusters). In nature there are various examples of clusters, mostly formed as a result of random processes. We'll start this lecture by discussing two different kinds of clusters formed by such random processes : Eden clusters and DLA (diffusion-limited aggregation) clusters. We'll first define them and discuss how they are formed and later we'll look at a fundamental difference between these to clusters regarding their dimensionality.

A. Eden clusters

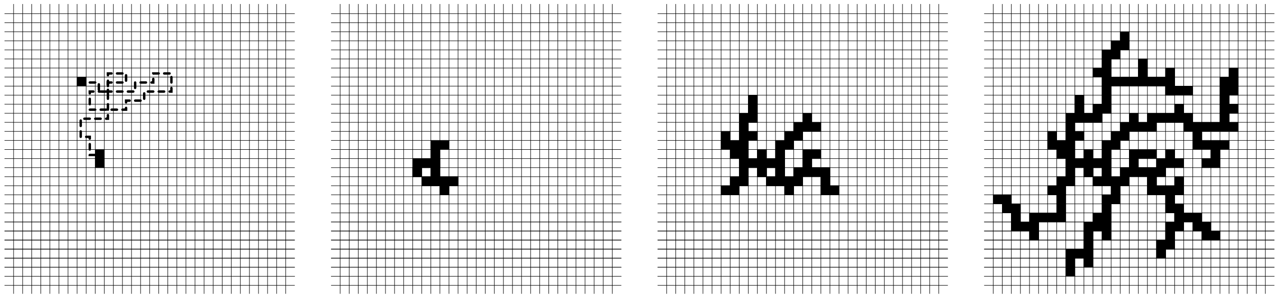
Suppose you have a regular grid in two dimensions like the one you see in the figure below. Select a cell at random and place a particle in that cell labeling it as occupied. Select a second cell in the grid and check whether that cell or any of its nearest neighbors (left, right, under or over) are occupied. If the cell is empty and it has at least one occupied neighbor, then place the particle in that cell and label it as occupied. If the cell has no neighbors, discard that particle and select a grid cell at random again. If you carry out this process for a large number of added particles, you obtain a more or less circular, compact cluster with few holes in the interior region as you can see in the figure below.



As you increase the number of particles in your cluster, you'll observe that holes in the innermost (oldest) regions will fill up. This is a result of the fact that in addition to the *perimeter* sites along the boundary of the clusters, particles have a finite probability of occuring in the internal empty sites. In a way these internal sites are treated on the same footing as the boundary sites. Examples to the use of such clusters include bacterial colonies, tumor growth, epidemic modelling and growth on surfaces of materials.

B. DLA (Diffusion Limited Aggregation) clusters

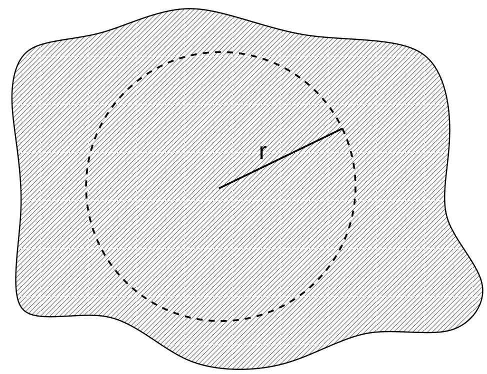
Let's bring back the grid from the previous section and start with the same seed particle in a random cell in the grid. This time place the second particle at random on a grid cell and let it exhibit a 2-dimensional random walk in the grid until it finds the adjacent sites of the first particle. Place the particle on that site and label it as occupied. Then release a third particle at a random site some distance away from the cluster and let it perform random walk again until it finds a perimeter site of the cluster. If a given particle happens to wander off outside of the grid, you may kill your that particle and generate a new one. A cluster grown in this manner is called a DLA cluster and might look like the following cartoon :



In contrast to Eden clusters, the inner regions of DLA clusters are less accessible to the diffusive particles attacking the cluster. For this reason, DLA clusters have a more open, dendritic structure than Eden clusters. Some examples include (among many others) electrodeposition, formation of snowflakes, soot particles, electric discharge models, dielectric breakdown models and sputter deposition of thin films.

II. FRACTALS AND FRACTAL DIMENSIONALITY

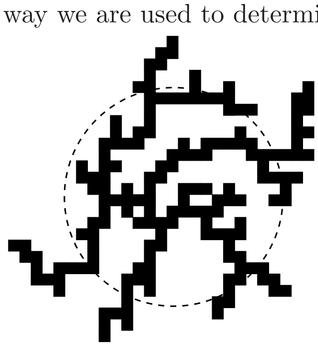
There is a fundamental difference in dimensionality between the two kinds of clusters described above. Although both seem to live in a two dimensional space, DLA clusters seem more like a sparse entangled piece of string than a compact two dimensional object. In other words, they neither seem to be one-dimensional nor two-dimensional. On the other hand, Eden clusters are doubtlessly two dimensional. In order to really determine the dimensionality of these to objects, we must first make a definition of *dimension*.



Consider the object to the right. If we cut out pieces from the interior region of this object in the form of circles of varying radii r the mass of the circular pieces will be proportional to r^2 as long as we stay within the boundary of the object. If instead of this object confined to this page, we had an equally compact object in space, such as a stone, then the same argument would lead to a mass, that would be proportional to r^3. Similarly, if we are talking about a line its mass would increase like r. In that case an operational definition of mass may be given in terms of the mass contained in a radius of r as

$$m(r) = Cr^d \tag{1}$$

where C is a constant and d is the dimensionality of the object. For objects like the disc or the stone, d is one of the three familiar numbers 1, 2 and 3. An Eden cluster is an object whose dimension is going to be one of these three numbers. In other words, the dimensionality of an Eden cluster may be unambigiously determined in the intuitive way we are used to determining the dimensionality of ordinary objects.



If you instead consider the object to the right, its mass as a function of radius does not go like r^2 and neither does it go like r but rather like a real power of r between 1 and 2. Such objects with noninteger dimension are called *fractals* and their dimensionality is called the *fractal dimensionality*. DLA clusters are fractal objects. Other examples from nature include coastlines, ferns, clouds and some rock formations.

The best way to determine fractal dimensionality is to find a property such as mass, length or area that is predicted to depend on an incremental entity such as radius or step length.

$$f(\alpha) = C\alpha^{d_f} \tag{2}$$

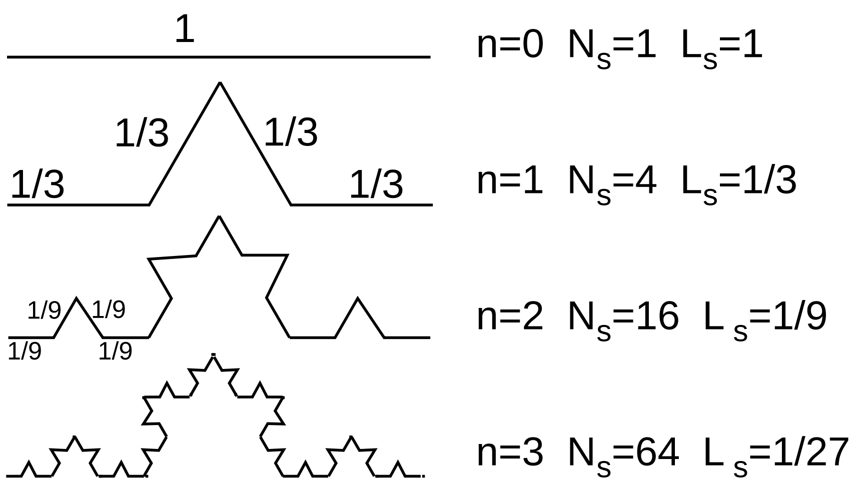
where $f(\alpha)$ takes the place of length, area etc., α replaces radius or step length, C is a constant and d_f is the fractal dimension. The slope of the $\log f$ versus $\log \alpha$ plot then gives us d_f.

All of the examples considered so far are examples of fractals generated by random processes. It is also possible to generate fractals using an analytical or deterministic rules. Such rules are generally based upon a recursive mathematical relation. We will now give an example of such an example of a mathematical fractal, namely the Koch fractal (a.k.a. the Koch snowflake). This is one of the simplest kinds of fractals and it is possible to calculate its dimensionality anaytically.

A Koch snowflake is generated as follows :

1. Start with a line of unit length.
2. Divide the line into three segments.
3. Transform the middle segment into an equilateral triangle by adding a fourth segment.
4. Repeat this procedure with all the line segments.
5. Iterate.

The Koch snowflake, in principle, can be iterated in this way infinitely many times but below is a figure that depicts the succession of four levels.



The effective length of any curve can be determined by taking steps of progressively smaller sizes along the curve and counting the number of steps needed to reach from one end to the other. This length is clearly given by the step length times the number of steps taken along the curve. We thus have the relation

$$L_{\text{eff}} = N_s L_s \propto L_s^{1-d_f} \tag{3}$$

where d_f is the fractal dimensionality.

At the zeroth level ($n = 0$) in the above figure, the step size is 1 and it takes a single step to travel from one end of the curve to the other, which is of unit length. At the next level ($n = 1$), we decrease the step size to $1/3$, which allows us to distinguish the finer structure of the curve and therefore we have to take a larger number of steps ($N_s = 4$) which increases the effective length. If we continue decreasing the length of steps by a factor of three in this manner, we see that number of steps necessary to reach the end increases by 4. This gives us the following relation from Eq. 3

$$N_s = 4^n, \quad L_s = 3^{-n} \Rightarrow n = -\frac{\log L_s}{\log 3} \Rightarrow N_s = 4^{-\log Ls/\log 3} \Rightarrow \log N_s = -\frac{\log 4}{\log 3} \log L_s. \tag{4}$$

Eq. 3 and Eq. 4 together yield the fractal dimensionality of the Koch curve to be $d_f = \frac{\log 4}{\log 3} \approx 1.2$.

As predicted, the dimension of a fractal is between 1 and 2. The reason for this is at every time scale (step size), the complexity of the curve remains the same. The smaller the step size, the more turns and triangles that we discover increasing the length of the curve. A common feature of fractals is this complexity, which shows itself at every length scale and is called *self-similarity*. The above Koch curve is self-similar because if we choose any small portion and zoom into that portion, the structure on this scale is an exact replica of the whole structure.

There are many different mathematical ways of creating fractals but they all rely on a recursive property. Two other well-known examples of such fractals is the Mandelbrot set and the Julia set. They are both generated by pixelizing the complex plane and coloring the pixel according to whether or not the number correspoding to the particular pixel diverges or not under the recursive rule,

$$z = z^2 + k \tag{5}$$

where z is the complex number corresponding to that pixel and k is a fixed complex number. There are several pages devoted to these beautiful geometric shapes on the web and several .m files are published. You can look them up and run them. In the lab this week we are going to write a code that calculates the fractal dimension of an arbitrary curve.