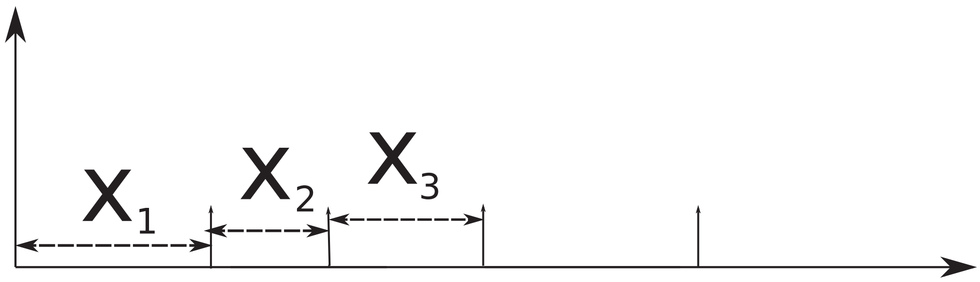
6.2 The Poisson Process

Consider arrivals (of busses, customers, photons, e-mails, etc) occurring at random points in time. We say that the arrival process is a Poisson Process if the times between arrivals are IID, Exponential random variables.



More precisely, let X_1, X_2, X_3, be the sequence of inter-arrival times as shown in the figure. The process is a Poisson process of rate λ if the $\{X_i\}$, $i \geq 1$ are independent and Exponential with rate λ. Note that the mean time between two arrivals is $\frac{1}{\lambda}$.

Ex: I am waiting for the bus, and bus arrivals are known to be a Poisson process at rate 1 bus per 10 minutes. Starting at time $t = 0$, what is the expected arrival time of the third bus?

Distribution of residual time: At an arbitrary time $t > 0$, let R be the duration until the next arrival. This is called the “residual time” because it is only part of the inter-arrival time that t falls into. It is easy to show that R has the same distribution as a regular inter-arrival time. This is a consequence of the Exponential being “memoryless”. It also implies that the Poisson process has the “fresh-start” property.

Ex: Given that I arrive at the bus-stop at $t = 19$ and learn that I have missed the second bus by two minutes, how much do I expect to wait?

Equivalent Definition of the Poisson Process:

An arrival process that satisfies the following is a Poisson process.

1. The probability $\mathrm{P}(k, \tau)$ that there are k arrivals in any time interval of size τ is given by:

$$P(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}.$$

   Note that this is the Poisson PMF, with mean $\lambda\tau$.

2. The numbers of arrivals in disjoint intervals are independent.

Ex: I get email according to a Poisson process at rate $\lambda = 0.1$ arrivals per minute. If I check my email every hour, what is the expected number of new messages I find in my inbox when I check my email? What is the probability that I find no messages? One message? Repeat for an e-mail checking period of two hours.

The time-reversed process is also Poisson: We can show that the reverse residual time distribution is the same as the inter-arrival time distribution.

Ex: In the bus problem, what is the expected number of people on the bus that I get on? (Hint: Consider the people that arrived in the two minutes before I arrived, as well as the people that arrive while I am waiting.)

The random incidence "paradox": When I arrive at random, the interval of time I arrive in has twice the expectation of a regular inter-arrival time. Recall the difference of interviewing bus drivers versus passengers, to understand how crowded a bus is on average.

Relationship to the Bernoulli process: Take a Poisson process at rate λ and discretize time finely, in chunks of size δ. Show that as $\delta \to 0$, the Poisson process can be approximated by a Bernoulli process.

The "Baby Bernoulli" definition of the Poisson process: We can equivalently define a Poisson process as a process where the probability of arrival in any time interval of size δ is $\lambda\delta + o(\delta)$, the probability of more than one arrival is, $o(\delta)$, and arrivals in disjoint intervals are independent.

6.2.1 Splitting and Merging Poisson Processes

Ex: Show that, when we send each arrival of a Poisson process at rate λ to a process A with probability p, and process B with probability $1-p$, the resulting processes A and B are Poisson with rates $p\lambda$ and $(1-p)\lambda$. Note also that processes A and B are independent of each other (this is unlike the Bernoulli case, where we can easily show that the split processes are not independent.) (Hint: Express the transform of the interarrival time as a geometric sum of exponentials.)

Ex: Show that when we merge two INDEPENDENT Poisson processes at rates λ_a and λ_b, we get a Poisson process at rate $\lambda_a + \lambda_b$. (Hint: consider two lightbulbs with exponential lifetimes running side by side. Find the distribution of the time that the first one burns out.)